

# Bounds to the first eigenvalues of weighted $p$ -Steklov and $(p,q)$ -Laplacian Steklov problems

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## Abstract

We consider the Steklov problem associated with the weighted  $p$ -Laplace operator and  $(p,q)$ -Laplacian on submanifolds with the boundary of Euclidean spaces and prove Reilly-type upper bounds for their first eigenvalues.

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## 1 Introduction

Let  $(M^n, g)$  be a compact Riemannian manifold with a possibly non-empty boundary  $\partial M$ . The triple  $(M, g, d\mu_g = e^{-f} dv)$  is called a smooth metric measure space, where  $f : M \rightarrow \mathbb{R}$  is a smooth real-valued function on  $M$  and  $dv$  is the Riemannian volume element related to  $g$ . We also call  $e^{-f}$  the density. For  $1 < p < \infty$  and any  $u \in W_0^{1,p}(M)$ , the  $p$ -Laplacian  $\Delta_p$  is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \operatorname{Hess}u(\nabla u, \nabla u),$$

where  $\operatorname{div}$  is the divergence operator,  $\nabla$  is the gradient operator, and  $\operatorname{Hess}u$  is the hessian of  $u$ . For  $p = 2$ , the  $p$ -Laplacian is the Laplace-Beltrami operator of  $(M^n, g)$ . Also, the weighted  $p$ -Laplacian is defined by

$$\Delta_{p,f} u = e^f \operatorname{div}(e^{-f} |\nabla u|^{p-2} \nabla u).$$

The spectrum of the weighted  $p$ -Laplacian has been studied on smooth metric measure spaces with Dirichlet or Neuman boundary conditions (see [10, 17, 18, 19]). In the present paper, we will consider the Steklov problem associated with the weighted  $p$ -Laplace operator and  $(p,q)$ -Laplacian on submanifolds with boundary of Euclidean spaces.

In following, we will consider the weighted  $p$ -Steklov problem on submanifolds with the boundary of the Euclidean space

$$\begin{cases} \Delta_{p,f} u = 0 & \text{in } M, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial M, \end{cases} \quad (1.1)$$

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where  $\frac{\partial u}{\partial \nu}$  is the derivative of the function  $u$  with respect to the outward unit normal  $\nu$  to the boundary  $\partial M$ . If  $f$  is a constant function then the weighted  $p$ -Steklov problem (1.1) reduces to the  $p$ -Steklov problem which has been studied in [16]. This problem arises from the following variational characterization of the first positive eigenvalue given by

$$\lambda_1(M) = \inf \left\{ \frac{\int_M |\nabla u|^p d\mu_g}{\int_{\partial M} |u|^p d\mu_h} \mid u \in W^{1,p}(M) \setminus \{0\}, \int_{\partial M} |u|^{p-2} u d\mu_h = 0 \right\} \tag{1.2}$$

where  $d\mu_h$  is the weighted measure on  $\partial M$ . Also, we consider a Steklov problem associated with the  $(p, q)$ -Laplacian as follows:

$$\begin{cases} \Delta_p u + \Delta_q u = 0 & \text{in } M, \\ (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \frac{\partial u}{\partial \nu} = \sigma |u|^{r-2} u & \text{on } \partial M, \end{cases} \tag{1.3}$$

where  $1 < p < q < r < \infty$ ,  $r \in (1, \frac{q(N-1)}{N-q})$  if  $p < N$  and  $r \in (1, \infty)$  if  $p \geq N$ . The first positive eigenvalue of the  $(p, q)$ -Steklov problem (1.3) defined as

$$\sigma_1(M) = \inf \left\{ \frac{\int_M (|\nabla u|^p + |\nabla u|^q) dv_g}{\int_{\partial M} |u|^r dv_h} \mid u \in W^{1,q}(M) \setminus \{0\}, \int_{\partial M} |u|^{r-2} u dv_h = 0 \right\}. \tag{1.4}$$

The  $(p, q)$ -Steklov problem has been studied in [4, 5, 6].

The aim of this paper is to obtain upper bounds for the first positive eigenvalue of the problems (1.1) and (1.3), for submanifolds of Riemannian manifolds, depending on the geometry of boundary in the spirit of the classical Reilly upper bounds for the Laplacian on closed hypersurfaces.

For the first positive eigenvalue  $\lambda_1$  of Laplacian, Reilly [12] proved the following well-known upper bound

$$\lambda_1 \leq \frac{n}{Vol(M)} \int_M H^2 dv_g,$$

where  $H$  is the mean curvature of the immersion. Also, he [12] showed that for  $r \in \{1, 2, \dots, n\}$ ,

$$\lambda_1 \left( \int_M H_{r-1} dv_g \right)^2 \leq Vol(M) \int_M H_r^2 dv_g,$$

where  $H_r$  is the  $r$ -th mean curvature of the immersion and defined by the  $r$ -th symmetric polynomial of the principal curvatures. Moreover, Reilly studied the equality cases and proved that equality holds in one of these inequalities, if and only if  $M$  is immersed in a geodesic sphere of radius  $\sqrt{\frac{n}{\lambda_1}}$ . More generally, he show that if  $(M^n, g)$  is isometrically immersed into  $\mathbb{R}^N$ ,  $N > n + 1$ , then

$$\lambda_1 \left( \int_M H_r dv_g \right)^2 \leq Vol(M) \int_M |H_{r+1}|^2 dv_g,$$

for any even  $r \in \{0, 1, \dots, n\}$  and equality holds if and only if  $M$  is minimally immersed in a geodesic sphere of  $\mathbb{R}^N$ . For codimension greater than 1,  $H_r$  is a function and  $\mathbf{H}_{r+1}$  is a normal vector field. These inequalities have been generalized for other ambient spaces and other operators (see [1, 2, 3, 7, 8, 11, 13, 14, 15, 16]).

Du and Mao [9] established the first positive eigenvalue of the  $p$ -Laplacian on closed submanifold of  $\mathbb{R}^N$  satisfies the following inequalities.

$$\lambda_1 \leq \frac{n^{\frac{p}{2}}}{(Vol(M))^p} \left( \int_M |H|^{\frac{p}{p-1}} dv_g \right)^{p-1} \begin{cases} N^{\frac{2-p}{2}} & \text{if } 1 < p \leq 2, \\ N^{\frac{p-2}{2}} & \text{if } p \geq 2. \end{cases}$$

In addition, equality holds if and only if  $p = 2$  and  $M$  is minimally immersed into a geodesic hypersphere. On the other hand, Roth [16] proved Reilly-type inequalities for the first eigenvalue of  $p$ -Steklov problem on submanifolds of  $\mathbb{R}^N$  and showed that

$$\lambda_1 \leq \frac{Vol(M)}{(Vol(\partial M))^p} n^{\frac{p}{2}} \left( \int_M |H|^{\frac{p}{p-1}} \right)^{p-1} \begin{cases} N^{\frac{2-p}{2}} & \text{if } 1 < p \leq 2, \\ N^{\frac{p-2}{2}} & \text{if } p \geq 2. \end{cases}$$

Moreover, equality holds if and only if  $p = 2$  and  $M$  is minimally immersed into  $B^N(\frac{1}{\lambda_1})$  such that  $X(\partial M) \subset \partial B^N(\frac{1}{\lambda_1})$ , where  $X$  is the isometric immersion.

## 2 Main results

Motivated by the above works, we prove that:

**Theorem 2.1.** Let  $(M^n, g, d\mu = e^{-f} dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $p \in (1, +\infty)$ . Assume that  $(M^n, g, d\mu = e^{-f} dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$ . If  $\lambda_1(M)$  is the eigenvalue of the weighted  $p$ -Steklov problem (1.1) then for  $1 < p \leq 2$  we have

$$\lambda_1(M) \leq 2^{\frac{1}{p-1}} n^{-\frac{p}{2}} N^{1-\frac{p}{2}} \left( \int_{\partial M} \left( |nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}} \right) d\mu_h \right)^{p-1} \frac{Vol_{\mu_g}(M)}{(Vol_{\mu_h}(\partial M))^p}$$

and for  $p \geq 2$  we get

$$\lambda_1(M) \leq 2^{\frac{1}{p-1}} n^{-\frac{p}{2}} N^{\frac{p}{2}-1} \left( \int_{\partial M} \left( |nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}} \right) d\mu_h \right)^{p-1} \frac{Vol_{\mu_g}(M)}{(Vol_{\mu_h}(\partial M))^p}.$$

where  $Vol_{\mu_g}(\partial M) = \int_M d\mu_g$  and  $Vol_{\mu_h}(\partial M) = \int_{\partial M} d\mu_h$ . Moreover,

- (i) If  $f$  is constant,  $H$  does not vanish identically then equality occurs in both inequalities if and only if  $p = 2$  and  $M$  is minimally immersed into  $B^N(\frac{1}{\lambda_1(M)})$  so that  $\partial M$  lies into geodesic hypersphere  $\partial B^N(\frac{1}{\lambda_1(M)})$ .
- (ii) If  $f$  is not constant and equality occurs then  $M$  is a self-shrinker for the mean curvature flow and  $f|_M = a - \frac{b}{2} r_p^2$  for some constants  $a, b$ , where  $r_p$  is the Euclidean distance to the center of mass  $p$  of  $M$ . In particular, if  $n = N - 1$  and  $H > 0$  or  $n = 2, N = 3$  and  $M$  is embedded and has genus 0, then  $M$  is a geodesic ball.

Let  $T$  be a symmetric positive definite and divergence-free  $(1, 1)$ -tensor on  $M$ . We associated with  $T$  the normal vector field  $H_T$  defined by

$$H_T = \sum_{i,j=1}^n \langle T e_i, e_j \rangle B(e_i, e_j),$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of  $T\partial M$  and  $B$  is the second fundamental form of the immersion of  $M$  into  $\mathbb{R}^N$ . We also, recall the generalized Hsiung-Minkowski formula [11, 14, 15] as

$$\int_{\partial M} (\langle X, H_T - T(\nabla f) \rangle + \text{tr}(T)) d\mu_h = 0. \tag{2.1}$$

In following theorem we extended the theorem 2.1 to estimates with higher order mean curvatures.

**Theorem 2.2.** Let  $(M^n, g, d\mu = e^{-f} dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $p \in (1, +\infty)$ . Assume that  $(M^n, g, d\mu = e^{-f} dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$  and let  $T$  be a symmetric and divergence-free  $(0, 2)$ -tensor on  $\partial M$ . If  $\lambda_1(M)$  is the eigenvalue of the weighted  $p$ -Steklov problem (1.1) then for  $1 < p \leq 2$  we have

$$\lambda_1(M) \left| \int_{\partial M} \text{tr}(T) d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{1-\frac{p}{2}} \left( \int_{\partial M} \left( |H_T|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}} \right) d\mu_h \right)^{p-1} Vol_{\mu_g}(M)$$

and for  $p \geq 2$  we get

$$\lambda_1(M) \left| \int_{\partial M} \text{tr}(T) d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{\frac{p}{2}-1} \left( \int_{\partial M} \left( |H_T|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}} \right) d\mu_h \right)^{p-1} Vol_{\mu_g}(M).$$

where  $Vol_{\mu_g}(\partial M) = \int_M d\mu_g$  and  $Vol_{\mu_h}(\partial M) = \int_{\partial M} d\mu_h$ . Moreover,

- (i) If  $f$  is constant,  $H_T$  does not vanish identically then equality occurs in both inequality if and only if  $p = 2$  and  $M$  is minimally immersed into  $B^N(\frac{1}{\lambda_1(M)})$  so that  $\partial M$  lies into geodesic hypersphere  $\partial B^N(\frac{1}{\lambda_1(M)})$ .
- (ii) If  $f$  is not constant and equality occurs then  $M$  is a self-shrinker for the mean curvature flow and  $f|_M = a - \frac{b}{2} r_p^2$  for some constants  $a, b$ , where  $r_p$  is the Euclidean distance to the center of mass  $p$  of  $M$ . In particular, if  $n = N - 1$  and  $H > 0$  or  $n = 2, N = 3$  and  $M$  is embedded and has genus 0, then  $M$  is a geodesic ball.

For  $r \in \{1, \dots, n\}$ , let

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \dots, i_r \\ j, j_1, \dots, j_r}} \varepsilon \binom{i, i_1, \dots, i_r}{j, j_1, \dots, j_r} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle e_i^* \otimes e_j^*$$

if  $r$  is even and

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \dots, i_r \\ j, j_1, \dots, j_r}} \varepsilon \binom{i, i_1, \dots, i_r}{j, j_1, \dots, j_r} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}} \rangle B_{i_r j_r} \otimes e_i^* \otimes e_j^*$$

if  $r$  is odd, where the  $B_{ij}$ 's are the coefficients of the second fundamental form  $B$  in a orthonormal frame  $\{e_1, \dots, e_n\}$  with the dual coframe  $\{e_1^*, \dots, e_n^*\}$  and  $\epsilon$  is the standard signature for permutations. The  $r$ -th mean curvature is defined as  $H_0 = 0$  and  $H_r = \frac{1}{(n-r)\binom{n}{r}} \text{tr}(T_r)$ . If  $r$  is even then  $H_r$  is a real function and if  $r$  is odd then  $H_r$  is a normal vector field, in this case, we will denote it  $\mathbf{H}_r$ . Also, the Hsiung-Minkowski formula becomes

$$\int_{\partial M} (\langle X, \mathbf{H}_{r+1} \rangle + H_r) d\mu_h = 0$$

for any even  $r \in \{0, 1, \dots, n\}$  if  $N > n + 1$ , and

$$\int_{\partial M} (\langle X, \nu \rangle H_{r+1} + H_r) d\mu_h = 0$$

for any  $r \in \{0, 1, \dots, n\}$  if  $N = n + 1$ , where  $\nu$  is the normal unit vector field on  $\partial M$  chosen to define the shape operator.

Now we obtain the following corollary from Theorem 2.2.

**Corollary 2.3.** Let  $(M^n, g, d\mu = e^{-f} dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $p \in (1, +\infty)$ . Assume that  $(M^n, g, d\mu = e^{-f} dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$ . If  $\lambda_1(M)$  is the eigenvalue of the weighted  $p$ -Steklov problem (1.1)

(1) If  $N > n + 1$ , and  $r \in \{0, \dots, n - 1\}$  is an even integer then we have

(a) If  $1 < p \leq 2$  we have

$$\lambda_1(M) \left| \int_{\partial M} H_r d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{1-\frac{p}{2}} \left( \int_{\partial M} (|\mathbf{H}_{r+1}|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h \right)^{p-1} Vol_{\mu_g}(M).$$

(b) If  $p \geq 2$  we have

$$\lambda_1(M) \left| \int_{\partial M} H_r d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{\frac{p}{2}-1} \left( \int_{\partial M} (|\mathbf{H}_{r+1}|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h \right)^{p-1} Vol_{\mu_g}(M).$$

Moreover, if  $f$  is constant,  $H_r$  does not vanish identically then equality occurs in both inequality if and only if  $p = 2$  and  $M$  is minimally immersed into  $B^N(\frac{1}{\lambda_1(M)})$  so that  $\partial M$  lies into geodesic hypersphere  $\partial B^N(\frac{1}{\lambda_1(M)})$ .

(2) If  $N = n + 1$ , and  $r \in \{0, \dots, n - 1\}$  is an even integer then we have

(a) If  $1 < p \leq 2$  we have

$$\lambda_1(M) \left| \int_{\partial M} H_r d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{1-\frac{p}{2}} \left( \int_{\partial M} (|H_{r+1}|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h \right)^{p-1} Vol_{\mu_g}(M).$$

(b) If  $p \geq 2$  we have

$$\lambda_1(M) \left| \int_{\partial M} H_r d\mu_h \right|^p \leq 2^{\frac{1}{p-1}} n^{\frac{p}{2}} N^{\frac{p}{2}-1} \left( \int_{\partial M} (|H_{r+1}|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h \right)^{p-1} Vol_{\mu_g}(M).$$

Moreover, if  $f$  is constant,  $H_{r+1}$  does not vanish identically then equality occurs in both inequality if and only if  $p = 2$  and  $X(M) = B^N(\frac{1}{\lambda_1(M)})$ .

In following we investigate the first eigenvalue of weighted  $p$ -Steklov problem on Riemannian products  $\mathbb{R} \times M$  where  $M$  is a complete Riemannian manifold.

**Theorem 2.4.** Let  $p \geq 2$  and  $(M^n, \bar{g})$  be a complete Riemannian manifold. Consider  $(\Sigma^n, g)$  a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times M, \tilde{g} = dt^2 \oplus \bar{g})$  with a density  $e^{-f}$ . Moreover, assume that  $\Sigma$  is mean-convex and bounds a domain  $\Omega$  in  $\mathbb{R} \times M$ . Let  $\lambda_1(M)$  be the first eigenvalue of the weighted  $p$ -Steklov problem on  $\Omega$ , then

$$\lambda_1(\Omega) \leq \left( \frac{\kappa_+(\Sigma) |H|_\infty}{\inf_\Sigma H} \right)^{\frac{p}{2}} \left( \frac{Vol_{\mu_g}(\Omega)}{Vol_{\mu_h}(\Sigma)} \right)^{1-\frac{p}{2}},$$

where  $\kappa_+(\Sigma) = \max\{\kappa_+(x) | x \in M\}$  with  $\kappa_+$  the biggest principal curvature of  $\Sigma$  at the point  $x$ .

Now, we obtain Reilly upper bounds for  $(p, q)$ -Steklov problem. Similar the theorem 2.1 we have

**Theorem 2.5.** Let  $(M^n, g, dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $1 < p < q < r < \infty$ . Assume that  $(M^n, g, dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$ . If  $\sigma_1(M)$  is the eigenvalue of the  $(p, q)$ -Steklov problem (1.3) then

(1) for  $1 < p < q < r \leq 2$  we have

$$\sigma_1(M) \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \frac{Vol(M)}{(Vol(\partial M))^r}.$$

(2) For  $1 < p < q \leq 2$  and  $r > 2$  we get

$$\sigma_1(M) \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \frac{Vol(M)}{(Vol(\partial M))^r}.$$

(3) For  $1 < p \leq 2$  and  $2 < q < r$  we get

$$\sigma_1(M) \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \frac{Vol(M)}{(Vol(\partial M))^r}.$$

(4) For  $2 \leq p < q < r$  we get

$$\sigma_1(M) \leq N^{\frac{r}{2}-1} \left( n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \frac{Vol(M)}{(Vol(\partial M))^r}.$$

In following theorem we extended the theorem 2.5 to estimates with higher order mean curvatures.

**Theorem 2.6.** Let  $(M^n, g, dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $1 < p < q < r$ . Assume that  $(M^n, g, dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$  and let  $T$  be a symmetric and divergence-free  $(0, 2)$ -tensor on  $\partial M$ . If  $\sigma_1(M)$  is the eigenvalue of the  $(p, q)$ -Steklov problem (1.3) then

(1) for  $1 < p < q < r \leq 2$  we have

$$\sigma_1(M) \left| \int_{\partial M} \text{tr}(T) dv_h \right|^r \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_T|^{\frac{r}{r-1}}) dv_h \right)^{r-1} Vol(M).$$

(2) For  $1 < p < q \leq 2$  and  $r > 2$  we get

$$\sigma_1(M) \left| \int_{\partial M} \text{tr}(T) dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_T|^{\frac{r}{r-1}}) dv_h \right)^{r-1} Vol(M).$$

(3) For  $1 < p \leq 2$  and  $2 < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} \text{tr}(T) dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_T|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(4) For  $2 \leq p < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} \text{tr}(T) dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_T|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

Also, we have

**Corollary 2.7.** Let  $(M^n, g, dv)$  be a compact connected and oriented Riemannian manifold with nonempty boundary  $\partial M$  and  $1 < p < q < r$ . Assume that  $(M^n, g, dv)$  isometrically immersed into the Euclidean space  $\mathbb{R}^N$  by  $X$ . If  $\sigma_1(M)$  is the eigenvalue of the  $(p, q)$ -Steklov problem (1.3)

(i) If  $N > n + 1$ , and  $s \in \{0, \dots, n - 1\}$  is an even integer then we have

(1) for  $1 < p < q < r \leq 2$  we have

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|\mathbf{H}_{s+1}|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(2) For  $1 < p < q \leq 2$  and  $r > 2$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|\mathbf{H}_{s+1}|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(3) For  $1 < p \leq 2$  and  $2 < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|\mathbf{H}_{s+1}|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(4) For  $2 \leq p < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|\mathbf{H}_{s+1}|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(ii) If  $N = n + 1$ , and  $s \in \{0, \dots, n - 1\}$  is an even integer then we have

(1) for  $1 < p < q < r \leq 2$  we have

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_s|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(2) For  $1 < p < q \leq 2$  and  $r > 2$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_s|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(3) For  $1 < p \leq 2$  and  $2 < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_s|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

(4) For  $2 \leq p < q < r$  we get

$$\sigma_1(M) \left| \int_{\partial M} H_s dv_h \right|^r \leq N^{\frac{r}{2}-1} \left( n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H_s|^{\frac{r}{r-1}}) dv_h \right)^{r-1} \text{Vol}(M).$$

In following we investigate the first eigenvalue of  $(p, q)$ -Steklov problem on Riemannian products  $\mathbb{R} \times M$  where  $M$  is a complete Riemannian manifold.

**Theorem 2.8.** Let  $2 \leq p < q < r$  and  $(M^n, \bar{g})$  be a complete Riemannian manifold. Consider  $(\Sigma^n, g)$  a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times M, \tilde{g} = dt^2 \oplus \bar{g})$ . Moreover, assume that  $\Sigma$  is mean-convex and bounds a domain  $\Omega$  in  $\mathbb{R} \times M$ . Let  $\sigma_1(M)$  be the first eigenvalue of the weighted  $p$ -Steklov problem on  $\Omega$ , then

$$\sigma_1(\Omega) \leq 2 \left( \frac{\kappa_+(\Sigma) |H|_\infty}{\inf_\Sigma H} \right)^{\frac{r}{2}} \left( \frac{Vol(\Omega)}{Vol(\Sigma)} \right)^{1-\frac{r}{2}},$$

where  $\kappa_+(\Sigma) = \max\{\kappa_+(x) | x \in M\}$  with  $\kappa_+$  the biggest principal curvature of  $\Sigma$  at the point  $x$ .

### 3 Proof of main results

In this section we give the proof of our main results.

**Proof .**[proof of theorem 2.1] For coordinates functions  $X^k$ , by replacing if needed,  $|X^i|^{p-2} X^i$  by

$$|X^i|^{p-2} X^i - \frac{\int_{\partial M} |X^i|^{p-2} X^i d\mu_h}{Vol_{\mu_h}(\partial M)}$$

we can assume without loss of generality,

$$\int_{\partial M} |X^i|^{p-2} X^i d\mu_h = 0$$

for all  $i \in \{1, 2, \dots, N\}$ . Thus, we can use the coordinates functions  $X^k$  as test functions.

The case  $1 < p \leq 2$ . By the definition of  $\lambda_1(M)$  we have

$$\lambda_1(M) \int_{\partial M} \sum_{i=1}^N |X^i|^p d\mu_h \leq \int_M \sum_{i=1}^N |\nabla X^i|^p d\mu_g. \tag{3.1}$$

Since  $p \leq 2$ , we get  $\left(\sum_{i=1}^N |X^i|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^N |X^i|^p\right)^{\frac{1}{p}}$ ,

$$|X|^p = \left(\sum_{i=1}^N |X^i|^2\right)^{\frac{p}{2}} \leq \sum_{i=1}^N |X^i|^p. \tag{3.2}$$

On the other hand, the concavity of  $y \rightarrow y^{\frac{p}{2}}$  yields

$$\sum_{i=1}^N |\nabla X^i|^p = \sum_{i=1}^N (|\nabla X^i|^2)^{\frac{p}{2}} \leq N^{1-\frac{p}{2}} \left(\sum_{i=1}^N |\nabla X^i|^2\right)^{\frac{p}{2}} = N^{1-\frac{p}{2}} n^{\frac{p}{2}}, \tag{3.3}$$

since we have  $\sum_{i=1}^N |\nabla X^i|^2 = n$  (see [14, Lemma 2.1]). Hence, we obtain

$$\lambda_1(M) \int_{\partial M} |X|^p d\mu_h \leq N^{1-\frac{p}{2}} n^{\frac{p}{2}} Vol_{\mu_g}(M). \tag{3.4}$$

On the other hand, using Hölder inequality we have

$$\begin{aligned} \int_{\partial M} \langle X, nH - \nabla f \rangle d\mu_h &\leq \left(\int_{\partial M} |X|^p d\mu_h\right)^{\frac{1}{p}} \left(\int_{\partial M} |nH - \nabla f|^{\frac{p}{p-1}} d\mu_h\right)^{\frac{p-1}{p}} \\ &\leq \left(\int_{\partial M} |X|^p d\mu_h\right)^{\frac{1}{p}} \left(2^{\frac{1}{p-1}} \int_{\partial M} (|nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h\right)^{\frac{p-1}{p}} \end{aligned}$$

With multiply both sides of (3.4) by  $\left(\int_{\partial M} (|nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h\right)^{p-1}$  and use the integral Hölder inequality, we conclude that

$$2^{-\frac{1}{p-1}} \lambda_1(M) \left| \int_{\partial M} \langle X, nH - \nabla f \rangle d\mu_h \right|^p \leq N^{1-\frac{p}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} (|nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h\right)^{p-1} Vol_{\mu_g}(M). \tag{3.5}$$

Now, using the Hsiung-Minkowski formula

$$\int_{\partial M} (\langle X, nH - \nabla f \rangle + n) d\mu_h = 0 \tag{3.6}$$

we infer

$$2^{-\frac{1}{p-1}} \lambda_1(M) (nVol_{\mu_h}(\partial M))^p \leq N^{1-\frac{p}{2}} n^{\frac{p}{2}} \left( \int_{\partial M} (|nH|^{\frac{p}{p-1}} + |\nabla f|^{\frac{p}{p-1}}) d\mu_h \right)^{p-1} Vol_{\mu_g}(M). \tag{3.7}$$

The inequality is proven. First, assume that  $f$  is constant,  $H$  does not vanish identically, and equality holds. Then the end of the proof is similar to the proof of Roth [16] for the  $p$ -Steklov problem. Now, assume that  $f$  is not constant. If equality occurs, then the end of the proof is similar to the proof of Roth [15].

The case  $p \geq 2$ . It is straightforward that

$$\sum_{i=1}^N |\nabla X^i|^p = \sum_{i=1}^N (|\nabla X^i|^2)^{\frac{p}{2}} \leq \left( \sum_{i=1}^N |\nabla X^i|^2 \right)^{\frac{p}{2}} = n^{\frac{p}{2}}. \tag{3.8}$$

On the other hand, using the fact that  $y \rightarrow y^{\frac{p}{2}}$  is convex, we obtain

$$\sum_{i=1}^N |X^i|^p \geq N^{1-\frac{p}{2}} \left( \sum_{i=1}^N |X^i|^2 \right)^{\frac{p}{2}} = N^{1-\frac{p}{2}} |X|^p. \tag{3.9}$$

Therefore, using the last two inequalities in the variational characterization of  $\lambda_1(M)$ , we get

$$\lambda_1(M) \int_{\partial M} |X|^p d\mu_h \leq N^{\frac{p}{2}-1} n^{\frac{p}{2}} Vol_{\mu_g}(M). \tag{3.10}$$

The end of the proof is the same that in case  $1 < p \leq 2$ .  $\square$

**Proof.** [Proof of theorem 2.2] Similar to the proof of Theorem 2.1, just enough to use the generalized Hsiung-Minkowski formula (2.1) instead of the classical one.  $\square$

**Proof.** [Proof of theorem 2.4] Similar to [16], we assume that the function  $t$  is a test function. Let  $v = \langle \partial_t, \nu \rangle = \langle \tilde{\nabla} t, \nu \rangle$ . Hence, we have  $\Delta t = -nHv$  and

$$\int_{\Sigma} |\nabla t|^2 d\mu_g = \int_{\Sigma} nHvt d\mu_g. \tag{3.11}$$

Also, since  $\nabla v = -S\nabla t$  we have

$$\int_{\Sigma} \langle S\nabla t, \nabla t \rangle d\mu_g = \int_{\Sigma} nHv^2 d\mu_g. \tag{3.12}$$

Then,

$$\begin{aligned} n \inf_{\Sigma} (H) \int_{\Sigma} v^2 d\mu_g &\leq \int_{\Sigma} nHv^2 d\mu_g \leq \int_{\Sigma} \langle S\nabla t, \nabla t \rangle d\mu_g \leq \kappa_+(\Sigma) \int_{\Sigma} |\nabla t|^2 d\mu_g \\ &\leq \kappa_+(\Sigma) \int_{\Sigma} nHvt d\mu_g \leq n\kappa_+(\Sigma) |H|_{\infty} \int_{\Sigma} vt d\mu_g \\ &\leq n\kappa_+(\Sigma) |H|_{\infty} \left( \int_{\Sigma} |t|^p d\mu_g \right)^{\frac{1}{p}} \left( \int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g \right)^{\frac{p-1}{p}}. \end{aligned} \tag{3.13}$$

From the Hölder inequality we have

$$\begin{aligned} \inf_{\Sigma} (H) \left( \int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g \right)^{\frac{2(p-1)}{p}} Vol_{\mu_g}(\Sigma)^{\frac{2-p}{p}} &\leq \inf_{\Sigma} (H) \int_{\Sigma} v^2 d\mu_g \\ &\leq \kappa_+(\Sigma) |H|_{\infty} \left( \int_{\Sigma} |t|^p d\mu_g \right)^{\frac{1}{p}} \left( \int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g \right)^{\frac{p-1}{p}}, \end{aligned}$$



therefore,

$$\frac{\left(\int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g\right)^{\frac{p-1}{p}}}{\left(\int_{\Sigma} |t|^p d\mu_g\right)^{\frac{1}{p}}} \leq \frac{\kappa_+(\Sigma)|H|_{\infty}}{\inf_{\Sigma}(H)} Vol_{\mu_g}(\Sigma)^{\frac{p-2}{p}}. \tag{3.14}$$

On the other hand, from the variational characterization of  $\lambda_1(M)$ , we have

$$\lambda_1(\Omega) \int_{\Sigma} |t|^p d\mu_g \leq \int_{\Omega} |\tilde{\nabla}t|^p d\mu_{\tilde{g}}. \tag{3.15}$$

Since  $|\tilde{\nabla}t| = 1$  and  $\tilde{\Delta}t = 0$  we have

$$\int_{\Omega} |\tilde{\nabla}t|^p d\mu_{\tilde{g}} = Vol_{\mu_{\tilde{g}}}(\Omega) = \left(\int_{\Omega} |\tilde{\nabla}t|^2 d\mu_{\tilde{g}}\right)^{\frac{p}{2}} Vol_{\mu_{\tilde{g}}}(\Omega)^{1-\frac{p}{2}} \tag{3.16}$$

and

$$\int_{\Omega} |\tilde{\nabla}t|^2 d\mu_{\tilde{g}} = \int_{\Sigma} \langle t\tilde{\nabla}t, \nu \rangle d\mu_g = \int_{\Sigma} tv d\mu_g. \tag{3.17}$$

By Hölder inequality we get

$$\int_{\Omega} |\tilde{\nabla}t|^2 d\mu_{\tilde{g}} \leq \left(\int_{\Sigma} |t|^p d\mu_g\right)^{\frac{1}{p}} \left(\int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g\right)^{\frac{p-1}{p}}, \tag{3.18}$$

thus, we obtain

$$\lambda_1(\Omega) \leq \frac{\left(\int_{\Sigma} |v|^{\frac{p}{p-1}} d\mu_g\right)^{\frac{p-1}{2}}}{\left(\int_{\Sigma} |t|^p d\mu_g\right)^{\frac{1}{2}}} Vol_{\mu_g}(\Omega)^{1-\frac{p}{2}}. \tag{3.19}$$

Therefore, substituting (3.14) in (3.19), we complete the proof of theorem.  $\square$

**Proof .**[Proof of theorem 2.5] For coordinates functions  $X^k$ , by replacing if needed,  $|X^i|^{r-2}X^i$  by

$$|X^i|^{r-2}X^i - \frac{\int_{\partial M} |X^i|^{r-2}X^i d\mu_h}{Vol_{\mu_h}(\partial M)}$$

we can assume without loss of generality,

$$\int_{\partial M} |X^i|^{r-2}X^i d\mu_h = 0$$

for all  $i \in \{1, 2, \dots, N\}$ . Thus, we can use the coordinates functions  $X^k$  as test functions.

(1) The case  $1 < p < q < r \leq 2$ . By the definition of  $\sigma_1(M)$  we have

$$\sigma_1(M) \int_{\partial M} \sum_{i=1}^N |X^i|^r dv_h \leq \int_M \sum_{i=1}^N (|\nabla X^i|^p + |\nabla X^i|^q) dv_g. \tag{3.20}$$

Since  $r \leq 2$ , we get  $\left(\sum_{i=1}^N |X^i|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^N |X^i|^r\right)^{\frac{1}{r}}$ , then

$$|X|^r = \left(\sum_{i=1}^N |X^i|^2\right)^{\frac{r}{2}} \leq \sum_{i=1}^N |X^i|^r. \tag{3.21}$$

On the other hand, the concavity of  $y \rightarrow y^{\frac{p}{2}}$  yields

$$\sum_{i=1}^N |\nabla X^i|^p = \sum_{i=1}^N (|\nabla X^i|^2)^{\frac{p}{2}} \leq N^{1-\frac{p}{2}} \left(\sum_{i=1}^N |\nabla X^i|^2\right)^{\frac{p}{2}} = N^{1-\frac{p}{2}} n^{\frac{p}{2}}. \tag{3.22}$$

Similarly,

$$\sum_{i=1}^N |\nabla X^i|^q \leq N^{1-\frac{q}{2}} n^{\frac{q}{2}},$$

Hence, we obtain

$$\sigma_1(M) \int_{\partial M} |X|^r dv_h \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) Vol(M), \tag{3.23}$$

with multiply by  $(\int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h)^{r-1}$  and use the integral Hölder inequality, we conclude that

$$\sigma_1(M) \left| \int_{\partial M} \langle X, H \rangle dv_h \right|^r \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} Vol(M). \tag{3.24}$$

Now, using the Hsiung-Minkowski formula

$$\int_{\partial M} (\langle X, H \rangle + 1) dv_h = 0 \tag{3.25}$$

we infer

$$\sigma_1(M) (Vol(\partial M))^r \leq \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) \left( \int_{\partial M} (|H|^{\frac{r}{r-1}}) dv_h \right)^{r-1} Vol(M). \tag{3.26}$$

(2) The case  $1 < p < q \leq 2$  and  $r > 2$ .

It is straightforward that

$$\sum_{i=1}^N (|\nabla X^i|^p + |\nabla X^i|^q) \leq N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}}. \tag{3.27}$$

On the other hand, using the fact that  $y \rightarrow y^{\frac{r}{2}}$  is convex, we obtain

$$\sum_{i=1}^N |X^i|^r \geq N^{1-\frac{r}{2}} \left( \sum_{i=1}^N |X^i|^2 \right)^{\frac{r}{2}} = N^{1-\frac{r}{2}} |X|^r. \tag{3.28}$$

Therefore, using the last two inequalities in the variational characterization of  $\sigma_1(M)$ , we get

$$\sigma_1(M) \int_{\partial M} |X|^r dv_h \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + N^{1-\frac{q}{2}} n^{\frac{q}{2}} \right) Vol(M). \tag{3.29}$$

(3) The case  $1 < p \leq 2$  and  $2 < q < r$ .

It is straightforward that

$$\sum_{i=1}^N |\nabla X^i|^p \leq N^{1-\frac{p}{2}} n^{\frac{p}{2}}, \tag{3.30}$$

and

$$\sum_{i=1}^N |\nabla X^i|^q = \sum_{i=1}^N (|\nabla X^i|^2)^{\frac{q}{2}} \leq \left( \sum_{i=1}^N |\nabla X^i|^2 \right)^{\frac{q}{2}} = n^{\frac{q}{2}}. \tag{3.31}$$

On the other hand, using the fact that  $y \rightarrow y^{\frac{r}{2}}$  is convex, we obtain

$$\sum_{i=1}^N |X^i|^r \geq N^{1-\frac{r}{2}} |X|^r. \tag{3.32}$$

Therefore, using the last two inequalities in the variational characterization of  $\sigma_1(M)$ , we get

$$\sigma_1(M) \int_{\partial M} |X|^r dv_h \leq N^{\frac{r}{2}-1} \left( N^{1-\frac{p}{2}} n^{\frac{p}{2}} + n^{\frac{q}{2}} \right) Vol(M). \tag{3.33}$$

(4) The case  $2 < p < q < r$ .  
 It is straightforward that

$$\sum_{i=1}^N (|\nabla X^i|^p + |\nabla X^i|^q) \leq n^{\frac{p}{2}} + n^{\frac{q}{2}}. \tag{3.34}$$

On the other hand, using the fact that  $y \rightarrow y^{\frac{r}{2}}$  is convex, we obtain

$$\sum_{i=1}^N |X^i|^r \geq N^{1-\frac{r}{2}} |X|^r. \tag{3.35}$$

Therefore, using the last two inequalities in the variational characterization of  $\sigma_1(M)$ , we get

$$\sigma_1(M) \int_{\partial M} |X|^r dv_h \leq N^{\frac{r}{2}-1} (n^{\frac{p}{2}} + n^{\frac{q}{2}}) Vol(M). \tag{3.36}$$

□

**Proof .**[Proof of theorem 2.5] Similar to the proof of Theorem 2.5, just enough to use the generalized Hsiung-Minkowski formula (2.1) instead of the classical one. □

**Proof .**[Proof of the Theorem 2.8] Similar to [16], we assume that the function  $t$  is a test function. Let  $v = \langle \partial_t, \nu \rangle = \langle \tilde{\nabla}t, \nu \rangle$ . Hence, we have  $\Delta t = -nHv$  and

$$\int_{\Sigma} |\nabla t|^2 dv_g = \int_{\Sigma} nHvt dv_g. \tag{3.37}$$

Also, since  $\nabla v = -S\nabla t$  we have

$$\int_{\Sigma} \langle S\nabla t, \nabla t \rangle dv_g = \int_{\Sigma} nHv^2 dv_g. \tag{3.38}$$

Then,

$$\begin{aligned} n \inf_{\Sigma} (H) \int_{\Sigma} v^2 dv_g &\leq \int_{\Sigma} nHv^2 d\mu_g \leq \int_{\Sigma} \langle S\nabla t, \nabla t \rangle dv_g \leq \kappa_+(\Sigma) \int_{\Sigma} |\nabla t|^2 dv_g \\ &\leq \kappa_+(\Sigma) \int_{\Sigma} nHvt d\mu_g \leq n\kappa_+(\Sigma) |H|_{\infty} \int_{\Sigma} vt dv_g \\ &\leq n\kappa_+(\Sigma) |H|_{\infty} \left( \int_{\Sigma} |t|^r dv_g \right)^{\frac{1}{r}} \left( \int_{\Sigma} |v|^{\frac{r}{r-1}} dv_g \right)^{\frac{r-1}{r}}. \end{aligned} \tag{3.39}$$

From the Hölder inequality we have

$$\begin{aligned} \inf_{\Sigma} (H) \left( \int_{\Sigma} |v|^{\frac{r}{r-1}} d\mu_g \right)^{\frac{2(r-1)}{r}} Vol_{v_g}(\Sigma)^{\frac{2-r}{r}} \\ \leq \inf_{\Sigma} (H) \int_{\Sigma} v^2 dv_g \\ \leq \kappa_+(\Sigma) |H|_{\infty} \left( \int_{\Sigma} |t|^r dv_g \right)^{\frac{1}{r}} \left( \int_{\Sigma} |v|^{\frac{r}{r-1}} dv_g \right)^{\frac{r-1}{r}}, \end{aligned}$$

therefore,

$$\frac{\left( \int_{\Sigma} |v|^{\frac{r}{r-1}} dv_g \right)^{\frac{r-1}{r}}}{\left( \int_{\Sigma} |t|^r dv_g \right)^{\frac{1}{r}}} \leq \frac{\kappa_+(\Sigma) |H|_{\infty} Vol(\Sigma)^{\frac{r-2}{r}}}{\inf_{\Sigma} (H)}. \tag{3.40}$$

On the other hand, from the variational characterization of  $\sigma_1(M)$ , we have

$$\sigma_1(\Omega) \int_{\Sigma} |t|^r dv_g \leq \int_{\Omega} (|\tilde{\nabla}t|^p + |\tilde{\nabla}t|^q) dv_{\tilde{g}}. \tag{3.41}$$

Since  $|\tilde{\nabla}t| = 1$  and  $\tilde{\Delta}t = 0$ , we have

$$\int_{\Omega} |\tilde{\nabla}t|^p dv_{\tilde{g}} = Vol(\Omega) = \left( \int_{\Omega} |\tilde{\nabla}t|^2 dv_{\tilde{g}} \right)^{\frac{r}{2}} Vol(\Omega)^{1-\frac{r}{2}} \quad (3.42)$$

and

$$\int_{\Omega} |\tilde{\nabla}t|^2 dv_{\tilde{g}} = \int_{\Sigma} \langle t\tilde{\nabla}t, \nu \rangle dv_g = \int_{\Sigma} t\nu dv_g. \quad (3.43)$$

By Hölder inequality we get

$$\int_{\Omega} |\tilde{\nabla}t|^2 dv_{\tilde{g}} \leq \left( \int_{\Sigma} |t|^r dv_g \right)^{\frac{1}{r}} \left( \int_{\Sigma} |v|^{\frac{r}{r-1}} dv_g \right)^{\frac{r-1}{r}}, \quad (3.44)$$

thus, we obtain

$$\sigma_1(\Omega) \leq 2 \frac{\left( \int_{\Sigma} |v|^{\frac{r}{r-1}} dv_g \right)^{\frac{r-1}{2}}}{\left( \int_{\Sigma} |t|^r dv_g \right)^{\frac{1}{2}}} Vol(\Omega)^{1-\frac{r}{2}}. \quad (3.45)$$

Therefore, substituting (3.40) in (3.45), we complete the proof of theorem.  $\square$

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