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New results for a complex-valued cellular neural networks model on time scales

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Abstract

In this study, we consider complex-valued cellular neural networks (CVCNNs) models on time scales. In contrast to earlier research, we employ a straightforward approach to arrive at our theoretical conclusion rather than breaking the model down into real-valued or complex-valued systems. Firstly, we use the Stepanov almost automorphy on time scales, the theory of time scale calculations, the Banach fixed point theorem, and by constructing an appropriate Lyapunov function to establish the existence, uniqueness, and Stepanov-stability of Stepanov almost automorphy solution for this class of CVCNNs on time scales via a direct method. Finally, an example with simulations is given to illustrate the feasibility of our results.

Keywords: Time scales, Complex-valued cellular neural networks, Stepanov almost automorphy
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1 Introduction

The neural networks (NNs) proposed by Chua and Yang [4] have attracted growing attention because of their broad range of applications, for example, function approximation, pattern recognition, associative memory, computing technology, nonlinear programming, and combinatorial optimization [5, 17, 18, 20]. On the one hand, the states, connection weights, and activation functions of complex-valued neural networks are complex-valued generalizations of real-valued neural networks. In general, complex-valued neural networks differ greatly from real-valued ones and exhibit more complex properties. This becomes urgently necessary as a result of their real-world uses in quantum, ultrasonic, and light-related physical networks [13, 26]. In reality, complex-valued neural networks (CVNNs) successfully handle many problems that real-valued neural networks cannot. For example, the detection of symmetry problems and the XOR problem may both be addressed by a single complex-valued neuron with orthogonal decision boundaries [15], but not by a single real-valued neuron. As a result, it is critical to investigate the dynamical behaviors of complex-valued cellular neural networks, particularly the stability problems of such networks. Recently, Zhang and Yu [27] studied a class of complex-valued Cohen-Grossberg neural networks with time delays and got some stability results. In [22], Song et al. investigated the global exponential stability of complex-valued neural networks with time-varying delays and an impulsive impact. In [23], Wang and Huang used the Lyapunov function approach and mathematical analysis methodology to achieve the stability criterion for complex-valued bidirectional associative memory (BAM) with time delay . Pan et al. [19] used a conjugate system of CVNNs, the fixed point theorem, the contraction mapping principle,

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and a delay differential inequality to determine the global exponential stability of a class of CVNNs with time-varying delays. The almost periodic dynamical behaviors for delayed complex-valued recurrent neural networks with discontinuous activation functions are discussed in [25] using differential inclusions theory, diagonal dominant principle, and nonsmooth analysis theory of the generalized Lyapunov function method. Recently, in [12], the authors discussed the existence and exponential stability of periodic solutions for Neutral-Type Complex-Valued Neural Networks.

On the other hand, as is well known, both continuous-time and discrete-time neural networks play important roles in theocratic research and applications. Furthermore, discrete-time neural networks are easier to compute and numerically simulate than continuous-time neural networks. As a result, we must investigate not only continuous-time neural networks but also discrete-time neural networks. Fortunately, Hilger's [14] theory of time scales, which he began in his PhD. thesis in 1988, can unify the continuous and discrete cases. The study of dynamic equations on time scales can help to reconcile the differential equation and difference equation cases. In recent years, the time scale theory has been widely discussed and rapidly developed [8]. Many authors have studied the dynamical behavior of neural networks on time scales [2, 10, 11, 16].

Very recently, M. Es-saiydy et al. [9] introduced the concept of Stepanov almost automorphy on time scales, which is a natural generalization of the concepts of almost periodic, almost automorphic, and Stepanov almost periodic, and much more general and plays a very important role in better understanding the almost periodicity. On the other hand, almost automorphy is a very important and powerful dynamic behavior of neural networks that has been extensively studied by a number of researchers [7, 24, 28, 29].

To the best of our knowledge, no such work has been done on the Stepanov-almost automorphic solution of real-valued RNNs or complex-valued CNNs on time scales. It is therefore a difficult and important problem in theory and applications. Motivated by the above analysis and discussion, our main contributions in this work are as follows:

1) We investigate an oscillation space that is never considered by various kinds of neural networks.

2) The existence and uniqueness of Stepanov-almost automorphic solution for complex-valued cellular neural networks (CVCNNs) on time scales are proved.

3) The exponential stability of a Stepanov-almost automorphic solution is demonstrated.

4) There are just a few publications in the literature on the dynamics of complex-valued cellular neural networks on time scales.

Further, our methods proposed in this paper can be used to study the problems of almost periodic solutions and Stepanov almost periodic solutions for other types of discrete-or continuous-CVCNNs such as complex-valued Hopfield NNs, Cohen-Grossberg NNs, and complex-valued BAM.

This paper is organized as follows: In Section 2, the CVCNNs on time scales are presented, and we introduce some necessary definitions and lemmas that are needed in later sections. In Section 3, we establish some sufficient conditions for the existence and uniqueness of Stepanov almost automorphic solution for CVCNNs on time scales. In Section 4, a numerical example with simulations is given to demonstrate the feasibility of our theoretical results.

2 Model description and Preliminaries

In this section, we shall first recall some fundamental definitions and lemmas, which are used in what follows. Throughout this paper we fix $p \geq 1$ and $(X, \|\,.\,\|)$ is a Banach space. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the set of positive integers, the set of integers, the set of real, and the set of complex numbers respectively. The skew field of the complex is determined by $\mathbb{C} := \{x; x = x^R + x^I i\}$, where x^R and x^I are real numbers and the element i is the imaginaire number.

In this paper, we consider the following complex-valued cellular neural networks (CVCNNs) with mixed time-varying delays on time scales:

$$x_l^{\Delta}(t) = -a_l(t)x_l(t) + \sum_{m=1}^{n} b_{lm}(t)f_m(x_m(t)) + I_l(t), \quad t \in \mathbb{T}. \tag{2.1}$$

Where $l \in \{1, 2, ..., n\}$, n corresponds to the number of units in neural networks, $\mathbb{T}$ is an almost periodic time scale; $\mathbb{C}$ is a complex algebra; $x_l(t) \in \mathbb{C}$ corresponds to the state of the lth unit at time t, $a_l(t) = diag(a_1(t), a_2(t), ..., a_n(t))$

represents the rate with which the ith neuron will reset its potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t. $f_m : \mathbb{C} \to \mathbb{C}$ is output transfer function, $b_{lm}(.)$ present the connection weights of the mth neuron on the l neuron. $I_l(.)$ denote the state bias of the lth neuron.

The initial condition of system (2.1) is of the form

$$x_l(s) = \psi_l(s), \quad s \in (-\infty, 0]_{\mathbb{T}},$$

where ψ_l is rd-continuous and $\psi_l \in L^p_{loc}((-\infty, 0]_{\mathbb{T}}, \mathbb{C})$ $l = 1, ..., n$.

Remark 2.1.
- If $\mathbb{T} = \mathbb{R}$, Then sys. (2.1) can be transformed into the form below :

$$x_l^{'}(t) = -a_l(t)x_l(t) + \sum_{m=1}^{n} b_{lm}(t)f_m(x_m(t)) + I_l(t), \quad t \in \mathbb{R}.$$

- If $\mathbb{T} = \mathbb{Z}$, then CVCNNs (2.1) reduces to:

$$x_l(k+1) - x_l(k) = -a_l(k)x_l(k) + \sum_{m=1}^{n} b_{lm}(k)f_m(x_m(k)) + I_l(k), \quad k \in \mathbb{Z}.$$

2.1 Time scales

Definition 2.2 ([6]). An arbitrary nonempty closed subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$ is called a time scale. The forward and backward jump operators $\sigma, \psi : \mathbb{T} \longrightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+$ are defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $\psi(t) = \sup\{s \in \mathbb{T} : s < t\}$, $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\psi(t) = t$, left-scattered if $\psi(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous or rd-continuous provided that it is continuous at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called continuous if and only if it is both left-dense continuous and right-dense continuous. A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called regressive provided if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T}; \mathbb{R})$. We define the set $\mathfrak{R}^+$ of all positively regressive elements by $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}) = \mathfrak{R}^+(\mathbb{T}; \mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Let $a, b \in \mathbb{T}$, with $a \leq b$, $[a, b]$, $[a, b)$, $(a, b]$, (a, b) being the usual intervals on the real line. The intervals $[a, a)$, $(a, a]$, (a, a) are understood as the empty set, and we use the following symbols :

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} \quad [a, b)_{\mathbb{T}} = [a, b) \cap \mathbb{T} \quad (a, b]_{\mathbb{T}} = (a, b] \cap \mathbb{T} \quad (a, b)_{\mathbb{T}} = (a, b) \cap \mathbb{T}.$$

Definition 2.3 ([6]). A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}; \forall t \in \mathbb{T}\} \neq \emptyset.$$

Definition 2.4 ([6]). For $f : \mathbb{T} \to X$ and $s \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, $f^{\Delta}(t) \in X$ is the delta derivative of f at s if for $\varepsilon > 0$, there is a neighborhood V of s such that for $t \in V$,

$$\| f(\sigma(s)) - f(t) - f^{\Delta}(s)(\sigma(s) - t) \| < \varepsilon \mid \sigma(s) - t \mid .$$

Moreover, f is delta differentiable on $\mathbb{T}$ provided that $f^{\Delta}(s)$ exists for $s \in \mathbb{T}$.

Definition 2.5 ([6]). If $p \in \mathfrak{R}$, then we define the exponential function by :

$$\hat{e}_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\},$$

for $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_m(z) = \begin{cases} \frac{\log(1+hz)}{h}, & if\ h \neq 0, \\ z, & if\ h = 0. \end{cases}$$

Definition 2.6 ([6]). If $p, q \in \mathfrak{R}$, then we define a circle plus addition by

$$(p \oplus q)(t) := p(t) + q(t) + p(t)q(t)\mu(t),$$

for all $t \in \mathbb{T}$. For $p \in \mathfrak{R}$, define a circle minus p by

$$\ominus p = -\frac{p}{1 + \mu p}.$$

Lemma 2.7 ([6]). Let $p, q \in \mathfrak{R}$, and $t, s, r \in \mathbb{T}$. Then,

1) $\hat{e}_0(t,s) = 1$ and $\hat{e}_p(t,t) = 1$;
2) $\hat{e}_p(\sigma(t), s) = (1 + p(t)\mu(t))\hat{e}_p(t,s)$;
3) $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus p}(s,t)$;
4) $\hat{e}_p(t,r)\hat{e}_p(r,s) = \hat{e}_p(t,s)$;
5) $(\hat{e}_p(t,s))^{\Delta} = p(t)\hat{e}_p(t,s)$;
6) If $a, b, c \in \mathbb{T}$. Then,
$$\int_a^b \hat{e}_p(c, \sigma(t))p(t)\Delta t = \hat{e}_p(c,a) - \hat{e}_p(c,b).$$
7) For $t_0 \in \mathbb{T}$, $\hat{e}_{\ominus\lambda}(t_0, .)$ is increasing on $(-\infty, t_0]_{\mathbb{T}}$.

Lemma 2.8 ([6]). Assume $p \in \mathfrak{R}$, and $t_0 \in \mathbb{T}$. If $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, then, $\hat{e}_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Definition 2.9 ([3]). $f : \mathbb{T} \to X$ is a $\Delta-$measurable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that, $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$.

Definition 2.10 ([3]). $f : \mathbb{T} \to X$ is a $\Delta-$integrable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$ and,

$$\lim_{k\to\infty} \int_{\mathbb{T}} \| f_k(s) - f(s) \| \Delta s = 0.$$

Then, the integral of f is defined as

$$\int_{\mathbb{T}} f(s)\Delta s = \lim_{k\to\infty} \int_{\mathbb{T}} f_k(s)\Delta s.$$

Definition 2.11 ([3]). For $p \geq 1$, $f : \mathbb{T} \to X$ is called locally L^p $\Delta-$integrable if f is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E \| f(s) \|^p \Delta s < \infty.$$

The set of all L^p $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; X)$.

Theorem 2.12. (Hölder's inequality)[1]. Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a,b] \to \mathbb{R}$ we have

$$\int_a^b | f(t)g(t) | \Delta t \leq \left(\int_a^b | f(t) |^p \Delta t \right)^{\frac{1}{p}} \left(\int_a^b | g(t) |^q \Delta t \right)^{\frac{1}{q}},$$

where $p > 1$ and $q = \frac{p}{p-1}$.

Theorem 2.13. (Minkowski's Inequality)[1]. Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a,b] \to \mathbb{R}$ we have

$$\int_a^b | (f+g)(t) | \Delta t \leq \left(\int_a^b | f(t) |^p \Delta t \right)^{\frac{1}{p}} + \left(\int_a^b | g(t) |^q \Delta t \right)^{\frac{1}{q}},$$

where $p > 1$ and $q = \frac{p}{p-1}$.

2.2 Stepanov almost automorphic functions on $\mathbb{T}$

This subsection is devoted to definitions, the important properties of Stepanov almost automorphic functions on time scales introduced by M. Es-saiydy and M. Zitane [9].

Definition 2.14 ([9]). We say that $f : \mathbb{T} \to \mathbb{C}$ is almost automorphic if from every sequence $\{A_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that :

$$g(t) = \lim_{n\to\infty} f(t+\tau_n)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in \mathbb{T}$. Denote by $AA(\mathbb{T}, \mathbb{C})$ the set of all such functions.

We set,

$$K = \begin{cases} \inf\{ \mid \tau \mid ; \tau \in \mathbb{T}, \tau \neq 0\}, & if\ \mathbb{T} \neq \mathbb{R}, \\ 1, & if\ \mathbb{T} = \mathbb{R}. \end{cases}$$

Let $f \in L^p_{loc}(\mathbb{T}, \mathbb{C})$, for $1 \le p < \infty$. Define :

- $\| \, . \, \|_{S^p} : L^p_{loc}(\mathbb{T}, \mathbb{C}) \to \mathbb{R}^+$ as : $\| f \|_{S^p} = \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid f(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}}$.
- $C_{rd}(\mathbb{T}; \mathbb{C}) = \{f : \mathbb{T} \to \mathbb{C} : f \text{ is rd-continuous}\}$.
- $BC_{rd}(\mathbb{T}; \mathbb{C}) = \{f : T \to \mathbb{C} : f \text{ is bounded and rd-continuous}\}$.
- $L^p_{loc}(\mathbb{T}; \mathbb{C}) = \{f : \mathbb{T} \to \mathbb{C} : f \text{ is locally } L^p\ \Delta - \text{integrable}\}$.
- $BS^p(\mathbb{T}; \mathbb{C}) = \{f \in L^p_{loc}(\mathbb{T}; \mathbb{C}) : \| f \|_{S^p} < \infty\}$.

Definition 2.15 ([2]). Let $f \in BS^p(\mathbb{T}, \mathbb{C})$ and $F \in BS^p(\mathbb{T} \times \mathbb{C}, \mathbb{C})$.

1) We say that $f : \mathbb{T} \to \mathbb{C}$ is Stepanov-like almost automorphic if for every sequence $\{A_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\| g(t) - f(t+\tau_n) \|_{S^p} \to 0, \quad as \quad n \to \infty,$$

is well defined for each $t \in \mathbb{T}$ and

$$\| g(t-\tau_n) - f(t) \|_{S^p} \to 0, \quad as \quad n \to \infty,$$

for each $t \in \mathbb{T}$. Denote by $S^pAA(\mathbb{T}, \mathbb{C})$ the set of all such functions.

2) A function $F : \mathbb{T} \times \mathbb{C} \to \mathbb{C}$, $(t, x) \to F(t, x)$ is said to be Stepanov-like almost automorphic if $t \to F(t, x)$ is Stepanov almost automorphic in $t \in \mathbb{T}$ uniformly for each $x \in \mathbb{C}$. Denote by $S^pAA(\mathbb{T} \times \mathbb{C}, \mathbb{C})$ the collection of such functions.

Lemma 2.16 ([2]). 1) If $h, g \in S^pAA(\mathbb{T}, \mathbb{C})$, then $h + g \in S^pAA(\mathbb{T}, \mathbb{C})$.

2) If $h \in S^pAA(\mathbb{T}, \mathbb{C})$ and $g \in S^pAP(\mathbb{T}, \mathbb{C})$, then $hg \in S^pAA(\mathbb{T}, \mathbb{C})$.

Proposition 2.17 ([2]). $(S^pAA(\mathbb{T}, \mathbb{C}), \| \, . \, \|_{S^p})$ is a Banach space.

3 Stepanov almost automorphic solution of (2.1) on time scales

In this section, we will study the existence and uniqueness of Stepanov almost automorphic solution of the system (2.1) on time scales. Now, we will list a few hypotheses that will be used for the rest of this paper.

(A_1) : For all $1 \leq l, m \leq n$, the functions $a_{lm}(.)$, $b_{lm}(.)$, $I_l(.) \in S^pAA(\mathbb{T}, \mathbb{C})$. And there exist positive constant L_l^f such that for any u, $v \in \mathbb{C}$, the activity function $f_l \in C_{rd}(\mathbb{C}, \mathbb{C})$ satisfy

$$\mid f_l(u) - f_l(v) \mid_{\mathbb{C}} \leq L_l^f \mid u - v \mid_{\mathbb{C}} .$$

Furthermore, we suppose that $f_l(0) = 0$.

(A_2) : $\varpi = \max_{1\leq l\leq n} \left\{ r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \times \left[\sum_{m=1}^{n} b_{lm}^* L_m^f \right] \right\} < 1.$

As a convenience, we have introduced these notations which simplify the writing of the equations:

$$b_{lm}^* = \sup_{t\in\mathbb{T}} \mid b_{lm}(t) \mid_{\mathbb{C}}, \quad \bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t),$$

$$\overline{a}_l = \inf_{t\in\mathbb{T}} a_l(t) \quad a_l^* = \sup_{t\in\mathbb{T}} a_l(t) > 0 \quad \check{a}_l = \inf_{t\in\mathbb{T}} \overline{a}_l + \inf_{t\in\mathbb{T}} a_l \quad l = 1, ..., n,$$

$$r_l(q) = \frac{2 + \overline{a}_l \bar{\mu} q}{\overline{a}_l q}, \quad \varrho_I = \max_{l=1,\dots,n} r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \parallel I \parallel_{S^p}, \quad d^* = \frac{\varpi \varrho_I}{1 - \varpi}.$$

Lemma 3.1. If a function $g \in C_{rd}(\mathbb{C}, \mathbb{C})$ satisfies condition (A_1) and $x_1 \in S^pAA(\mathbb{T}, \mathbb{C})$, then $g \circ x_1 \in S^pAA(\mathbb{T}, \mathbb{C})$.

Proof . Since $x_1 \in S^pAA(\mathbb{T}, \mathbb{C})$, then for every sequence $\{A_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid x_1(s + \tau_n) - \varphi(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

We set $\varphi_1(.) := g \circ \varphi$, therefore,

$$\begin{aligned} &\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid g \circ x_1(s + \tau_n) - \varphi_1(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}} \\ &= \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid g \circ x_1(s + \tau_n) - g \circ \varphi(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}}, \\ &\leq L^g \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid x_1(s + \tau_n) - \varphi(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}}. \end{aligned}$$

Then,

$$\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid g \circ x_1(s + \tau_n) - \varphi_1(s) \mid_{\mathbb{C}}^p \Delta s \right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

In conclusion, $g \circ x_1 \in S^pAA(\mathbb{T}, \mathbb{C})$. □

Theorem 3.2. Let $\psi = (\psi_1, ..., \psi_n) \in S^pAA(\mathbb{T}, \mathbb{C})$. Under assumptions (A_1)-(A_4), the nonlinear operator defined by:

$$(\Pi_\psi)_l(t) = \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t, \sigma(s)) J_l(s) \Delta s, \quad l = 1, ..., n.$$

Where

$$J_l(t) = \sum_{m=1}^{n} b_{lm}(t) f_m(\psi_m(t)) + I_l(t)$$

maps $S^pAA(\mathbb{T}, \mathbb{C})$ into itself.

Proof . It's easy to see that $(\Pi_\psi)_l$ is well defined and continuous. Again from Lemma (3.1), we have $J_l(.)$ belongs to $S^pAA(\mathbb{T},\mathbb{C})$. Now, we will prove that

$$(F_\psi)_l(t) = \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))J_l(s)\Delta s \in S^pAA(\mathbb{T},\mathbb{C}).$$

Since $J_l(.) \in S^PAA(\mathbb{T},\mathbb{C})$, then for every sequence $\{A_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\sup_{t_1\in\mathbb{T}} \left(\frac{1}{K}\int_{t_1}^{t_1+K} \mid J_l(t+\tau_n) - \varphi_l(t) \mid_{\mathbb{C}}^p \Delta t\right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n\to\infty.$$

We set $(\mathcal{U}_\psi)_l(t) = \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))\varphi_l(s)\Delta s$. Thus,

$$\begin{aligned}
&|(F_\psi)_l(t+\tau_n) - (\mathcal{U}_\psi)_l(t) \mid_{\mathbb{C}} \\
&= \left|\int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))J_l(s+\tau_n)\Delta s - \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))\varphi_l(s)\Delta s\right|_{\mathbb{C}}, \\
&\le \int_{-\infty}^{t} \hat{e}_{\ominus \overline{a}_l}(t,\sigma(s))\big|J_l(s+\tau_n) - \varphi_l(s)\big|_{\mathbb{C}}\Delta s, \\
&\le \int_{-\infty}^{0} \hat{e}_{\ominus \overline{a}_l}(0,\sigma(s))\big|J_l(t+s+\tau_n) - \varphi_l(t+s)\big|_{\mathbb{C}}\Delta s, \\
&\le r_l^{\frac{1}{q}}(q)\left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s))\big|J_l(t+s+\tau_n) - \varphi_l(t+s)\big|_{\mathbb{C}}^p\Delta s\right)^{\frac{1}{p}}.
\end{aligned}$$

Fubini's theorem implies that

$$\begin{aligned}
&\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}\Big|(F_\psi)_l(t+\tau_n) - (\mathcal{U}_\psi)_l(t)\Big|_{\mathbb{C}}^p\Delta t\right)^{\frac{1}{p}} \\
&\le \sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K} r_l^{\frac{p}{q}}.\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s))\Big|J_l(s+t+\tau_n) - \varphi_l(s+t)\Big|_{\mathbb{C}}^p\Delta s\Delta t\right)^{\frac{1}{p}}, \\
&\le r_l^{\frac{1}{q}}(q).\left(\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s))\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\Big|J_l(t+\tau_n) - \varphi_l(t)\Big|_{\mathbb{C}}^p\Delta t\Delta s\right)^{\frac{1}{p}}, \\
&\le r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p).\left(\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\Big|J_l(t+\tau_n) - \varphi_l(t)\Big|_{\mathbb{C}}^p\Delta t\Delta s\right)^{\frac{1}{p}}, \\
&< \max_{1\le l\le n} r_l^{\frac{1}{q}}(q)r_l^{\frac{1}{p}}(p)\left(\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\Big|J_l(t+\tau_n) - \varphi_l(t)\Big|_{\mathbb{C}}^p\Delta t\Delta s\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}\Big|(F_\psi)_l(t+\tau_n) - (\mathcal{U}_\psi)_l(t)\Big|_{\mathbb{C}}^p\Delta t\right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n\to\infty.$$

Which means that $(F_\psi)_l(.) \in S^pAA(\mathbb{T},\mathbb{C})$. □

Theorem 3.3. Assume that the conditions (A_1)-(A_2) are satisfied. Then, system (2.1) has a unique S^p-almost automorphic solution in the region $\mathbb{H} = \{\psi : \psi \in S^pAA(\mathbb{T},\mathbb{C}), \parallel \psi - \psi_0 \parallel_{S^p} \le d^*\}$.

Proof . First step : At first, we show that $(\Pi_\psi)_l$ is a self-mapping from $\mathbb{H}$ to $\mathbb{H}$. Let $\psi \in \mathbb{H}$, by using Hölder's and

Minkowski's inequality we can obtain

$$
\begin{aligned}
\mid (\Lambda_\psi)_l(t) - \psi_0(t) \mid_{\mathbb{C}} &= \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(z)) \times \left[\sum_{m=1}^{n} b_{lm}(z) f_m\left(\psi_m(z)\right) \right] \Delta z \right|_{\mathbb{C}}, \\
&\leq r_l^{\frac{1}{q}}(q) \times \left[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \left| \sum_{m=1}^{n} b_{lm}(z+t) f_m\left(\psi_m(z+t)\right) \right|_{\mathbb{C}}^{p} \Delta z \right]^{\frac{1}{p}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\| (\Pi_\psi)_l(t) - \psi_0(t) \|_{S^p} &= \sup_{t_1\in\mathbb{T}} \left[\frac{1}{K} \int_{t_1}^{t_1+K} \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(z)) \times \left(\sum_{m=1}^{n} b_{lm}(z) f_m\left(\psi_m(z)\right) \right] \Delta z \right|_{\mathbb{C}}^{p} \Delta t \right]^{\frac{1}{p}}, \\
&\leq r_l^{\frac{1}{q}}(q) \sup_{t_1\in\mathbb{T}} \left[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \times \frac{1}{K} \int_{t_1}^{t_1+K} \left| \sum_{m=1}^{n} b_{lm}(z+t) f_m\left(\psi_m(z+t)\right) \Delta s \right|_{\mathbb{C}}^{p} \Delta(z+t)\Delta z \right]^{\frac{1}{p}}, \\
&\leq r_l^{\frac{1}{q}}(q) \sup_{t_2\in\mathbb{T}} \left[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \times \frac{1}{K} \int_{t_2}^{t_2+K} \left| \sum_{m=1}^{n} b_{lm}(\hat{t}) f_m\left(\psi_m(\hat{t})\right) \right|_{\mathbb{C}}^{p} \Delta\hat{t}\Delta z \right]^{\frac{1}{p}}. \\
&\leq r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \times \left(\sup_{t_2\in\mathbb{T}} \left[\frac{1}{K} \int_{t_2}^{t_2+K} \sum_{m=1}^{n} \mid b_{lm}(\hat{t}) \mid_{\mathbb{C}}^{p} \mid f_m\left(\psi_m(\hat{t})\right) \mid_{\mathbb{C}}^{p} \Delta\hat{t} \right]^{\frac{1}{p}} \right), \\
&\leq \max_{1\leq l\leq n} \left\{ r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \left[\sum_{m=1}^{n} b_{lm}^{*} L_m^{f} \right] \right\} \times \| \psi \|_{S^p}, \\
&\leq \varpi \| \psi \|_{S^p}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\| \psi_0(t) \|_{S^p} &= \sup_{t_1\in\mathbb{T}} \left(\frac{1}{K} \int_{t_1}^{t_1+K} \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(z)) I_l(z) \right|_{\mathbb{C}}^{p} \Delta z \Delta t \right)^{\frac{1}{p}}, \\
&\leq r_l^{\frac{1}{q}}(q) \sup_{t_1\in\mathbb{T}} \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \frac{1}{K} \int_{t_1}^{t_1+K} \mid I_l(z+t) \mid_{\mathbb{C}}^{p} \Delta t \Delta z \right)^{\frac{1}{p}}, \\
&\leq r_l^{\frac{1}{q}}(q) \sup_{t_2\in\mathbb{T}} \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \frac{1}{K} \int_{t_2}^{t_2+K} \mid I_l(\overline{t}) \mid_{\mathbb{C}}^{p} \Delta\overline{t}\Delta z \right)^{\frac{1}{p}}, \\
&\leq r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \| I \|_{S^p} < \varrho_I. \quad \text{(Hölder's inequality)}
\end{aligned}
$$

So, for any $\psi \in \mathbb{H}$ we have

$$
\| \psi \|_{S^p} \leq \| \psi - \psi_0 \|_{S^p} + \| \psi_0 \|_{S^p} \leq \frac{\varpi \varrho_I}{1-\varpi} + \varrho_I = \frac{\varrho_I}{1-\varpi}.
$$

Hence,

$$
\| (\Pi_\psi)_l(t) - \psi_0(t) \|_{S^p} \leq \frac{\varpi \varrho_I}{1-\varpi},
$$

which implies that $(\Pi_\psi)_l \in \mathbb{H}$.

Second step : we shall prove that $(\Pi_\psi)_l$ is a contraction mapping. In fact, For $\psi, \phi \in \mathbb{H}$, we get

$$
\begin{aligned}
&\| (\Pi_\psi)_l(t) - (\Pi_\phi)_l(t) \|_{S^p} \\
&= \sup_{t_1\in\mathbb{T}} \left[\frac{1}{K} \int_{t_1}^{t_1+K} \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(z)) \left(\sum_{m=1}^{n} b_{lm}(z)(f_m(\psi_m(z)) - f_m(\phi_m(z))) \right) \Delta z \right|_{\mathbb{C}}^{p} \Delta t \right]^{\frac{1}{p}} \\
&\leq \max_{1\leq l\leq n} \left\{ r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \left[\sum_{m=1}^{n} b_{lm}^{*} L_m^{f} \right] \right\} \times \| \psi - \phi \|_{S^p}, \\
&\leq \varpi \| \psi - \phi \|_{S^p} < 1.
\end{aligned}
$$

Hence, we obtain that $(\Pi_\psi)_l$ is a contraction mapping. Then, system (2.1) has a unique S^p-almost automorphic solution in the region $\mathbb{H}$. □

Remark 3.4. There have been no outcomes on the almost automorphic solution, Stepanov almost periodic ones, and Stepanov almost automorphic solution for complex-valued cellular neural networks on time scales until now.

4 Stepanov exponential stability

In this section, we applied with a suitable Lyapunov function so that some sufficient criteria are achieved to guarantee the S^p-exponential stability of the Stepanov almost automorphic solution on time scales.

Lemma 4.1 ([21]). For any $x, y \in \mathbb{C}$, if $M \in \mathbb{C}^{n\times n}$ is a positive-definite Hermitian matrix, then

$$\bar{x}y + \bar{y}x \leq \bar{x}Mx + \bar{y}M^{-1}y.$$

Definition 4.2. The dynamical networks (2.1) is said to be S^p-globally exponentially stable, if there exist positive constants α with $\ominus\alpha \in \mathfrak{R}^+$ and $C > 0$ such that

$$\| L(t) - u(t) \|_{S^p} \leq C\hat{e}_{\ominus\alpha}(t, 0), \quad \forall t \in (0, \infty)_{\mathbb{T}}.$$

Where $u(.) = (u_1(.), u_2(.), ..., u_n(.))$ is a Stepanov-like almost automorphic solution of CVCNNs (2.1) on $\mathbb{T}$ and $L(.) = (v_1(.), v_2(.), ..., v_n(.))$ is an arbitrary solution of CVCNNs (2.1) on $\mathbb{T}$.

Theorem 4.3. Suppose that assumptions (A_1)-(A_2) hold, and $\varpi < 1$. Then the unique Stepanov-like almost automorphic solution of system (2.1) is S^p-globally exponentially stable on $\mathbb{T}$ whenever

$$(A_3) \ : \ \Theta_l = \frac{1}{a_l^*} - \check{a}_l \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 < 0.$$

Proof .

Let $u(.)$ be the S^p-almost automorphic solution on $\mathbb{T}$ and let $y(.)$ be an arbitrary solution of sys.(2.1), $X(.) = u(.) - v(.)$, $F(X(.)) = f(u(.)) - f(v(.))$. Let $x \in [0, \infty)$, we define the function $x \mapsto E_l(x)$ as follows:

$$E_l(x) = x + \frac{1}{a_l^*} - \check{a}_l + \exp(x(\bar{\mu})) \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 < 0.$$

By (A_3), we have $E_l(0) < 0$. Since the function $E_l(.)$ is continuous on $[0, \infty)$. Then, we can choose the positive constant $0 < \alpha < \min_{1\leq l\leq n} \overline{a}_l$, such that $E_l(\alpha) < 0$.
Now, construct a Lyapunov function as follows :

$$L(t) = \sum_{l=1}^{n} | X_l(t) |_{\mathbb{C}}^2 \hat{e}_\alpha(t, 0) + \exp(\alpha(\bar{\mu})) \sum_{l=1}^{n} \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 \int_0^t | X_l(z) |_{\mathbb{C}}^2 \hat{e}_\alpha(\sigma(z), 0)\Delta z.$$

Computing the $\Delta-$derivative of $L(.)$, we get

$$\begin{aligned} L^\Delta(t) &= \alpha\hat{e}_\alpha(t, 0) \sum_{l=1}^{n} | X_l(t) |_{\mathbb{C}}^2 + \hat{e}_\alpha(t, 0) \sum_{l=1}^{n} X_l^\Delta(t)\overline{X_l(t)} + \exp(\alpha(\bar{\mu})) \sum_{l=1}^{n} \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 \\ &\times \left[\hat{e}_\alpha(t, 0) | X_l(t) |_{\mathbb{C}}^2 - \exp(-\alpha)\hat{e}_\alpha(t, 0) | X_l(t) |_{\mathbb{C}}^2\right] \end{aligned}$$

According to Lemma (4.1) we get

$$\begin{aligned} &L^\Delta(t) \leq \alpha\hat{e}_\alpha(t, 0) \sum_{l=1}^{n} | X_l(t) |_{\mathbb{C}}^2 - \hat{e}_\alpha(t, 0) \sum_{l=1}^{n} \check{a}_l | X_l(t) |_{\mathbb{C}}^2 \\ &+ \hat{e}_\alpha(t, 0) \sum_{l=1}^{n} \left(\sum_{m=1}^{n} \overline{b_{lm}(t)F_m(X_m(t - \xi_m(t)))} b_{lm}(t)F_m(X_m(t))a_l^* + \frac{X_l(t)\overline{X_l(t)}}{a_l^*} \right) \\ &+ \exp(\alpha\bar{\mu}) \sum_{l=1}^{n} \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 \left(\exp(\alpha)\hat{e}_\alpha(t, 0) | X_l(t) |_{\mathbb{C}}^2 - \hat{e}_\alpha(t, 0) | X_l(t) |_{\mathbb{C}}^2 \right) \\ &\leq \hat{e}_\alpha(t, 0) \sum_{l=1}^{n} \left(\alpha + \frac{1}{a_l^*} - \check{a}_l + \exp(\alpha(\bar{\mu})) \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 \right) | X_l(t) |_{\mathbb{C}}^2, \\ &< 0. \end{aligned}$$

Otherwise, $\sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{C}}^2 \leq \hat{e}_{\ominus\alpha}(t,0)L(0), \quad \text{with} \quad \ominus\,\alpha \in \mathfrak{R}^+, \ \ l=1,...,n.$
It follows that

$$\sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{C}}^p \leq \hat{e}_{\ominus p\alpha}(t,0)L(0)^p, \ \ p \geq 2.$$

Hence,

$$\sum_{l=1}^{n} \frac{1}{K} \int_{t-\bar{\xi}}^{t-\bar{\xi}+K} \mid X_l(z) \mid_{\mathbb{C}}^p \Delta z \leq \sum_{l=1}^{n} \frac{1}{K} \int_{t}^{t+K} \mid X_l(z) \mid_{\mathbb{C}}^p \Delta z \leq \int_{t}^{t+K} \frac{\hat{e}_{\ominus p\alpha}(t,0)L(0)^p}{K}.$$

Consequently,

$$\sum_{l=1}^{n} \frac{1}{K} \int_{t}^{t+K} \mid X_l(z) \mid_{\mathbb{C}}^p \Delta z \leq \frac{L(0)^p \hat{e}_{\ominus\alpha p}(t,0)}{K} \frac{(\exp(-\alpha pK)-1)}{\ominus\alpha p}.$$

According to the previous inequality, we can obtain

$$\max_{l=1,...,n} \sup_{t_1\in\mathbb{T}} \left(\frac{1}{K} \int_{t_1}^{t_1+K} \mid X_l(z) \mid_{\mathbb{C}}^p \Delta z \right)^{\frac{1}{p}} \leq \frac{L(0)\hat{e}_{\ominus\alpha}(t,0)}{K^{\frac{1}{p}}} \left(\frac{(\exp(-\alpha pK)-1)}{\ominus\alpha p} \right)^{\frac{1}{p}}.$$

We claim that

$$\| X \|_{S^p} \leq \frac{L(0)\hat{e}_{\ominus\alpha}(t,0)}{K^{\frac{1}{p}}} \left(\frac{(\exp(-\alpha pK)-1)}{\ominus\alpha p} \right)^{\frac{1}{p}} \leq C\hat{e}_{\ominus\alpha}(t,0).$$

Where $C = \frac{L(0)}{K^{\frac{1}{p}}} \left(\frac{(\exp(-\alpha pK)-1)}{\ominus\alpha p} \right)^{\frac{1}{p}}$. Therefore, the unique Stepanov-like almost automorphic solution of CVCNNs (2.1) is S^p-globally exponentially stable on time scales. This completes the proof. □

Remark 4.4. To the best of our knowledge, the Stepanov almost automorphic solution on time scales for complex valued-neural networks has not yet yielded any results. The inquiry techniques employed in this paper may also be utilized to examine Stepanov almost periodic and almost automorphic on time scale solution for certain other types of neural networks, such as the renowned class of Lotka-Volterra neural networks. As a consequence, the obtained results are fundamentally novel.

5 Numerical Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results derived in the previous sections.

Example 5.1. Let us consider the following model of CVCNNs for $\mathbb{T} = \mathbb{R}$ and $n = 2$:

$$x_l'(t) = -a_l(t)x_l(t) + \sum_{m=1}^{2} b_{lm}(t)f_m(x_m(t)) + I_l(t), \quad t > 0. \tag{5.1}$$

Let $p = q = 2$, and the coefficients are taken as follows:

- $f_m(x_m(.)) = \frac{1}{7}\left(i\sin(x_m^I(.))\cos(x_m^I(.) - x_m^R(.))\right),$
- $a_1(.) = 8 + 2i\cos(\sqrt{2}t), \ \ a_2(.) = 7 + i\sin(t),$
- $I_l(t) = \frac{2+\exp(it)}{|2+\exp(it)+\exp(i\sqrt{5}t)|},$
- $b_{lm} = \frac{1}{30}\begin{bmatrix} \sin\left(\frac{1}{2+\sin(t)+\sin(\sqrt{7}t)}\right) & i\cos(\sqrt{3}t) \\ \sin(\sqrt{3}t) + i\cos(\pi t) & \sin(t) \end{bmatrix}.$

Then, we have

- $\overline{a}_1 = 6, \;\; \overline{a}_2 = 6, \;\; \check{a}_1 = 10, \;\; \check{a}_2 = 8, \;\; a_1^* = 13, \;\; a_2^* = 7,$
- $L_m^f = \frac{1}{7},$
- $b_{lm}^* = \frac{1}{30}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},$
- $d_{lm}^* = \frac{1}{20}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}.$

Hence, by a simple calculation we get

- $\varpi \simeq \max\{0.8231;\; 0.7045\} = 0.8231,$
- $\Theta_l = (-3.0147, \; -4.0098) < 0.$

Therefore, all of the conditions of Theorems (3.3) and (4.3) are satisfied. Then, system (5.1) has a unique S^p-almost automorphic solution which is S^p-globally exponential stable.

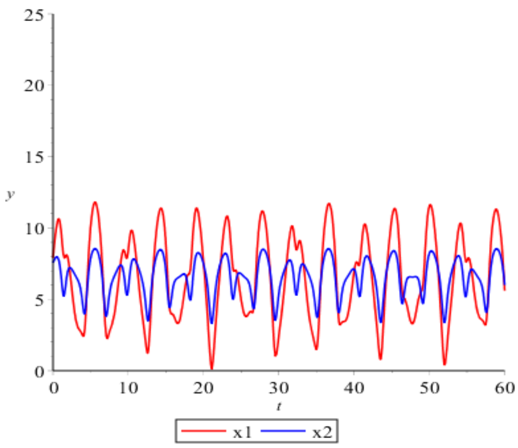

Figure 1: Behavior of the state variables x_1^R and x_1^I of (5.1) on $\mathbb{T} = \mathbb{R}$.

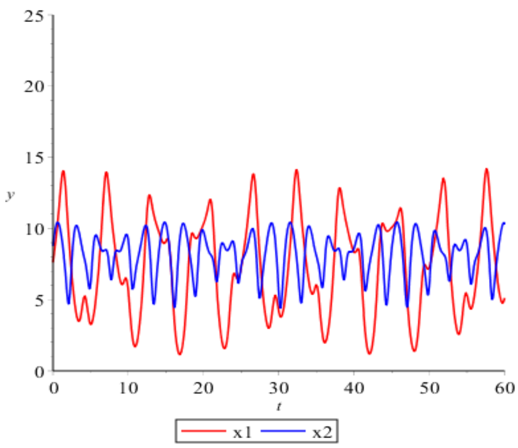

Figure 2: Behavior of the state variables x_2^R and x_2^I of (5.1) on $\mathbb{T} = \mathbb{R}$.

Conclusion

In this research, we presented a class of complex-valued cellular neural networks on time scales. Also, we provide certain necessary criteria on the existence, uniqueness, and global exponential stability of Stepanov almost automorphic solution for complex-valued cellular neural networks based on Banach's fixed point theorem and the theory of calculus on time scales. Finally, an example with simulations has been provided to show the efficacy of our outcomes. To our knowledge, this is the first time that the Stepanov almost automorphic solution for complex-valued cellular neural networks on time scales has been studied.

References

[1] R. Agrwal, M. Bohner and A. Peterson, *Inequalities on Time Scales: A Survey*, Math. Inequal. Appl. **4** (2003), no. 4.

[2] A. Arbi, J. Cao, M Es-saiydy, M. Zarhouni and M. Zitane, *Dynamics of delayed cellular neural networks in the Stepanov pseudo almost automorphic space*, Discrete Continuous Dyn. Systems-S **15** (2022), no. 11, 3097–3109.

[3] A. Cabada and D. Vivero, *Expression of the Lebesgue $\Delta-$Integral on Time Scales as a Usual Lebesgue Intregral, Application to the Calculus of D-Antiderivatives*, Math. Comput. Modell. **43** (2006), 194–207.

[4] L.O. Chua and L. Yang, *Cellular neural networks: theory*, IEEE Trans. Circuits Syst. **35** (1988), no. 10, 1257–1272.

[5] W.H. Chen, S. Luo and W.X. Zheng, *Impulsive synchronization of reaction-diffusion neural networks with mixed delays and its application to image encryption*, IEEE Trans. Neural Networks Learn. Syst. **27** (2016), 2696–2710.

[6] M. Bohner, V.S. Hari Rao and S. Sanyal, *Global Stability of Complex-Valued Neural Networks on Time Scales*, Differ. Equ. Dyn. Syst. **19** (2011), 3–11.

[7] M. Es-saiydy and M. Zitane, *Weighted Stepanov-like pseudo almost periodicity on time scales and applications*, Differ. Equ. Dyn. Syst. (2020). https://doi.org/10.1007/s12591-020-00543-7

[8] M. Es-saiydy and M. Zitane, *A new composition theorem for weighted Stepanov-like pseudo almost periodic functions on time scales and applications*, Bol. Soc. Parana. Math., to appear.

[9] M. Es-saiydy, M. Zarhouni and M. Zitane, *Stepanov-like pseudo almost automorphy on time scales : New developments and applications*, Asia Pac. J. Math. **9** (2022).

[10] M. Es-saiydy, I. Oumadane and M. Zitane, (2021): *Stepanov stability for delayed Lotka-Volterra recurrent neural networks on time scales*, Appl. Anal. DOI: 10.1080/00036811.2021.1967330

[11] M. Es-saiydy and M. Zitane, *Dynamic behavior af a class of delayed Lotka-Volterra recurrent neural networks on time scales*, Russian Math. **65** (2021), 59–75.

[12] S. Guo and B. Du, *Global exponential stability of periodic solution for Neutral-type complex-valued neural networks*, **2016** (2016), Article ID 1267954, 10 pages.

[13] A. Hirose, *Dynamics of fully complex-valued neural networks*, Eletron Lett. **28** (1992), no. 16, 1492—1494.

[14] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.

[15] S. Jankowski, A. Lozowski and J. Zurada, *Complex-valued multistate neural associative memory*, IEEE Trans. Neural Netw. **7** (1996), no. 6, 1491—1496.

[16] M. Khuddush and K.R. Prasad, *Permanence and stability of multi-species nonautonomous Lotka-Volterra competitive systems with delays and feedback controls on time scales*, Khayyam J. Math. **7** (2021), no. 2, 241–256.

[17] T. Lindblad and J.M. Kinser, *Image Processing Using Pulse-Coupled Neural Networks*, Springer, 2005.

[18] P.D. McNelis, *Neural Networks in Finance: Gaining Predictive Edge in the Market*, Academic Press, 2005.

[19] J. Pan, X. Liu and W. Xie, *Exponential stability of a class of complex-valued neural networks with time-varying delays*, Neurocomput. **164** (2015), 293—299.

[20] J.R. Rabunal and J. Dorrado, *Artificial Neural Networks in Real-life Applications*, Idea Group Publishing, 2005.

[21] Q. Song and X. Chen, *Multistability analysis of quaternion-valued neural networks with time delays*, IEEE Trans. Neural Netw. Learn. Syst. **29** (2018), 5430–5440.

[22] Q. Song, H. Yan, Z. Zhao and Y. Liu, *Global exponential stability of complex-valued neural networks with both time-varying delays and impulsive effects*, Neural Netw. **79** (2016), 108–116.

[23] Z. Wang and L. Huang, *Global stability analysis for delayed complex-valued BAM neural networks*, Neurocomput. **173** (2016), 2083—2089.

[24] C. Xu and Y. Wu, *On almost automorphic solutions for cellular neural networks with time-varying delays in leakage terms on time scales*, J. Intelligent and Fuzzy Syst. **30** (2016), 423–436.

[25] M. Yan, J. Qiu, X. Chen, X. Chen, C. Yang and A. Zhang, *Almost periodic dynamics of the delayed complex-valued recurrent neural networks with discontinuous activation functions*, Neural Comput. Appl. **30** (2018), no. 11, 3339-–3352.

[26] S. Yang, Z. Guo Z and J. Wang, *Global synchronization of multiple recurrent neural networks with time delays via impulsive interactions* IEEE Trans Neural Netw. Learn. Syst. **15** (2016), no. 2, 318-–328.

[27] Z. Zhang and S. Yu, *Global asymptotic stability for a class of complex-valued Cohen-Grossberg neural networks with time delays*, Neurocomput. **171** (2016), 1158-–1166.

[28] H. Zhu, Q. Zhu, X. Sun and H. Zhou, *Existence and exponential stability of pseudo almost automorphic solutions for Cohen-Grossberg neural networks with mixed delays*, Adv. Differ. Equ. **2016** (2016), 120.

[29] M. Zitane, *A generalization of weighted Stepanov-like pseudo-almost automorphic space*, New Zealand J. Math. **48** (2018), 129–155.