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Some of the graph energies of zero-divisor graphs of finite commutative rings

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Abstract

In this paper, we investigate some of the graph energies of the zero-divisor graph $\Gamma(R)$ of finite commutative rings R. Let $Z(R)$ be the set of zero-divisors of a commutative ring R with non-zero identity and $Z^*(R) = Z(R) \setminus \{0\}$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set in $Z^*(R)$ and two vertices u and v are adjacent if and only if $uv = vu = 0$. We investigate some energies of $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers.

Keywords: Commutative ring, Zero-divisor graph, Line graph, Minimum edge dominating energy, Laplacian energy
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1 Introduction

Assume that $G = (V, E)$ is a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. The number of edges incident to vertex u in G is denoted $deg_G(u) = d(u)$. The isolated vertex and pendant vertex are the vertices with degrees zero and 1 in graph G, respectively.

The adjacency matrix of G, $A(G) = (a_{ij})$ is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E$ and $a_{ij} = 0$ otherwise. The eigenvalues of graph G are the eigenvalues of its adjacency matrix $A(G)$ [17]. The energy of a graph G was introduced in the 1970s as $E(G) = \sum_{i=1}^{n} |\lambda_j(G)|$ in which $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A(G)$ [13]. The edge energy of a graph G is defined as the sum of the absolute values of eigenvalues of $A(L_G)$ [6] in which L_G is the line graph of G. The line graph L_G of G is the graph that each vertex of it represents an edge of G and two vertices of L_G are adjacent if and only if their corresponding edges are incident in G [17].

Let $D(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree of the vertex v_i of the graph G. Then the matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian matrix, respectively, of the graph G. If $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ are, respectively, the eigenvalues of the matrices $L(G)$ and $L^+(G)$, then the Laplacian energy of G is defined as [14]

$$LE = LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|,$$

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and the signless Laplacian energy is defined as follows [15]

$$LE^{+} = LE^{+}(G) = \sum_{i=1}^{n} |\mu_i^{+} - \frac{2m}{n}|.$$

Details on the properties and results of Laplacian and signless Laplacian energies and energy of a line graph can be found in [14, 15, 10, 11, 24].

A subset D of V is the dominating set of graph G if every vertex of $V \setminus D$ is adjacent to some vertices in D. Any dominating set with minimum cardinality is called a minimum dominating set [18]. The minimum dominating energy of graph G, denoted by $E_D(G)$, is introduced as the sum of the absolute values of eigenvalues of the minimum dominating matrix [25]. The minimum dominating matrix $A_D(G)$ is as following

$$A_D(G) := (a_{ij}) = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \textit{ and } v_i \in D \\ 0 & \textit{otherwise} \end{cases}$$

A set F of edges in G is the edge dominating set if every edge in $E \setminus F$ is adjacent to at least one edge in F. The edge domination number, denoted by γ', is the minimum the cardinalities of the edge dominating sets of G [12]. Note that F is the minimum edge dominating set of G or the minimum dominating set of L_G. The minimum edge dominating matrix of G is the $m \times m$ matrix defined by $A_F(G) := (a_{ij})$ in which

$$A_F(G) := (a_{ij}) = \begin{cases} 1 & \text{if } e_i \textit{ and } e_j \textit{ are adjacent}, \\ 1 & \text{if } i = j \textit{ and } e_i \in F, \\ 0 & \textit{otherwise.} \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [3] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A_F(G)$. Note the minimum edge dominating energy of graph G is a minimum dominating energy for its line graph L_G. Details on the properties and results of the minimum dominating energy of a graph and its line graph can be found in [25, 3, 21, 22, 23, 19].

Let R be a ring and $Z(R)$ denotes the set of all zero-divisors of R. The zero-divisor graph of R is a simple graph $\Gamma(R)$ with vertex set $Z(R) \setminus \{0\}$ such that distinct vertices x and y are adjacent if and only if $xy = 0$ [4].

In [7] some energies of graphs $\Gamma(R)$ and line graph $\Gamma(R)$ for the commutative ring $\mathbb{Z}_n$ are investigated. In this paper, we investigate graph energy, Laplacian energy, signless Laplacian energy, edge energy and the minimum edge dominating energy of $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers.

2 Preliminaries

In this section, we state some previous results that will be used in the next section. First, we recall the definition of the Zagreb index of graph G. The Zagreb index $M(G)$ is defined as $M(G) = \sum_{i=1}^{n} d_i^2$ such that the vertices have the degree d_i for $i = 1, 2, \ldots, n$ [16].

Lemma 2.1. [21] Let G be a graph with m edges. If F is the minimum edge dominating set of graph G, then

$$EE_F(G) \leq M(G) - m,$$

where $M(G)$ is the Zagreb index of graph G.

Lemma 2.2. [21] Let G be a graph of the order n with m edges. If F is the minimum edge dominating set of graph G with cardinality k, then

$$EE_F(G) \leq 4m - 2n + k.$$

Lemma 2.3. [21] Let G be a connected graph with n vertices and $m(\geq n)$ edges. Then

$$EE_F(G) \geq 4(m-n+s)+2p,$$

where p and s are the number of pendant vertices and isolated vertices in G.

Lemma 2.4. [15] Let G be a graph of order n with m edges. Then

$$\sqrt{2M(G)-4m} \leq E(L_G) \leq M(G)-2m,$$

where L_G and $M(G)$ are the line graph and the Zagreb index of graph G.

Lemma 2.5. [11] Let G be a connected graph of order n. Then

$$E(L_G) \geq 2\big(E(G)-2v^+\big),$$

where v^+ is the number of positive eigenvalues.

Lemma 2.6. [15] Let G be a graph with n vertices and m edges such that $m > n$. Then

$$E(L_G) < LE^+(G)+4(m-n).$$

Lemma 2.7. [9] Let G be a graph of order n with $m \geq \frac{n}{2}$ edges and the maximum degree Δ. Then

$$2\big(\Delta+1-\frac{2m}{n}\big) \leq LE(G) \leq 4m-2\Delta-\frac{4m}{n}+2.$$

Lemma 2.8. [21] Let G be a simple graph and L_G the line graph of G. If F is the minimum edge dominating set with $|F|=k$, then

$$EE_F(G) \leq EE(G)+k.$$

Lemma 2.9. [27, 29] For a graph G with n vertices and m edges,

$$\frac{4m}{n} \leq LE(G) \leq 4m\big(1-\frac{1}{n}\big).$$

Lemma 2.10. [1] Let G be a graph with n vertices and m edges. Then

$$LE^+(G) \leq 4m\big(1-\frac{1}{n}\big).$$

Lemma 2.11. [5] Let G be a graph with n vertices and m edges. Then, $\gamma' \leq \lfloor \frac{n}{2} \rfloor$.

3 Main Results

In this section, we study energies of the zero-divisor graphs $\Gamma(R)$ such as the edge energy, the minimum edge dominating energy, the Laplacian energy and the signless Laplacian energy of the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers. Firstly, we investigate these energies of the zero divisor graph $\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)$ where is defined as follows.

For $x \in \mathbb{Z}_{p^2}$ and $y \in \mathbb{Z}_q$, $(x,y) \notin V\Big(\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)\Big)$ if and only if $x \neq p, 2p, \ldots, (p-1)p$ and $y \neq 0$. According to the structure of graph $\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)$, the number of vertices is equal to $p^2+pq-p-1$ [2]. Authors in [2], characterized the vertices of graph $\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)$ according to their degree as follows.

$$\begin{aligned}
A &= \big\{(0,y) : y \in \{1,2,\ldots,q-1\}\big\}\ ,\ |A| = q-1,\\
B &= \big\{(x,y) : x = p, 2p, \ldots, (p-1)p \textit{ and } y \in \{1,2,\ldots,q-1\}\big\}\ ,\ |B| = (p-1)(q-1),\\
C &= \big\{(x,0) : x \in \mathbb{Z}_{p^2} \setminus \{0, p, 2p, \ldots, (p-1)p\}\big\}\ ,\ |C| = p-1,\\
D &= \big\{(x,0) : x = p, 2p, \ldots, (p-1)p\big\}\ ,\ |D| = p(p-1).
\end{aligned}$$

Also, they obtained the degree sequence DS of this graph as follows.

$$DS\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)=\big\{(p^2-1)^{[q-1]},(p-1)^{[(p-1)(q-1)]},(pq-2)^{[p-1]},(q-1)^{[(p^2-p)]}\big\}. \tag{3.1}$$

Therefore, the number of edges in this graph is equal to $m=\frac{(p-1)(4pq-3p-2)}{2}$ [2].

Theorem 3.1. Let $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$ for prime numbers $p,q>2$ be the zero-divisor graph of size m.

i) If $p>q$, then $2\big(p^2-\alpha\big)\le LE\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)\le 4(m+1)-(2p^2+\alpha)$,

ii) If $p<q$, then $2\big(pq-\alpha-1\big)\le LE\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)\le 4(m+1)-2(pq+\alpha-1)$.

where $\alpha=\frac{2(p-1)(4pq-3p-2)}{p^2+pq-p-1}$.

Proof . Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$ for prime numbers $p,q>2$ such that the number of vertices and the number of edges are $n=p^2+p(q-1)-1$ and $m=\frac{(p-1)(4pq-3p-2)}{2}$, respectively. According to the sequence degree (3.1) of the zero-divisor graph G, we consider two following cases.

Case 1: If $p>q$, then the maximum degree of graph G is $\Delta=p^2-1$. Using Lemma 2.7, we have

$$\begin{aligned}LE(G)&\le 4m-2\Delta-\frac{4m}{n}+2\\&=4m-2(p^2-1)+2-\frac{4(p-1)(4pq-3p-2)}{2(p^2+p(q-1)-1)}\\&=4m+4-2p^2-\frac{2(p-1)(4pq-3p-2)}{(p^2+p(q-1)-1)}.\end{aligned}$$

With putting $\alpha=\frac{2(p-1)(4pq-3p-2)}{p^2+pq-p-1}$, the result holds for the upper bound. For the lower bound, using Lemma 2.7, we have

$$\begin{aligned}LE(G)&\ge 2\big(\Delta+1-\frac{2m}{n}\big)\\&=2\Big(p^2-1+1-\frac{2(p-1)(4pq-3p-2)}{(p^2+p(q-1)-1)}\Big).\end{aligned}$$

So, the result holds.

Case 2: If $p<q$, then $\Delta=pq-2$. Similar to the proof of case 1, the result completes. □

Theorem 3.2. Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R\simeq\mathbb{Z}_{p^2}\times\mathbb{Z}_q$ for prime numbers $p,q>2$. Then

$$\sqrt{2(p-1)(\alpha-\beta)}\le E\big(L_{\Gamma(R)}\big)\le(p-1)(\alpha-\beta),$$

in which $\alpha=p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta=4pq-3p-2$.

Proof . We suppose $\Gamma(R)$ be the zero-divisor graph of the ring $R\simeq\mathbb{Z}_{p^2}\times\mathbb{Z}_q$ for prime numbers $p,q>2$ of order $n=p^2+p(q-1)-1$ and size $m=\frac{(p-1)(4pq-3p-2)}{2}$. Using the sequence degree (3.1) of graph $\Gamma(R)$ where $R\simeq\mathbb{Z}_{p^2}\times\mathbb{Z}_q$ and the definition of the Zagreb index, we get

$$\begin{aligned}M(G)&=\sum_{i=1}^{p^2+p(q-1)-1}d_i^2\\&=(q-1)(p^2-1)^2+(p-1)^3(q-1)+(pq-2)^2(p-1)+(q-1)^2(p^2-p)\\&=(p-1)\big[p(q-1)\big(q+p(p+2)-4\big)+pq-2\big].\end{aligned}$$

Thus, using Lemma 2.4, we have

$$\begin{aligned}E\big(\Gamma(G)\big)&\le M(G)-2m\\&=(p-1)\big[p(q-1)\big(q+p(p+2)-4\big)+pq-2\big]-2\Big(\frac{(p-1)(4pq-3p-2)}{2}\Big)\\&=(p-1)\Big(\big[p(q-1)\big(q+p(p+2)-4\big)+pq-2\big]-(4pq-3p-2)\Big).\end{aligned}$$

With setting $\alpha = p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta = 4pq-3p-2$ in the above relation, the result holds for the upper bound. By applying Lemma 2.4 and similar to the above discussion, the lower bound follows. □

Theorem 3.3. Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for prime numbers $p, q > 2$. If F is the minimum edge dominating set of $\Gamma(R)$, then

$$EE_F\big(\Gamma(R)\big) \leq \frac{(p-1)(2\alpha-\beta)}{2},$$

in which $\alpha = p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta = 4pq-3p-2$.

Proof . According to the proof of Theorem 3.2 and using Lemma 2.1, the result completes. □

In the following, we are interested to investigate some energies of the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for a prime $p > 2$. To do this, we need the following known result.

Lemma 3.4. [8] Let G be a simple graph of the order n and size m. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of graph G, then

$$\sum_{i=1}^{n} \lambda_i^2 = 2m.$$

First, we consider the connected graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ of order $n = 3p(p-1)$. In the following theorem, we compute the energy of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$.

Theorem 3.5. Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then

$$E\big(\Gamma(R)\big) = 2(p-1)\big(\sqrt{4p-3}+\sqrt{p}\big).$$

Proof . Suppose that $\Gamma(R)$ is the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ with the number of vertices $n = 3p(p-1)$. According to the structure of the zero-divisor graph $\Gamma(R)$ in [20], the spectrum of $\Gamma(R)$ is as follows.

$$\begin{aligned} Spec\big(\Gamma(R)\big) = \Big\{ & \frac{1}{2}\big((p-1)(\sqrt{4p-3}-1)\big)^{[2]}, \frac{1}{2}\big((1-p)(\sqrt{4p-3}+1)\big)^{[2]}, \\ & \big((p-1)(1+\sqrt{p})\big)^{[1]}, \big((p-1)(1-\sqrt{p})\big)^{[1]}, 0^{[3(p+1)(p-2)]} \Big\}. \end{aligned} \tag{3.2}$$

Therefore, the energy of graph $\Gamma(R)$ equals

$$\begin{aligned} E\big(\Gamma(R)\big) &= \sum_{i=1}^{3p(p-3)} |\lambda_i| \\ &= 2\Big|\frac{1}{2}\big((p-1)(\sqrt{4p-3}-1)\big)\Big| + 2\Big|\frac{1}{2}\big((1-p)(\sqrt{4p-3}+1)\big)\Big| \\ &+ \big|(p-1)(1+\sqrt{p})\big| + \big|(p-1)(1-\sqrt{p})\big| \\ &= (p-1)(\sqrt{4p-3}-1) + (p-1)(\sqrt{4p-3}+1) \\ &+ (p-1)(1+\sqrt{p}) + (p-1)(\sqrt{p}-1) \\ &= 2(p-1)\sqrt{4p-3} + 2(p-1)\sqrt{p} \\ &= 2(p-1)\big(\sqrt{4p-3}+\sqrt{p}\big). \end{aligned}$$

□

Theorem 3.6. Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$4(p-1) \leq LE\big(\Gamma(R)\big) \leq 4\big(p-1\big)(3p^2-3p-1\big).$$

Proof . Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\big)$ of order $n=3p(p-1)$ and size m. Suppose that $\lambda_1,\dots,\lambda_n$ are the eigenvalues of graph G. Using Lemma 3.4 and the spectrum of graph $\Gamma(R)$ in (3.2), we have

$$\begin{aligned} m &= \frac{\sum_{i=1}^{n}\lambda_i^2}{2} \\ &= \frac{6p(p-1)^2}{2} = 3p(p-1)^2. \end{aligned}$$

By applying Lemma 2.9, we have

$$\begin{aligned} LE(G) &\le 4m-\frac{4m}{n} \\ &= 4\big(3p(p-1)^2\big)-\frac{4\big(3p(p-1)^2\big)}{3p(p-1)} \\ &= 12p(p-1)^2-4\big(p-1\big) \\ &= 4(p-1)\big(3p^2-3p-1\big). \end{aligned}$$

And for the lower bound,

$$\begin{aligned} LE(G) &\ge \frac{4m}{n} \\ &= \frac{4\big(3p(p-1)^2\big)}{3p(p-1)} \\ &= 4(p-1). \end{aligned}$$

□

The following result is obtained directly from Lemma 2.10 and the proof of Theorem 3.6.

Corollary 3.7. For the zero-divisor graph $\Gamma\big(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\big)$ where $p>2$ is a prime,

$$LE^+\big(\Gamma\big(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\big)\big)\le 4\big(p-1\big)(3p^2-3p-1\big).$$

Theorem 3.8. Let $\Gamma(R)$ be the zero-divisor graph of the ring $R\simeq\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$ for prime number $p>2$ of order n. Then

$$E\big(L_{\Gamma(R)}\big)<4(p-1)\big(6p^2-9p-1\big).$$

Proof . The size of graph $\Gamma(R)$ is $m=3p(p-1)^2$. So using Lemmas 2.6 and 2.10,

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &< LE^+\big(\Gamma(R)\big)+4(m-n) \\ &\le 4m-\frac{4m}{n}+4m-4n \\ &= 24p(p-1)^2-\frac{12p(p-1)^2}{3p(p-1)}-12p(p-1) \\ &= 24p(p-1)^2-4(p-1)-12p(p-1) \\ &= 4(p-1)\big(6p^2-9p-1\big). \end{aligned}$$

□

Theorem 3.9. Let $\Gamma(R)$ be the zero-divisor graph where $R\simeq\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$ for prime number $p>2$. Then

$$E\big(L_{\Gamma(R)}\big)\ge 4\Big(\big((p-1)\sqrt{4p-3}+\sqrt{p}\big)-3\Big).$$

Proof . According to the spectrum of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\big)$ in the proof of Theorem 3.5, the number of positive eigenvalues of $\Gamma(R)$ is $v^+=3$. Therefore using Lemma 2.5 and Theorem 3.5, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &\ge 2E\big(\Gamma(R)\big)-4v^+ \\ &= 4(p-1)\Big(\sqrt{4p-3}+\sqrt{p}\Big)-12 \\ &= 4\Big(\big((p-1)\sqrt{4p-3}+\sqrt{p}\big)-3\Big). \end{aligned}$$

□

Theorem 3.10. Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$4n\big(p - \frac{13}{8}\big) \leq EE_F\big(\Gamma(R)\big) \leq 4n(p-2).$$

Proof . Assume that G is the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ where $p > 2$ is a prime. Let F be the minimum edge dominating set of graph G. Since G is a connected graph of order $n = 3(p^2 - p)$ with $m = 3p(p-1)^2$ edges without any isolated and pendant vertex for $p > 2$, then using Lemma 2.3, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\geq 4(m - n + s) + 2p \\ &= 4\big((3p(p-1)^2) - n\big) \\ &= 4\big(n(p-1) - n\big) \\ &= 4n(p-2). \end{aligned}$$

Using Lemma 3.4, $\gamma' = |F| \leq \lfloor \frac{n}{2} \rfloor$. Then by applying Lemma 2.2, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\leq 4m - 2n + |F| \\ & 4(3p(p-1)^2) - 2n + \lfloor \frac{n}{2} \rfloor \\ &\leq 4n(p-1) - 2n + \frac{n}{2} \\ &= 4n\big(p - 1 - \frac{12}{8}\big) \\ &= 4n\big(p - \frac{13}{8}\big). \end{aligned}$$

Therefore, the result completes. □

Now, we consider the connected graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where $p > 2$ is a prime. The graph $\Gamma(R)$ is a connected graph of order $n = 2(p-1)(2p^2 - p + 1)$ [20].

Theorem 3.11. Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then

$$E\big(\Gamma(R)\big) = 14p^2 - 21p + 8.$$

Proof . In [20], the spectrum of graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is obtained as follows.

$$\begin{aligned} Spec\big(\Gamma(R)\big) = \Big\{ &\Big((p-1)^2\Big)^{[5]}, (-p^2 + p - 1)^{[1]}, \frac{-1}{2}\Big((p-1)\big((2p-1) \pm \sqrt{4p-3}\big)\Big)^{[3]}, \\ &\frac{1}{2}\Big((p-1)\big((2p+1) \pm \sqrt{12p-3}\big)\Big)^{[1]}, 0^{[(p^3+p^2+5p+7)(p-2)]}\Big\}. \end{aligned}$$

Therefore, the energy of graph $\Gamma(R)$ equals

$$\begin{aligned} E\big(\Gamma(R)\big) = &\sum_{i=1}^{(2p-2)(2p^2-p+1)} |\lambda_i| \\ &= 5(p-1)^2 + (p^2 - p + 1) + \frac{3}{2}(p-1)\big((2p-1) + \sqrt{4p-3}\big) \\ &+ \frac{3}{2}(p-1)\big((2p-1) - \sqrt{4p-3}\big) + \frac{1}{2}(p-1)\big((2p+1) + \sqrt{12p-3}\big) \\ &+ \frac{1}{2}(p-1)\big((2p+1) - \sqrt{12p-3}\big). \end{aligned}$$

With the simplification of the above relation, the result follows. □

Theorem 3.12. Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$\frac{4p\big(5n+\alpha\big)+8}{n} \leq LE\big(\Gamma(R)\big) \leq \frac{\big(4p(5n+\alpha)+8\big)\big(n-1\big)}{n},$$

where $\alpha = -6p^3 + p + 4$.

Proof . The zero-divisor graph $\Gamma(R)$ for $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ has $n = 2(p-1)(2p^2-p+1)$ vertices. Using Lemma 3.4 and the spectrum of zero-divisor graph $\Gamma(R)$ in the proof Theorem 3.11, the number of edges of graph $\Gamma(R)$ is $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$. Using Lemma 2.9, we get

$$\begin{aligned} LE\big(\Gamma(R)\big) &\leq 4m\big(1-\frac{1}{n}\big) \\ &= \frac{2(14p^4-30p^3+21p^2-6p+2\big)\big(4p^3-6p^2+4p-3\big)}{2p^3-3p^2+2p-1} \\ &= \frac{2\big(p\big(5(4p^3-6p^2+4p-2)-6p^3+p+4\big)+2\big)(n-1)}{\frac{n}{2}}. \end{aligned}$$

With putting $n = 4p^3 - 6p^2 + 4p - 2$ and $\alpha = -6p^3 + p + 4$, the upper bound for the Laplacian energy of $\Gamma(R)$ follows. For the lower bound, we get

$$\begin{aligned} LE\big(\Gamma(R)\big) &\geq \frac{4m}{n} \\ &= \frac{4(14p^4-30p^3+21p^2-6p+2)}{n} \\ &= \frac{4\big(p\big(5(4p^3-6p^2+4p-2)-6p^3+p+4\big)+2\big)}{n} \\ &= \frac{4\big(p\big(5n-6p^3+p+4\big)+2\big)}{n}. \end{aligned}$$

With putting $\alpha = -6p^3 + p + 4$ in the above relation, the result completes. □

The following result is obtained directly from Lemma 2.10 and the proof of Theorem 3.12.

Corollary 3.13. Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$LE^+\big(\Gamma(R)\big) \leq \frac{\big(4p(5n+\alpha)+8\big)\big(n-1\big)}{n},$$

where $\alpha = -6p^3 + p + 4$.

Theorem 3.14. Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$E\big(L_{\Gamma(R)}\big) < \frac{\big(4p(5n+\alpha)+8\big)(2n-1)-4n^2}{n},$$

where $\alpha = -6p^3 + p + 4$.

Proof . Since the number of edges of graph $\Gamma(R)$ is $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$, using Lemmas 2.6 and 2.10, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &< LE^+\big(\Gamma(R)\big) + 4(m-n) \\ &\leq 4m - \frac{4m}{n} + 4m - 4n \\ &= 8(14p^4-30p^3+21p^2-6p+2) - \frac{4(14p^4-30p^3+21p^2-6p+2)}{n} - 4n. \end{aligned}$$

with considering $n = 4p^3 - 6p^2 + 4p - 2$ and $\alpha = -6p^3 + p + 4$ and the similar to the discussion in proof of Theorem 3.12, the result follows. □

Theorem 3.15. Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then

$$E\big(L_{\Gamma(R)}\big) \geq 2\big(14p^2 - 21p - 4\big).$$

Proof . According to the spectrum of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ in the proof of Theorem 3.11, the number of positive eigenvalues of $\Gamma(R)$ is $v^+ = 6$. Therefore using Lemma 2.5 and Theorem 3.11, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &\geq 2E\big(\Gamma(R)\big) - 4v^+ \\ &= 2\big(14p^2 - 21p + 8\big) - 24 \\ &= 28p^2 - 42p - 8. \end{aligned}$$

□

Theorem 3.16. Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then

$$56p^4 - 136p^3 + 108p^2 - 56p + 16 \leq EE_F\big(\Gamma(R)\big) \leq 56p^4 - 126p^3 + 93p^2 - 36p + 11.$$

Proof . Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ where $p > 2$ is a prime. Let F be the minimum edge dominating set of graph G.

Since G is a connected graph of order $n = 4p^3 - 6p^2 + 8p - 2$ with $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$ edges without any isolated and pendant vertex for $p > 2$, using Lemma 2.3, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\geq 4(m - n + s) + 2p \\ &= 4\big((14p^4 - 30p^3 + 21p^2 - 6p + 2) - (4p^3 - 6p^2 + 8p - 2)\big) \\ &= 4\big(14p^4 - 34p^3 + 27p^2 - 14p + 4\big). \end{aligned}$$

Using Lemma 3.4, $\gamma' = |F| \leq \lfloor \frac{n}{2} \rfloor$, and Lemma 2.2, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\leq 4m - 2n + |F| \\ &\leq 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - 2n + \lfloor \frac{n}{2} \rfloor \\ &\leq 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - 2n + \frac{n}{2} \\ &= 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - \frac{3(4p^3 - 6p^2 + 8p - 2)}{2} \\ &= 56p^4 - 126p^3 + 93p^2 - 36p + 11. \end{aligned}$$

Therefore, the result completes. □

References

[1] N. Abreu, D. M. Cardoso, I. Gutman, E. A. Martins and M. Robbiano, *Bounds for the signless Laplacian energy*, Linear Alg. Appl. **435** (2011), no. 10, 2365–2374.

[2] A. Ahmad, R. Hasni, N. Akhter and K. Elahi, *Analysis of distance-based topological polynomials associated with zero-divisor graphs*, Comput. Mater. Continua Henderson **70** (2022), no. 2, 2895–2904.

[3] M. H. Akhbari, K. K. Choong and F. Movahedi, *A note on the minimum edge dominating energy of graphs*, J. Appl. Math. Comput. **63** (2020), 295–310.

[4] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **167** (1999), 434–447.

[5] S. Arumugam and S. Velammal, *Edge Domination in Graphs*, Taiwanese J. Math. **2** (1998), no. 2, 173–179.

[6] S. B. Bozkurt and D. Bozkurt, *On Incidence Energy*, MATCH Commun. Math. Comput. Chem. **72** (2014), 215–225.

[7] S. Chokani, F. Movahedi and S.M. Taheri, *Results on some energies of the Zero-divisor graph of the commutative ring* $\mathbb{Z}_n$, 14th Iran. Int. Group Theory Conf., 2022, pp. 138–146.

[8] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, New York, 1980.

[9] K.C. Das and S.A. Mojallal, *On Laplacian energy of graphs*, Discrete Math. **325** (2014), 52–64.

[10] K.C. Das and S.A. Mojallal, *Relation between signless Laplacian energy of graph and its line graph*, Linear Alg. Appl. **493** (2016), 91–107.

[11] K.C. Das, S.A. Mojallal and I. Gutman, *On energy of line graphs*, Linear Alg. Appl. **499** (2016), 79–89.

[12] R.P. Gupta, *Independence and covering numbers of line graphs and total graphs. In: Harary, F. (ed.) Proof Techniques in Graph Theory, pp. 61-62*, Academic press, New York, 1969.

[13] I. Gutman, *The energy of a graph*, Ber. Math-Statist. Sekt. Forschungsz. Graz **103** (1978), 1–22.

[14] I. Gutman and B. Zhou, *Laplacian energy of a graph*, Linear Alg. Appl. **414** (2006), 29–37.

[15] I. Gutman, M. Robbiano, E.A. Martins, D.M. Cardoso, L. Medina and O. Rojo, *Energy of line graphs*, Linear Alg. Appl. **433** (2010), 1312–1323.

[16] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total* π*-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), 535–538.

[17] F. Harary, *Graph Theory*, Addison-Wesley. Reading, 1972.

[18] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[19] I.Z. Milovanoviić and E.I. Milovanoviić, *Remarks on the energy and the minimum dominating energy of a graph*, MATCH Commun. Math. Comput. Chem, **75** (2016), 305–314.

[20] K. Mönius, *Eigenvalues of zero-divisor graphs of finite commutative rings*, J. Alg. Combin.**54**, (2021), 787–802.

[21] F. Movahedi, *The relation between the minimum edge dominating energy and the other energies*, Discrete Math. Algorithms Appl. **12** (2020), no. 6, 2050078 (14 pages).

[22] F. Movahedi, *Bounds on the minimum edge dominating energy of induced subgraphs of a graph*, Discrete Math. Algorithms Appl. **13** (2021), no. 06, 2150080.

[23] F. Movahedi and M.H. Akhbari, *New results on the minimum edge dominating energy of a graph*, J. Math. Ext. **16** (2022), no. 5, 1–17.

[24] N. Palanivel and A.V. Chithra, *Signless Laplacian energy, distance Laplacian energy and distance signless Laplacian spectrum of unitary addition Cayley graphs*, Linear Multilinear Alg. (2021), 1–22. doi.org/10.1080/03081087.2021.1996524

[25] M.R. Rajesh Kanna, B.N. Dharmendra and G. Sridhara, *The minimum dominating energy of a graph*, Int. J. Pure App. Math. **85** (2013), 707–718.

[26] B.S. Reddy, R. Jain and N. Laxmikanth, *Eigenvalues and Wiener index of the zero-divisor graph* $\Gamma(\mathbb{Z}_n)$, arXiv preprint arXiv:1707.05083 (2017).

[27] M. Robbiano, R. Jiménez, *Applications of a theorem by* K_y *Fan in the theory of Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem. **62** (2009), 537–552.

[28] S. Suthar and O. Prakash, *Adjacency Matrix and Energy of the Line Graph of* $\Gamma(\mathbb{Z}_n)$, arXiv: Combinatorics (2018), arXiv:1806.08944.

[29] B. Zhou, *More on energy and Laplacian energy*, MATCH Commun. Math. Comput. Chem. **64** (2010), 75–84.