Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1347–1357
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24614.2782

Generalized iterative scheme for a generalized spectral problem

Ammar Khellaf[a,*], Hamza Guebbai[b], Wassim Merchela[c], Mohamed zine Aissaoui[b]

[a]*Preparatory Class Department,National Polytechnic College of Constantine (Engineering College), Algeria*

[b]*Department of Mathematics, Faculty of Mathematics and Computer Science and Material Sciences, University 8 May 1945 of Guelma, Algeria*

[c]*Derzhavin Tambov State University, 33 internatsionalnaya St., Tambov 392000, Russia*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this work, we define an iterative scheme for a generalized spectral problem associated with two operators defined on a Banach space of infinite dimension. We show that under the norm convergence, the generalized approximated eigenvalues and eigenvectors converge to the exact eigenpairs. As a numerical application, we tackle a generalized eigenvalue problem associated with integral operators, where the accuracy and efficiency are illustrated in some numerical examples.

Keywords: Generalized spectrum approximation, Operator pencil, Iterative scheme
2020 MSC: 65J15, 45G10, 35P05

1 Introduction

Let X be a Banach space. The space $BL(X)$ is the set of bounded linear operators from X into X. This space is provided with the subordinate standard norm defined as, for $T \in BL(X)$

$$\|T\|_{BL(X)} = \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}.$$

Let T and S be two operators in $BL(X)$, we recall that the generalized spectrum $sp(T, S)$ is the set

$$sp(T, S) = \{\lambda \in \mathbb{C} : (T - \lambda S) \text{ not invertible}\}.$$

Thus, the generalized resolving set $re(T, S)$ is given as

$$re(T, S) = \mathbb{C} \setminus sp(T, S).$$

Then the generalized point spectrum $sp_p(T, S)$ is defined by

$$sp_p(T, S) = \{\lambda \in \mathbb{C} : \exists \varphi \in X \setminus \{0\},\ T\varphi = \lambda S\varphi\}.$$

*Corresponding author
Email addresses: amarlasix@gmail.com (Ammar Khellaf), guebaihamza@yahoo.fr (Hamza Guebbai), merchela.wassim@gmail.com (Wassim Merchela), aissaouizine@gmail.com (Mohamed zine Aissaoui)

Received: September 2021 *Accepted:* November 2022

Let $\lambda \in sp_p(T,S)$ be a generalized eigenvalue. We say that λ has a finite algebraic multiplicity, if there exists a positive integer l such that

$$\dim Ker(T-\lambda S)^l < \infty,$$

in this case, λ is called a generalized eigenvalue of finite type. If λ is a nonzero generalized eigenvalue of couple (T,S) and Θ is a closed Jordan curve in $re(T,S)$ isolating λ, then

$$P = -\frac{1}{2\pi i}\int_{\Theta}(T-zS)^{-1}Sdz : X \to X$$

defines the generalized spectral projection at λ and

$$Q = -\frac{1}{2\pi i}\int_{\Theta}(T-zS)^{-1}(\lambda-z)^{-1}dz : X \to X$$

is the generalized reduced resolvent at λ (see the book [1]).

In recent papers [2], [3], [4], [6], [5] and [7] the authors have studied the numerical resolution of the generalized spectral problems:

$$\text{Find } (\varphi,\lambda) \in X \times \mathbb{C}: \quad T\varphi = \lambda S\varphi, \quad \varphi \neq 0. \tag{1.1}$$

Problem (1.1) is approximated by a discreted version:

$$\text{Find } (\varphi_n,\lambda_n) \in X_n \times \mathbb{C}: \quad T_n\varphi_n = \lambda_n S_n\varphi_n, \quad \varphi_n \neq 0, \tag{1.2}$$

where X_n is a subspace of X of finite dimension and T_n and S_n result from a projection method. Hence, the matrices representing T_n and S_n are commonly full and the numerical solution of (1.2) is not feasible for large n because it may become very expensive to get in computer time or storage.

The purpose of the present paper is to provide a method of attaining high precision without having to solve a generalized matrix eigenvalue problem of a very large size. The idea of the method presented in this paper is to refine iteratively the generalized eigenelements (λ_n,φ_n) obtained when solving (1.2) with a relatively small n.

Let φ_n be a generalized eigenvector of (T_n,S_n) corresponding to a simple generalized eigenvalue of finite type λ_n and φ_n^* be the generalized eigenvector of (T_n^*,S_n^*) corresponding to it simple generalized eigenvalue $\bar{\lambda}_n$ such that

$$\langle \varphi_n, \varphi_n^* \rangle = 1.$$

We define our Generalized Elementary Iteration (G.E.I) through the successive iterates:

$$(E): \quad \begin{cases} \varphi_n^{(0)} = \varphi_n, \text{ and for } k = 1,2,... \\[2mm] \lambda_n^{(k)} = \dfrac{\langle T\varphi_n^{(k-1)}, \varphi_n^* \rangle}{\langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle}, \quad \langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle \neq 0, \\[2mm] \varphi_n^{(k)} = \varphi_n^{(k-1)} + Q_n\big(\lambda_n^{(k)}S\varphi_n^{(k-1)} - T\varphi_n^{(k-1)}\big), \end{cases}$$

where Q_n is defined as:

$$Q_n = -\frac{1}{2\pi i}\int_{\Theta}(T_n-zS_n)^{-1}(\lambda-z)^{-1}dz : X \to X.$$

We notice that when $S = I$, our method becomes the elementary iteration method defined in [8].

In the next section, we prove the convergence results and the errors analysis of the G.E.I by employing mathematical induction. In the last section, we illustrate these results with a numerical application showing the accuracy and efficiency of our algorithms.

2 Framework

We state in this section a set of theorems which will be needed in the proof of our main Theorem 2.3.

Theorem 2.1. Assume that there exist two sequences of bonded operators $(T_n)_{n\geq 1}$ and $(S_n)_{n\geq 1}$ converging on the norm to T and S respectively, i.e.

$$\text{i) } T_n \xrightarrow{\text{n}} T \qquad \text{ii) } S_n \xrightarrow{\text{n}} S.$$

If $\lambda_0 \in re(T,S)$, then for all large n, $\lambda_0 \in re(T_n,S_n)$, $\big(\big(\|(T_n-\lambda_0 S_n)^{-1}\|\big)\big)$ is uniformly bounded independent on n and

$$(\lambda_0 S_n - T_n)^{-1}S_n \xrightarrow{\text{n}} (\lambda_0 S - T)^{-1}S,$$

$$\|(\lambda_0 S_n - T_n)^{-1}S_n - (\lambda_0 S - T)^{-1}S\| \leq C\big(\|T_n - T\| + \|S - S_n\|\big).$$

Proof . Let $\lambda_0 \in re(T,S)$, we can see

$$\begin{aligned} T_n - \lambda_0 S_n &= (T-\lambda_0 S) - \big((T-T_n) - \lambda_0(S-S_n)\big) \\ &= \Big[I - \Big((T-T_n) - \lambda_0(S-S_n)\Big)(T-\lambda_0 S)^{-1}\Big](T-\lambda_0 S). \end{aligned}$$

As, $T_n \xrightarrow{n} T$ and $S_n \xrightarrow{n} S$ then there is a positive integer n_0 such that for all $n \geq n_0$

$$\Big\|\Big((T-T_n) - \lambda_0(S-S_n)\Big)(T-\lambda_0 S)^{-1}\Big\| \leq \frac{1}{2}.$$

So, using Neumann Series Lemma, we can find that $\lambda_0 \in re(T_n,S_n)$ and

$$\begin{aligned} \|(T_n-\lambda_0 S_n)^{-1}\| &= \Big\|(T-\lambda_0 S)^{-1}\sum_{k=0}^{\infty}\Big[\Big((T-T_n) - \lambda_0(S-S_n)\Big)(T-\lambda_0 S)^{-1}\Big]^k\Big\| \\ &\leq 2\|(T-\lambda_0 S)^{-1}\| = c. \end{aligned}$$

Now, let $n \geq n_0$, then

$$\begin{aligned} \|(\lambda_0 S_n - T_n)^{-1}S_n - (\lambda_0 S - T)^{-1}S\| &= \Big\|(\lambda_0 S_n - T_n)^{-1}(S_n - S) + \\ &\quad \Big((\lambda_0 S_n - T_n)^{-1} - (\lambda_0 S - T)^{-1}\Big)S\Big\| \\ &\leq c\|S_n - S\| + \\ &\quad \|S\|\,\Big\|(\lambda_0 S_n - T_n)^{-1}\Big((T_n - T) - \lambda_0(S_n - S)\Big) \times \\ &\quad (\lambda_0 S - T)^{-1}\Big\| \\ &\leq c\big(\|T_n - T\| + \|S_n - S\|\big) + \\ &\quad \|S\|\,\frac{c^2}{2}\max\{1,\lambda_0\}\big(\|T_n - T\| + \|S_n - S\|\big) \\ &\leq C\big(\|T_n - T\| + \|S_n - S\|\big), \end{aligned}$$

where $C = c + \|S\|\,\dfrac{c^2}{2}\max\{1,\lambda_0\}$. □

Theorem 2.2. Let λ be a simple generalized eigenvalue of (T,S) and φ be a corresponding eigenvector.

$$\text{i) } T_n \xrightarrow{\text{n}} T \qquad \text{ii) } S_n \xrightarrow{\text{n}} S.$$

Then for n large enough, the couple (T_n,S_n) has a simple generalized eigenvalue λ_n such that

$$\lambda_n \longrightarrow \lambda.$$

Let φ_n be a generalized eigenvector of (T_n,S_n) corresponding to λ_n and φ_n^* be the generalized eigenvector of (T_n^*,S_n^*) corresponding to it simple generalized eigenvalue $\overline{\lambda_n}$ such that

$$\big\langle \varphi_n, \varphi_n^* \big\rangle = 1.$$

Then $\langle \varphi, \varphi_n^* \rangle \neq 0$ for all large n. Further, if we note

$$\varphi_{(n)} = \frac{\varphi}{\langle \varphi, \varphi_n^* \rangle},$$

then for all large n, we have

$$\max \left\{ |\lambda_n - \lambda|, \frac{\|\varphi_n - \varphi_{(n)}\|}{\|\varphi_n\|} \right\} \leq l\big(\|T_n - T\| + \|S_n - S\|\big),$$

where l is a constant independent of n.

Proof . Let λ be a simple generalized eigenvalue of (T, S) and φ be a corresponding generalized eigenvector. If $\varepsilon > 0$ is small enough, then by Theorem 6 of [7], there is a positive integer n_0 such that for each $n \geq n_0$, we have a unique $\lambda_n \in sp(T_n, S_n)$ satisfying $|\lambda_n - \lambda| < \varepsilon$. Further, λ_n is a simple eigenvalue of (T_n, S_n) corresponding to the generalized eigenvector φ_n, where $\lambda_n \longrightarrow \lambda$.
Fix $\lambda_0 \in re(T, S)$, where for any Cauchy contour Θ associated with λ, $\lambda_0 \notin \Theta$. We can prove that $(\lambda_0 - \lambda)^{-1}$ is a simple eigenvalue of $(\lambda_0 S - T)^{-1} S$ corresponding to the eigenvector φ. Indeed,

$$\begin{aligned}
\varphi \in \mathrm{Ker}(T - \lambda S) \quad &\Rightarrow \quad (T - \lambda S)\varphi = 0 \\
&\Rightarrow \quad (\lambda_0 S - T)^{-1}(\lambda_0 S - T + T - \lambda S)\varphi = \varphi \\
&\Rightarrow \quad (\lambda_0 S - T)^{-1} S u = (\lambda_0 - \lambda)^{-1}\varphi \\
&\Rightarrow \quad \varphi \in \mathrm{Ker}\big((\lambda_0 S - T)^{-1} S - (\lambda_0 - \lambda)^{-1} I\big).
\end{aligned}$$

Further, we reverse the last process to find

$$\mathrm{Ker}(T - \lambda S) = \mathrm{Ker}\big((\lambda_0 S - T)^{-1} S - (\lambda_0 - \lambda)^{-1} I\big).$$

Now, as $T_n \overset{\mathrm{n}}{\longrightarrow} T$ and $S_n \overset{\mathrm{n}}{\longrightarrow} S$ then according to Theorem 2.1, we find that, for all large n, $\lambda_0 \in re(T_n, S_n)$. Hence, with the same technics, we can also prove that $(\lambda_0 - \lambda_n)^{-1}$ is a simple eigenvalue of $(\lambda_0 S_n - T_n)^{-1} S_n$ corresponding to the eigenvector φ_n, and that

$$\mathrm{Ker}(T_n - \lambda S_n) = \mathrm{Ker}\big((\lambda_0 S_n - T_n)^{-1} S_n - (\lambda_0 - \lambda)^{-1} I\big).$$

We constate. also that $\overline{\lambda}_n$ is a simple generalized eigenvalue of (T_n^*, S_n^*) corresponding to the generalized eigenvector φ_n^* if, and only if $\overline{(\lambda_0 - \lambda_n)^{-1}}$ is a simple eigenvalue of $(\overline{\lambda}_0 S_n^* - T_n^*)^{-1} S_n^*$ corresponding to the eigenvector φ_n^*.On the other hand, by Theorem 2.1, we have

$$(\lambda_0 S_n - T_n)^{-1} S_n \overset{\mathrm{n}}{\longrightarrow} (\lambda_0 S - T)^{-1} S.$$

So, according to Theorem 3.10 of [5], which shows the convergence of the approximate eigenvectors towards the exact eigenvectors, $\varphi_n \to \varphi$, then we have $\langle \varphi, \varphi_n^* \rangle \neq 0$. Thus

$$\max \left\{ |(\lambda_0 - \lambda_n)^{-1} - (\lambda_0 - \lambda)^{-1}|, \frac{\|\varphi_n - \varphi_{(n)}\|}{\|\varphi_n\|} \right\} \leq l_1\big(\|(\lambda_0 S_n - T_n)^{-1} S_n - (\lambda_0 S - T)^{-1} S\|\big),$$

where l_1 is a constant independent of n. So, we can find easily that,

$$\max \left\{ |\lambda_n - \lambda|, \frac{\|\varphi_n - \varphi_{(n)}\|}{\|\varphi_n\|} \right\} \leq l\big(\|T_n - T\| + \|S_n - S\|\big),$$

where l is a constant independent of n. □ Now let's analyse the Generalized Elementary Iteration (G.E.I) given as follow

$$(E): \qquad \begin{cases} \varphi_n^{(0)} = \varphi_n, \text{ and for } k = 1, 2, ... \\[2ex] \lambda_n^{(k)} = \dfrac{\langle T\varphi_n^{(k-1)}, \varphi_n^* \rangle}{\langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle}, \quad \langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle \neq 0, \\[2ex] \varphi_n^{(k)} = \varphi_n^{(k-1)} + Q_n\big(\lambda_n^{(k)} S\varphi_n^{(k-1)} - T\varphi_n^{(k-1)}\big). \end{cases}$$

where Q_n is defined as:

$$Q_n = -\frac{1}{2\pi i}\int_\Theta (T_n - zS_n)^{-1}(\lambda - z)^{-1}dz : X \to X.$$

Next, we build three essential relations about the G.E.I

First, using the proprieties of the operator projections Q_n and P_n given in Theorem 1.1 page 50 of [1], where

$$P_n = -\frac{1}{2\pi i}\int_\Theta (T_n - zS_n)^{-1}S_n dz : X \to X$$

and P_n^* designates the adjoint of P_n. We have, for $x \in X$

$$\langle Q_n x, \varphi_n^* \rangle = \langle Q_n x, P_n^* \varphi_n^* \rangle = \langle P_n Q_n x, \varphi_n^* \rangle = \langle 0, \varphi_n^* \rangle = 0.$$

Further, since

$$\langle \varphi_n^{(0)}, \varphi_n^* \rangle = \langle \varphi_n, \varphi_n^* \rangle = 1$$

and for $k = 1, 2, ...$

$$\begin{aligned}\langle \varphi_n^{(k)}, \varphi_n^* \rangle &= \langle \varphi_n^{(k-1)}, \varphi_n^* \rangle + \langle Q_n\big(\lambda_n^{(k)} S\varphi_n^{(k-1)} - T\varphi_n^{(k-1)}\big), \varphi_n^* \rangle \\ &= \langle \varphi_n^{(k-1)}, \varphi_n^* \rangle,\end{aligned}$$

we get the first relation

$$(E1): \qquad \langle \varphi_n^{(k)}, \varphi_n^* \rangle = 1 \;\text{ for all }\; k = 0, 1, 2, ...$$

We note that $(E1)$ is equivalent to

$$P_n \varphi_n^{(k)} = \varphi_n, \text{ for all } k = 0, 1, 2, ...$$

Next, for all $x \in X$, we have

$$\langle T_n x, \varphi_n^* \rangle = \langle x, T_n^* \varphi_n^* \rangle = \langle x, \overline{\lambda_n} S_n^* \varphi_n^* \rangle = \langle \lambda_n S_n x, \varphi_n^* \rangle,$$

therefore

$$\Big\langle \big(T_n - \lambda_n S_n\big)x, \varphi_n^* \Big\rangle = 0,$$

and as

$$\lambda = \frac{\langle T\varphi_{(n)}, \varphi_n^* \rangle}{\langle S\varphi_{(n)}, \varphi_n^* \rangle},$$

then using the notation

$$\widetilde{x} = \frac{x}{\langle Sx, \varphi_n^* \rangle},$$

we obtain,

$$\begin{aligned}\lambda_n^{(k)} - \lambda &= \frac{\langle T\varphi_n^{(k-1)}, \varphi_n^* \rangle}{\langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle} - \frac{\langle T\varphi_{(n)}, \varphi_n^* \rangle}{\langle S\varphi_{(n)}, \varphi_n^* \rangle} \\ &= \Big\langle T\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle \\ &= \Big\langle (T - T_n)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle + \Big\langle T_n\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle \\ &= \Big\langle (T - T_n)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle + \lambda_n \Big\langle S_n\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle \\ &= \Big\langle (T - T_n)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle + \lambda_n \Big\langle (S_n - S)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle \\ &\quad + \lambda_n \Big(\langle S\widetilde{\varphi}_n^{(k-1)}, \varphi_n^* \rangle - \langle S\widetilde{\varphi}_{(n)}, \varphi_n^* \rangle \Big) \\ &= \Big\langle \big((T - T_n) - \lambda_n(S - S_n)\big)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle \\ &\quad + \lambda_n \Big(\frac{\langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle}{\langle S\varphi_n^{(k-1)}, \varphi_n^* \rangle} - \frac{\langle S\varphi_{(n)}, \varphi_n^* \rangle}{\langle S\varphi_{(n)}, \varphi_n^* \rangle} \Big) \\ &= \Big\langle \big((T - T_n) - \lambda_n(S - S_n)\big)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle.\end{aligned}$$

Thus, we get the second relation

$$(E2): \qquad \lambda_n^{(k)} - \lambda = \Big\langle \big((T - T_n) - \lambda_n(S - S_n)\big)\big(\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\big), \varphi_n^* \Big\rangle.$$

Finally for $k = 1, 2, ...$ we have

$$\begin{aligned}
\varphi_n^{(k)} &= \varphi_n^{(k-1)} + Q_n\big(\, \lambda_n^{(k)} S \varphi_n^{(k-1)} - T\varphi_n^{(k-1)} \,\big) \\
&= \varphi_n^{(k-1)} + Q_n\big(\, (\lambda_n^{(k)} S - \lambda_n S_n)\varphi_n^{(k-1)} + (T_n - T)\varphi_n^{(k-1)} \,\big) \\
&\quad - Q_n\big(\, (T_n - \lambda_n S_n)\varphi_n^{(k-1)} \,\big).
\end{aligned}$$

Then, by Theorem 1.1 page 50 of [1] (relation(12)),

$$\begin{aligned}
Q_n(T_n - \lambda_n S_n)\varphi_n^{(k-1)} &= (I - P_n)\varphi_n^{(k-1)} \\
&= \varphi_n^{(k-1)} - P_n\varphi_n^{(k-1)} \\
&= \varphi_n^{(k-1)} - \varphi_n.
\end{aligned}$$

Therefore,

$$\varphi_n^{(k)} = \varphi_n + Q_n\big(\, (\lambda_n^{(k)} S - \lambda_n S_n)\varphi_n^{(k-1)} + (T_n - T)\varphi_n^{(k-1)} \,\big),$$

but since $T\varphi_{(n)} = \lambda S\varphi_{(n)}$, we have also

$$\begin{aligned}
\varphi_{(n)} &= P_n\varphi_{(n)} + (I - P_n)\varphi_{(n)} \\
&= \varphi_n + Q_n\big((T_n - \lambda_n S_n)\varphi_{(n)}\big) \\
&= \varphi_n + Q_n\big(\, (T_n - T)\varphi_{(n)} + (\lambda S - \lambda_n S_n)\varphi_{(n)} + (T - \lambda S)\varphi_{(n)} \,\big) \\
&= \varphi_n + Q_n\big(\, (T_n - T)\varphi_{(n)} + (\lambda S - \lambda_n S_n)\varphi_{(n)} \,\big).
\end{aligned}$$

So, the third relation

$$(E3): \quad \begin{cases}
\text{for } k = 1, 2, \quad ... \\
\varphi_n^{(k)} - \varphi_{(n)} \quad = Q_n\Big[\, \big(T_n - T\big)\,(\varphi_n^{(k-1)} - \varphi_{(n)}) \\
\qquad\qquad + \big(\lambda_n^{(k)} S - \lambda_n S_n\big)\,\varphi_n^{(k-1)} - \big(\lambda S - \lambda_n S_n\big)\,\varphi_{(n)} \,\Big] \\
\qquad\qquad = Q_n\Big[\, \big((T_n - T) - \lambda_n(S_n - S)\big)(\varphi_n^{(k-1)} - \varphi_{(n)}) \\
\qquad\qquad + (\lambda_n^{(k)} - \lambda)S\varphi_n^{(k-1)} + (\lambda - \lambda_n)S\,(\varphi_n^{(k-1)} - \varphi_{(n)}) \,\Big].
\end{cases}$$

These equations (E1), (E2) and (E3) are essential in convergence proofs and error estimating of the G.E.I sheme (E)

Theorem 2.3. Let assume that,

$$\text{i) } T_n \xrightarrow{\text{n}} T, \qquad \text{ii) } S_n \xrightarrow{\text{n}} S. \tag{2.1}$$

For each large n, we chose φ_n such that the sequence $(\|\varphi_n\|)$ is bounded and also bounded from zero. Then there is a positive integer n_1 such that for all $n \geq n_1$ and for all $k = 1, 2, ...$

$$\max\big\{\, |\lambda_n^{(k)} - \lambda|, \|\varphi_n^{(k)} - \varphi_{(n)}\| \,\big\} \leq \Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^{k+1},$$

where β is a constant independent of n and k.

Proof . By Theorem 2.2, we can find that the sequence $(\|\varphi_{(n)}\|)$ and $(\|\varphi_n^*\|)$ are bounded. Further, since the sequence $(\|\varphi_n\|)$ is bounded and also bounded from zero. Also, the sequences $(\|T_n\|)$ and $(\|S_n\|)$ are bounded. Hence there are constants

$$\|\varphi_n\| \leq \gamma, \quad \|\varphi_n^*\| \leq p, \quad \|Q_n\| \leq a, \quad |\lambda_n| \leq c, \quad \|S\| = s, \quad |\langle S\varphi_{(n)}, \varphi_n^* \rangle| = \alpha.$$

Let $\gamma_1 = \max\{1, \gamma\}$, $c_1 = \max\{1, c\}$, and $\beta_1 = \max\Big\{\dfrac{2}{\alpha}, \dfrac{2(\gamma s p)}{\alpha^2}\Big\}$.

Now, by (2.1) and according to Theorem 2.2, there is a positive integer n_0, and there is a constant l such that for all $n \geq n_0$

$$\max\{|\lambda_n - \lambda|, \|\varphi_n - \varphi_{(n)}\|\} \leq l\gamma_1\big(\|T_n - T\| + \|S_n - S\|\big).$$

Let

$$\beta = \max\{l\gamma_1, \beta_1, a(c_1 + (1+q)c_1 p\beta_1 s + l\gamma_1)s\}.$$

We choose $n_1 \geq n_0$ such that

$$\beta\big(\|T_n - T\| + \|S_n - S\|\big) \leq \frac{\alpha}{2(s\,p+1)(1+\alpha)}.$$

Then, we fix $n \geq n_1$, we prove by induction on k that

$$\max\{|\lambda_n^{(k)} - \lambda|, \|\varphi_n^{(k)} - \varphi_{(n)}\|\} \leq \Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^{k+1}, \text{ for } k = 0, 1, 2...$$

Since $\lambda_n^{(0)} = \lambda_n$, $\varphi_n^{(0)} = \varphi_n$ and $l\gamma_1 \leq \beta$ and for $n_1 \geq n_0$, we find that the expected inequality remains if $k = 0$.

Next, we assume that the expected inequality is true for a given $k \geq 0$, then we demonstrate that is true with k replaced by $k+1$.

Using the equation (E2), we have

$$|\lambda_n^{(k)} - \lambda| \leq \big(\|T_n - T\| + |\lambda_n|\|S_n - S\|\big)\|\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\|\|\varphi_n^*\|, \tag{2.2}$$

where

$$\begin{cases} \widetilde{\varphi}_n^{(k-1)} = \dfrac{\varphi_n^{(k-1)}}{\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle}, \\[2ex] \widetilde{\varphi}_{(n)} = \dfrac{\varphi_{(n)}}{\big\langle S\varphi_{(n)}, \varphi_n^*\big\rangle}. \end{cases}$$

On the other hand, we have

$$\begin{aligned} \|\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\| &\leq \|\frac{\varphi_n^{(k-1)}}{\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle} - \frac{\varphi_{(n)}}{\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle}\| + \|\varphi_{(n)}\|\Big|\frac{1}{\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle} - \frac{1}{\big\langle S\varphi_{(n)}, \varphi_n^*\big\rangle}\Big| \\ &\leq \frac{\|\varphi_n^{(k)} - \varphi_{(n)}\|}{|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle|} + \|\varphi_{(n)}\|\frac{|\big\langle S(\varphi_n^{(k-1)} - \varphi_{(n)}), \varphi_n^*\big\rangle|}{|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle||\big\langle S\varphi_{(n)}, \varphi_n^*\big\rangle|} \\ &\leq \Big[\frac{1}{|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle|} + \frac{\|S\|\|\varphi_{(n)}\|\|\varphi_n^*\|}{|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle||\big\langle S\varphi_{(n)}, \varphi_n^*\big\rangle|}\Big]\|\varphi_n^{(k)} - \varphi_{(n)}\|. \end{aligned} \tag{2.3}$$

Proving now that $|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle| \neq 0$ and that

$$\frac{1}{|\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle|} \leq \frac{2}{\alpha}.$$

Indeed, we remark that

$$\begin{aligned} \||\big\langle S\varphi_n^{(k-1)}, \varphi_n^*\big\rangle| - |\big\langle S\varphi_{(n)}, \varphi_n^*\big\rangle|\| &\leq \|S\|\|\varphi_n^*\|\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \\ &\leq \|S\|\|\varphi_n^*\|\Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^k \\ &\leq p\,s\,\big(\frac{\alpha}{2(s\,p+1)(1+\alpha)}\big)^k \\ &\leq p\,s\,\big(\frac{\alpha}{2(s\,p+1)(1+\alpha)}\big) \leq \frac{\alpha}{2}. \end{aligned}$$

Since $|\langle S\varphi_{(n)}, \varphi_n^*\rangle| = \alpha \neq 0$, then $|\langle S\varphi_n^{(k-1)}, \varphi_n^*\rangle| \neq 0$ and

$$|\langle S\varphi_{(n)}, \varphi_n^*\rangle| \leq |\langle S\varphi_n^{(k-1)}, \varphi_n^*\rangle| + \frac{\alpha}{2} = |\langle S\varphi_n^{(k-1)}, \varphi_n^*\rangle| + \frac{1}{2}|\langle S\varphi_{(n)}, \varphi_n^*\rangle|,$$

which implies that

$$\frac{1}{|\langle S\varphi_n^{(k-1)}, \varphi_n^*\rangle|} \leq \frac{2}{\alpha}.$$

According to (2.3), we have

$$\|\widetilde{\varphi}_n^{(k-1)} - \widetilde{\varphi}_{(n)}\| \quad \leq \quad \big(\frac{2}{\alpha} + 2\frac{\gamma ps}{\alpha^2}\big)\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \leq \beta_1\|\varphi_n^{(k-1)} - \varphi_{(n)}\|,$$

and then by inserting the previous inequality in (2.2), we obtain

$$|\lambda_n^{(k)} - \lambda| \leq cp\beta_1\big(\|T_n - T\| + \|S_n - S\|\big)\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \leq \Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^{k+1}.$$

Next, by the equation (E3), we have

$$\begin{aligned}\|\varphi_n^{(k)} - \varphi_{(n)}\| \quad &\leq \quad \|Q_n\|\big(\|T_n - T\| + |\lambda_n|\|S_n - S\|\big)\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \\ &+ \quad |\lambda_n^{(k)} - \lambda|\|S\|\|\varphi_n^{(k-1)}\| + |\lambda_n - \lambda|\|S\|\|\varphi_n^{(k-1)} - \varphi_{(n)}\|.\end{aligned}$$

Note that

$$|\lambda_n - \lambda| \leq l\gamma_1\big(\|T_n - T\| + |\lambda_n|\,\|S_n - S\|\big),$$

and

$$|\lambda_n^{(k)} - \lambda| \leq cp\beta_1\big(\|T_n - T\| + \|S_n - S\|\big)\|\varphi_n^{(k-1)} - \varphi_{(n)}\|,$$

and since

$$\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \leq \Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^{k+1} \leq 1,$$

which implies that

$$\|\varphi_n^{(k-1)}\| \leq \|\varphi_n^{(k-1)} - \varphi_{(n)}\| + \|\varphi_{(n)}\| \leq (1 + q).$$

Hence,

$$\begin{aligned}\|\varphi_n^{(k)} - \varphi_{(n)}\| \quad &\leq \quad a(c_1 + (1+q)c_1p\beta_1 s + c_1\gamma_1 s)\big(\|T_n - T\| + \|S_n - S\|\big)\|\varphi_n^{(k-1)} - \varphi_{(n)}\| \\ &\leq \quad \Big(\beta\big(\|T_n - T\| + \|S_n - S\|\big)\Big)^{k+1}.\end{aligned}$$

Thus the expected inequality is true for k and the induction is complete. □

3 Numerics

In this section, we study the following generalized spectral problem:

$$\text{Find } (\varphi, \lambda) \in X \times \mathbb{C} : \ \varphi + T\varphi = \lambda S\varphi,$$

where T and S are two integral operators defined on $X = \mathcal{C}([0, a])$. So, T and S are given by:

$$Tu(x) = \int_0^a k_1(x, y)u(y)dy, \quad Su(x) = \int_0^a k_2(x, y)u(y)dy, \quad u \in \mathcal{C}([0, a]).$$

We assume that in the following, the functions k_1 and k_2 are continuous.

Let $(x_i)_{1\leq i\leq n}$ a grid in $[0, a]$,

$$h_n = \frac{a}{n-1}, \ x_i = (i-1)h_n, \ 1 \leq i \leq n.$$

Then we establish the canonical basis of the hat functions on $(x_i)_{1\leq i\leq n}$ as

$$e_i(x) = \begin{cases} 1-\dfrac{|x-x_i|}{h_n} & \text{for } x_{i-1}\leq x\leq x_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$e_1(x) = \begin{cases} \dfrac{x_2-x}{h_n} & \text{for } x_1\leq x\leq x_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$e_n(x) = \begin{cases} \dfrac{x-x_{n-1}}{h_n} & \text{for } x_{n-1}\leq x\leq x_n, \\ 0 & \text{otherwise.} \end{cases}$$

We put,

$$w_{1,i}(x) = \int_0^a k_1(x,y)e_i(y)dy,\ w_{2,i}(x) = \int_0^a k_2(x,y)e_i(y)dy,\ 1\leq i\leq n.$$

Thus, we consider the matrices $A_n, B_n \in \mathbb{C}^{n\times n}$ such that

$$A_n(i,j) = w_{1,i}(x_j),\ B_n(i,j) = w_{2,i}(x_j).$$

Apply Kantorovich's projection method (see [8]), i.e. we change the formula of the operators T and S by $\pi_n T$ and $\pi_n S$ respectively. So, we get for all $x\in[0,a]$

$$u_n(x) + \sum_{i=1}^{n}(\int_0^a k_1(x_i,y)u_n(y)dy)e_i(x)$$
$$= \lambda_n \sum_{i=1}^{n}(\int_0^a k_2(x_i,y)u_n(y)dy)e_i(x).$$

Multiplying first by $k_1(x_j,x)$ then by $k_2(x_j,x)$, and integrating over $[0,a]$, so these equations lead to the implementation matrix of generalized eigenvalue problem as:

$$\begin{bmatrix} A_n + I_{n\times n} & O_{n\times n} \\ B_n & I_{n\times n} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \lambda_n \begin{bmatrix} O_{n\times n} & A_n \\ O_{n\times n} & B_n \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

where, $\beta_1,\beta_2\in\mathbb{C}^n$. The generalized eigenvector u_n associated to λ_n is given by:

$$u_n(x) = \sum_{i=1}^{n}(\lambda_n\beta_2(i) - \beta_1(i))e_i(x).$$

Kantorovich's projection method is norm-convergent as proved in [8].

The developed scheme (E) requires the evaluation of T and S at certain points of X; In particular T and S are not used for this purpose, and approximate operators T_m and S_m are preferred, where m is large enough than n. The implementation of refinement scheme (E) involves the following matrices P and R. For $n,m\in\mathbb{N}$, where $m>n$, we define a grid $(y_i)_{1\leq i\leq m}$ on $[0,a]$ as previously,

$$h_m = \frac{a}{m-1},\ y_i = (i-1)h_m,\ 1\leq i\leq m.$$

Let $(e_{i,m})_{1\leq i\leq m}$ be the canonical basis of the hat functions given on $(y_i)_{1\leq i\leq m}$. We assume that the two grids $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ are uniform, i.e., the number

$$r_0 = \frac{m-1}{n-1}$$

is a positive integer.

Let us now define the extension matrix $P\in\mathbb{C}^{m\times n}$, for all $k=1,...,m$ and for $j=1,...,n$

$$P(k,j) = e_{n,j}(y_k),$$

then we define the restriction matrix $R \in \mathbb{C}^{n\times m}$, for all $k = 1, ..., m$ and for $j = 1, ..., n$

$$R(j,k) = \begin{cases} 1 & \text{if } k = (j-1)r_0 + 1, \\ 0 & \text{else.} \end{cases}$$

Finally, let us denote by

$$D^T = A_m P,\ C^T = RA_m,\ D^S = B_m P,\ C^S = RB_m.$$

Algorithm

▷ Construction of A_n, B_n, A_m, B_m, D^T, D^S, C^T and C^S.

▷ $\beta = (\beta_1, \beta_2)$ and $\lambda \longleftarrow$ solutions of

$$\begin{bmatrix} A_n + I_{n\times n} & O_{n\times n} \\ B_n & I_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \lambda_n \begin{bmatrix} O_{n\times n} & A_n \\ O_{n\times n} & B_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

- $u_n^{(0)} = -\beta_1 + \lambda_n \beta_2$

▷ $u_m^{(0)} = (\lambda_n D^S - D^T) u_n^{(0)}$

▷ $E_{n,m}^{(0)} \longleftarrow \dfrac{\|u_m^{(0)} + A_m u_m^{(0)} - \lambda_n B_m u_m^{(0)}\|}{\|u_m^{(0)}\|}$

▷ $\lambda_n^{(k)} = \dfrac{v' \cdot [\lambda_n C^S - C^T] D^T u_n^{(0)} + 1}{v' \cdot [\lambda_n C^S - C^T] D^S u_n^{(0)}}$

- $b_n^{(k)} = (\lambda_n C^S - C^T)(\lambda_n^{(k)} D^S - D^T u_n^{(0)}) u_n^{(0)}$
- $b_m^{(k)} = (\lambda_n B_m - A_m)(\lambda_n^{(k)} D^S - D^T u_n^{(0)}) u_n^{(0)}$

▷ $w_n^{(k)} \longleftarrow$ solution of $\begin{cases} (I_n + A_n - \lambda_n B_n) w_n^{(k)} = b_n^{(k)}, \\ w_n^{(k)} \cdot v' = 0 \end{cases}$

- $u_m^{(k)} = u_m^{(0)} + (\lambda_n D^S - D^T) w_n^{(k)} + b_m^{(k)}$

▷ $E_{n,m}^{(k)} \longleftarrow \dfrac{\|u_m^{(k)} + A_m u_m^{(k)} - \lambda_n^{(k)} B_m u_m^{(k)}\|}{\|u_m^{(k)}\|}$

- $u_n^0 = u_n^{(k)} + w_n^{(k)}$.

For the numerical results, we use the kernels

$$k_1(x,y) = (x+y)^2, \quad k_2(x,y) = y^2(x+y)^2.$$

We applied our algorithm on the second approximated generalized eigenvalue $\lambda_{n,2}$ which are ordered in ascending order of the absolute values. We note (see Tab. 1) that the convergence is established, where we have chose $n = 10$ and $m = 100$.

4 Final remarks

As a general conclusion, through this work, we laid the first stone for constructing a generalized iterative scheme for the generalized spectrum problem related to two bounded operators in an infinite Banach space. However, as an open problem, we are trying to address the same generalized iterative schema but for more general unbounded operators (T, S), and also in a two-dimensional or three-dimensional spatial context.

Table 1: The numerical results, where $n = 10$ and $m = 100$

n=10 m=100	$E^k_{n,m}$		
k=0	0.1318271192004 e-02	k=5	0.1161093961081 e-02
k=1	0.1300527287167 e-02	k=6	0.1105596608060 e-02
k=2	0.1276578541196 e-02	k=7	0.1040187537319e-02
k=3	0.01245814226672 e-02	k=8	0.963895915666 e-03
k=4	0.1207564368813 e-02	k=9	0.875656398268 e-03

References

[1] I. Gohberg, S. Goldbery and M.A. Kaashoek, *Classes of Linear Operators Vol I*, Springer Basel AG, Birkhuser Basel, 1990.

[2] A. Khellaf, W. Merchela and H. Guebbai, *New sufficient conditions for the computation of generalized eigenvalues*, Russian Math. **65** (2021), 65–68.

[3] H. Guebbai, *Generalized spectrum approximation and numerical computation of eigenvalues for Schrödinger's operators*. Lobachevskii J. Math. **34** (2013), 45–60.

[4] A. Khellaf, H. Guebbai, S. Lemita and M. Z. Aissaoui, *Eigenvalues computation by the generalized spectrum method of Schrödinger's operator*, Comput. Appl. Math. **37** (2018), 5965–5980.

[5] A. Khellaf, S. Benarab, H. Guebbai, W. Merchela, *A class of strongly stable approximation for unbounded operators*, Vestnik Tambovskogo Univer. Seriya: Estest. Tekh. Nauki-Tambov Univ. Rep. Ser. Nat. Tech. Sci. **24** (2019), 218–234 .

[6] A. Khellaf, *New sufficient conditions in the generalized spectrum approach to deal with spectral pollution*, Vestnik Tambovskogo Univer. Seriya: Estest. Tekh. Nauki-Tambov Univ. Rep. Ser. Nat. Tech. Sci. **23** (2018), 595–604.

[7] A. Khellaf and H. Guebbai, *A Note on genralized spectreum approximation*, Lobachevskii J. Math. **39** (2018), 1388–1395.

[8] M. Ahues, A. Largillier and B.V Limaye, *Spectral computations for bounded operators*, Appl. Math. 18, Chapman and Hall-CRC, Boca Raton 2001.