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# The use of the trapezoidal method for solving the Tacoma Narrows Bridge model

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**Abstract**

In this paper, an efficient method is developed for the approximate solution of a benchmark non-smooth dynamical system. In the proposed method, the trapezoidal method is utilized for solving the Tacoma Narrows Bridge equation. For this purpose, at first, the integral form of the dynamical equation is considered. Afterwards, the obtained integral equation is discretized by the trapezoidal method. The accuracy and performance of the proposed method are examined by means of some numerical experiments.

Keywords: Non-smooth dynamical system, Integral equation, Numerical method
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## 1 Introduction

One of the most challenging problems in engineering which has been attracted the attention of many engineers, physicists, and mathematicians is the Tacoma Narrows Bridge problem. The Tacoma Narrows Bridge was a suspension bridge in the State of Washington which suffered collapse in a strong wind on the morning of November 7, 1940 (see Fig. 1). In many physics textbooks, the event is presented as an example of elementary forced mechanical resonance, but it was more complicated in reality. Accordingly, many researchers considered the reasons of the collapse which we can refer to the remarkable reference [1] in this regard. This paper will not discuss the reasons of this collapse, but instead a simple mathematical model of the problem is considered. It should be noted that the presented model is a very simplified one-dimensional model which can not consider all of the role-playing parameters of the problem. The interested readers are referred to [6] for more complicated models.

Now, consider the following Tacoma Narrows Bridge equation which is taken from [5, 3]. The problem is

$$\begin{aligned} &m\ddot{y}(t) = g(t) + F(y), \quad 0 \le t \le 3\pi, \\ &y(0) = 0, \; \dot{y}(0) = 1, \end{aligned}$$

where

$$F(y) = \begin{cases} -\alpha y, & y \ge 0, \\ -\beta y, & y < 0, \end{cases}$$

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Figure 1: Collapse of the Tacoma Narrows Bridge.

and the constant parameters are considered as $m = 1$, $\alpha = 4$, $\beta = 1$ and $g(t) = \sin(4t)$. For more details on the history and how to model this problem, the interested readers are referred to [4, 6, 3]. The problem has the following analytical solution

$$y(t) = \begin{cases} (\frac{2}{3} - \frac{1}{6}\cos(2t))\sin(2t), & 0 \le t \le \frac{\pi}{2}, \\ (\frac{7}{5} - \frac{4}{15}\sin(t)\cos(2t))\cos(t), & \frac{\pi}{2} \le t \le \frac{3\pi}{2}, \\ (-\frac{11}{15} - \frac{1}{6}\cos(2t))\sin(2t), & \frac{3\pi}{2} \le t \le 2\pi, \\ (-\frac{23}{15} - \frac{4}{15}\cos(t)\cos(2t))\sin(t), & 2\pi \le t \le 3\pi. \end{cases}$$

It is noted that, this problem is actually a non-smooth dynamical system which has a smooth solution. It is simple to show that, by using a substitution

$$y_1 = y,$$
$$y_2 = \dot{y},$$

the order of the problem is reduced to one and the following system of first-order initial value problems is derived

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = \frac{1}{m}(g(t) - \begin{cases} \alpha y_1(t), & y_1(t) \ge 0, \\ \beta y_1(t), & y_1(t) < 0, \end{cases} , \\ y_1(0) = 0, \quad y_2(0) = 1. \end{cases} \tag{1.1}$$

The non-smooth dynamical systems whose right hand side of their dynamical systems or trajectories may not be differentiable everywhere are utilized to model a wide variety of phenomenon, especially in mechanical and control systems [7, 9, 8]. It is necessary to mention that, due to non-smoothness in the right hand side of their dynamical systems or in their solution, the numerical approximation of non-smooth dynamical systems is a very difficult task. The aim of this paper is to present a method for an efficient numerical solution of the non-smooth dynamical equation of the Tacoma Narrows Bridge model. The next section is about introducing this method.

## 2 The proposed approach

Let's go back to the non-smooth initial value problem (1.1). In particular, the non-smooth initial value problem (1.1) can be considered as an initial value problem of the form

$$\begin{cases} \dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t)), & t_0 \leq t \leq t_f, \\ \mathbf{y}(t_0) = \mathbf{y}_0, \end{cases} \tag{2.1}$$

where, $\mathbf{y} = [y_1, \ldots, y_p]^T : [t_0, t_f] \to \mathbb{R}^p$ and $\mathbf{f} = [f_1, \ldots, f_p]^T : [t_0, t_f] \times \mathbb{R}^p \to \mathbb{R}^p$. It is worthwhile to note that, in the Eq. (2.1), the function $\mathbf{f}(t, \mathbf{y}(t))$ is a non-smooth function with respect to $t$ or $\mathbf{y}$. Furthermore, it is assumed that, the Eq. (2.1) has a unique solution. Now, by integrating the dynamical equations in the Eq. (2.1) from $t_0$ to $t$, the equivalent system of Volterra integral equations is induced as

$$\mathbf{y}(t) = \mathbf{y}(t_0) + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau, \quad t_0 \leq t \leq t_f. \tag{2.2}$$

In the following, the trapezoidal integral formula for approximating the Volterra integral equations (2.2) is reviewed. For this purpose, at first, an equally spaced grid

$$t_i = ih, \qquad i = 0, 1, \ldots, N,$$

is considered, where, $hN \leq t_f$ and $h(N+1) > t_f$. Now, for $n > 0$, we can write

$$\mathbf{y}(t_n) = \mathbf{y}_n = \mathbf{y}_0 + \int_{t_0}^{t_n} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau, \quad n = 1, 2, \ldots, N. \tag{2.3}$$

As a general approach, the integral term in the Eq. (2.3) can be approximated by the numerical integration such as

$$\int_{t_0}^{t_n} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau \simeq h \sum_{j=0}^{n} w_{n,j} \mathbf{f}(t_j, \mathbf{y}(t_j)), \quad n = 1, 2, \ldots, N, \tag{2.4}$$

where, the quadrature weights $hw_{n,j}$ are allowed to vary with the grid point $t_n$. So, the Eq. (2.3) is approximated by

$$\mathbf{y}_n \simeq \mathbf{y}_0 + h \sum_{j=0}^{n} w_{n,j} \mathbf{f}(t_j, \mathbf{y}(t_j)), \quad n = 1, 2, \ldots, N. \tag{2.5}$$

Obviously, the Eq. (2.5), defines $\mathbf{y}_n$ implicitly. In other words, the Eq. (2.5) is a set of algebraic equations which can be solved by the root finding methods.

It is noted that, in this paper, the Eq. (2.5) is solved by the simple fixed point iteration method [2] where $h$ is supposed to be sufficiently small. As we can see in the numerical illustrations section, using the fixed point iterations greatly increased the speed of the method. So, as a result, the Eq. (2.5) will find the following form

$$\mathbf{y}_n^{(k+1)} \simeq \mathbf{y}_0 + h \sum_{j=0}^{n-1} w_{n,j} \mathbf{f}(t_j, \mathbf{y}_j) + h w_{n,n} \mathbf{f}(t_n, \mathbf{y}_n^{(k)}), \quad k = 0, 1, \ldots \tag{2.6}$$

for some given initial estimation of $\mathbf{y}_n^{(0)}$. There are many possible schemes for being in the Eq. (2.5). In this paper, the fantastic trapezoidal numerical method will be used. It is worthwhile to note that, the trapezoidal rule has the form

$$\int_{\alpha}^{\alpha+h} F(s)ds \simeq \frac{h}{2}[F(\alpha) + F(\alpha + h)].$$

So, in Eq. (2.4) we have

$$\int_{t_0}^{t_n} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau \simeq \frac{h}{2}\mathbf{f}(t_0, \mathbf{y}_0) + h \sum_{j=1}^{n-1} \mathbf{f}(t_j, \mathbf{y}_j) + \frac{h}{2}\mathbf{f}(t_n, \mathbf{y}_n),$$

and consequently looking at the Eq. (2.6), the trapezoidal method will lead to the following iterative equation

$$\mathbf{y}_n^{(k+1)} \simeq \mathbf{y}_0 + \frac{h}{2}\mathbf{f}(t_0, \mathbf{y}_0) + h \sum_{j=1}^{n-1} \mathbf{f}(t_j, \mathbf{y}_j) + \frac{h}{2}\mathbf{f}(t_n, \mathbf{y}_n^{(k)}), \quad k = 0, 1, \ldots \tag{2.7}$$

It is emphasized again that, the Eq. (2.7) is solved by the simple fixed point iteration method where $h$ is supposed to be sufficiently small. This leads the proposed method being very fast.

## 3 Numerical illustrations

This section is devoted to the numerical illustrations and the effectiveness of the presented method is shown. Noted that, all computations are performed on a 2.53 GHz Core i5 PC Laptop with 4 GB of RAM running in Matlab R2016a.

Now, consider the non-smooth initial value problem (1.1) again. This problem is solved by using the presented method. The approximated solution for $N = 6000$ discretization points is shown in Figure 2 alongside the exact solution, and the absolute error of the approximated solution on the interval $0 \leq t \leq 3\pi$. Moreover, the approximated solutions for different values of $y$ and different values of discretization points $N$, are shown in Table 1. Also, for exploring the dependence of the error of the approximated solution on the parameter $N$, the presented method is applied on this problem for various values of $N$. In Figure 3, an overview of the rate of convergence by plotting the Euclidean norm of error, $E_N$, as a function of $N$ can be seen. Obviously, if $N$ increases, then the Euclidean norm of error will become smaller. Furthermore to better show the efficiency of the method, the CPU time of the presented method versus $N$ is shown in Figure 4 and the log-linear graph for better vision, is plotted. As we can see, the presented method has desirable speed.

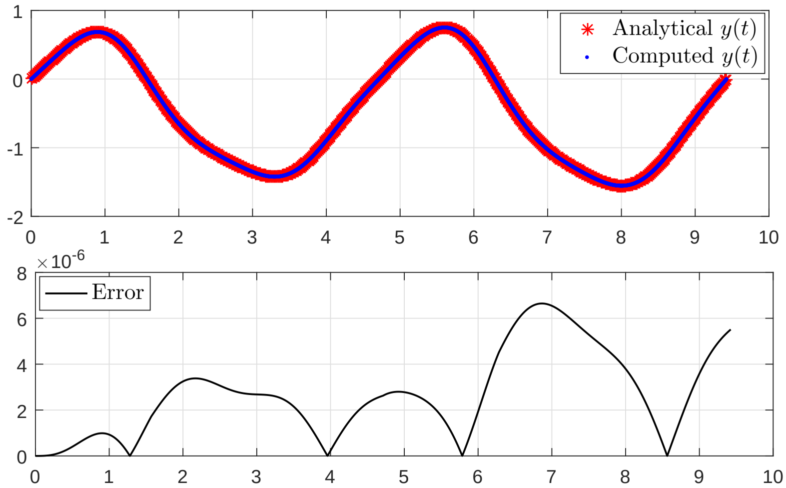


Figure 2: Solution of the Tacoma Narrows Bridge equation.

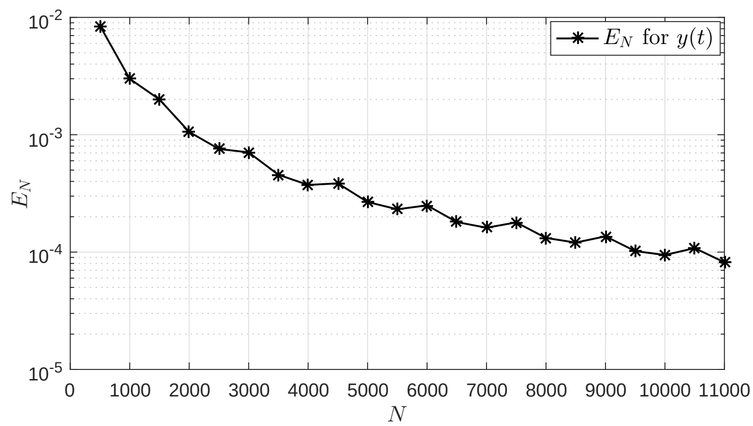


Figure 3: Euclidean norm of error versus $N$.

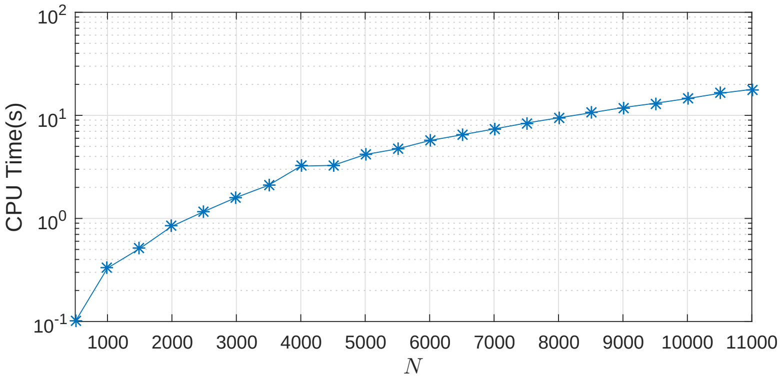


Figure 4: CPU time of the presented method versus $N$.

Table 1: Comparison the different values of $y$, between the proposed method and the analytical solution.

| $t$ | $N$ | The proposed method | CPU time(s) | Analytical solution | Absolute error |
|---|---|---|---|---|---|
| 0 | 500 | 0 | 0.087 | 0 | 0 |
| $\pi/3$ | 1000 | 0.64948838 | 0.276 | 0.64951905 | 3.0673e-05 |
| $\pi/4$ | 1500 | 0.66665205 | 0.493 | 0.66666667 | 1.4620e-05 |
| $3\pi/4$ | 2000 | -0.98992890 | 0.786 | -0.98994949 | 2.0590e-05 |
| $3\pi/2$ | 2500 | -0.00001513 | 1.166 | 0 | 1.5128e-05 |
| $\pi$ | 3000 | -1.39998939 | 1.583 | -1.4 | 1.0610e-05 |
| $3\pi/5$ | 3500 | -0.49602001 | 2.032 | -0.49602756 | 7.5500e-06 |
| $6\pi/5$ | 4000 | -1.17180962 | 2.553 | -1.17180948 | 1.400e-07 |
| $2\pi$ | 4500 | 0.00000804 | 3.108 | 0 | 8.040e-06 |
| $3\pi/8$ | 5000 | 0.55473719 | 3.778 | 0.55473785 | 6.6000e-07 |
| $9\pi/4$ | 5500 | -1.08422513 | 4.517 | -1.08423040 | 5.2700e-06 |
| $3\pi$ | 6000 | -0.00000551 | 5.397 | 0 | 5.5100e-06 |
| $3\pi/4$ | 6500 | -0.98994754 | 6.098 | -0.98994949 | 1.9500e-06 |
| $3\pi/2$ | 7000 | -0.00000193 | 6.974 | 0 | 1.9300e-06 |
| $3\pi/5$ | 7500 | -0.49602565 | 7.816 | -0.49602756 | 1.9100e-06 |
| $6\pi/5$ | 8000 | -1.17180951 | 8.805 | -1.17180948 | 3.0000e-08 |
| $9\pi/4$ | 8500 | -1.08422819 | 9.918 | -1.08423040 | 2.2100e-06 |
| $2\pi$ | 9000 | 0.00000201 | 11.098 | 0 | 2.0097e-06 |
| $9\pi/5$ | 9500 | 0.74642344 | 12.203 | 0.74642355 | 1.1000e-07 |
| $12\pi/5$ | 10000 | -1.39488200 | 13.478 | -1.39488289 | 8.9000e-07 |
| $\pi$ | 10500 | -1.39999913 | 14.863 | -1.4 | 8.7000e-07 |
| $3\pi$ | 11000 | -0.00000164 | 15.831 | 0 | 1.6400e-06 |

## 4 Conclusion

In this paper, the fantastic trapezoidal method is proposed for the numerical solution of the non-smooth Tacoma Narrows Bridge equation. The method can be easily applied to any type of non-smooth initial value problems. According to the numerical illustrations, the accuracy and the speed of the method is satisfactory. Furthermore, by using the simple iteration method for solving the root finding problem appeared in the method, the CPU time of the method is significantly reduced. Further research in the usage of the presented method to solve the non-smooth boundary value problems will be interesting.

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