Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 585–595
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27013.3475

Constructing exact solutions to systems of reaction-diffusion equations

M. O. Aibinu[a,c,d,*], S. Moyo[b,d]

[a]Institute for Systems Science and KZN e-Skills CoLab, Durban University of Technology, South Africa
[b]Department of Applied Mathematics and School for Data Science and Computational Thinking, Stellenbosch University, South Africa
[c]DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa
[d]National Institute for Theoretical and Computational Sciences (NITheCS), South Africa

(Communicated by Haydar Akca)

Abstract

Many generalizations have been considered on how to construct the exact solutions of one-component Reaction-Diffusion (RD) equations. Two-component RD systems of equations allow for the study of a wider range of physical phenomena as well as dynamical processes than their counterpart one-component RD equations. The most suitable and best way to study certain models is by using two-component RD systems of equations. Moreover, the presence of delay in nonlinear Partial Differential Equations (PDEs) makes them more difficult to study than those without delay. This study introduces some new exact solutions associated with a generalized form of two-component RD systems of equations with delay. The exact solutions for more complex multidimensional reaction-diffusion systems of equations are also derived. Solutions to RD systems of equations with a delay which are presented in this study are applicable for the formulation of test problems to verify the efficiency of numerical methods which are being used to obtain the solutions of nonlinear delay PDEs.

Keywords: Reaction-diffusion, Exact solutions, Delay differential equations, Fundamental matrix, Systems of equations
2020 MSC: Primary 35K55, 35B10; Secondary 35B40, 35K57

1 Introduction

Several physical phenomena are governed by mathematical models and can be represented by the corresponding Reaction-Diffusion (RD) equations. RD equations are essential for the description of dynamical processes in science, engineering and other fields of study such as social science, to mention just a few. In general, RD systems of equations take the form

$$\mathbf{u}_t = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{G}(\mathbf{u}),$$

where $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$ is the unknown vector function with $\mathbf{r}$ and t denoting the position vector and time respectively, $\mathbf{D}$ is the diagonal matrix of diffusion coefficients, ∇^2 is the Laplace operator which acts on the vector $\mathbf{u}$ componentwise

*Corresponding author
Email addresses: moaibinu@yahoo.com / mathewa@dut.ac.za (M. O. Aibinu), smoyo@sun.ac.za (S. Moyo)

Received: April 2022 *Accepted:* November 2022

and **G** accounts for all local reactions. Many generalizations have been considered on how to construct the exact solutions of one-component RD equations which are of the form,

$$u_t = au_{xx} + g(u), \tag{1.1}$$

where $u = u(x,t)$, the diffusion coefficient is $a > 0$, and the arbitrary function is $g(u)$, which stands for local dynamics (See e.g, [24, 25, 31, 23, 22, 17, 19, 21, 1] and references therein). Construction of exact solutions in terms of the Lambert function was dealt with in [34]. The bilinear approach for constructing the exact soliton solutions is discussed in [12]. Moreover, there are additional works on approximate analytical and numerical solutions for equations of the form (1.1) (See e.g., [11, 33, 16]). Two-component RD systems allow for the study of a wider range of physical phenomena as well as dynamical processes than their counterpart one-component RD equations. Predator-prey, competition of species for a common food source and symbiotic relationships between two species are examples of models which can best be studied by using two-component RD systems. Two-component RD systems of equations of the form

$$\begin{aligned} u_t &= a_1 u_{xx} + g_1(u,v), \\ v_t &= a_2 v_{xx} + g_2(u,v), \end{aligned} \tag{1.2}$$

with $u = u(x,t), v = v(x,t)$, were considered in [9, 8, 18, 20]. In (1.2), $g_1(u,v)$ and $g_2(u,v)$ are arbitrary functions which account for local dynamics, and $a_i > 0$ are diffusion constants, with $i = 1, 2$. The presence of delay in nonlinear Partial Differential Equations (PDEs) makes them more difficult to study than their counterparts without delay. Let $\tau > 0$ denote the delay time while $u = u(x,t), v = v(x,t)$, $\bar{u} = u(x, t-\tau)$ and $\bar{v} = v(x, t-\tau)$. Two-component RD systems of equations with delay which are of the form,

$$\begin{aligned} u_t &= a_1 u_{xx} + G_1(u,\bar{u},v,\bar{v}), \\ v_t &= a_2 v_{xx} + G_2(u,\bar{u},v,\bar{v}), \end{aligned} \tag{1.3}$$

were considered in [28, 29]. The arbitrary functions are $G_1(u,\bar{u},v,\bar{v})$ and $G_2(u,\bar{u},v,\bar{v})$.

In connection with nonlinear PDEs, exact solution signifies where the solution can be displayed in (See e.g, [31, 23, 22]):

(i) respect to elementary functions;

(ii) the form of definite or/and indefinite integrals;

(iii) respect to solutions of Ordinary Differential Equations (ODEs) or systems of ODEs.

Two or more of the cases listed above could also be combined. A more general form of (1.2) occurs when the time derivatives u_t and v_t, and also the kinetic functions G_1 and G_2 are associated with the nonlinear terms. Such systems are weighty in nature and hard to deal with. An example of such systems that is more general than (1.3) is given by

$$\begin{aligned} r_1(x)u_t &= a_1 u_{xx} + b_1 u_x + c_1(x)G_1(u,\bar{u},v,\bar{v}), \\ r_2(x)v_t &= a_2 v_{xx} + b_2 v_x + c_2(x)G_2(u,\bar{u},v,\bar{v}), \end{aligned} \tag{1.4}$$

with $u = u(x,t), v = v(x,t), \bar{u} = u(x,t-\tau), \bar{v} = v(x,t-\tau)$ and τ denotes the delay time. The state variables concentration at position $x \in \mathbb{R}$ and time t are $u(x,t)$ and $v(x,t)$, respectively, e.g, densities of the prey and the predator. Let $i = 1, 2$, the diffusion constants are $a_i > 0$, while b_i are the advection rates which are real numbers (See e.g, [35]). Time rate of change of concentration of state variables are respectively associated with variable coefficients $r_i(x)$. Some functions $c_i(x)$ are respectively associated with the arbitrary functions $G_i(u,\bar{u},v,\bar{v})$. The functions $G_i(u,\bar{u},v,\bar{v})$ account for local dynamics.

This study considers two-component RD systems of equations with delay which are of the form (1.4). Some new exact solutions are introduced. The exact solutions for more complex multidimensional RD systems are also derived. Several numerical methods such as spectral methods [13, 6, 15, 4], adomian decomposition [2, 10], Tan-Cot [3], residual power series [30], perturbation [7, 14], have been used to obtain the solutions of PDEs. However, those numerical methods are frequently accompanied with problems of certain forms. This present study introduces the methods for obtaining exact solutions of RD systems that are associated with delay and for formulation of test problems for the purpose of evaluating the efficiency of numerical methods.

2 Solutions of two-component RD systems with delay

The exact solutions of (1.4) are desired to be in the form

$$\begin{array}{ll} u = U(y), & y = t + \int f(x)dx, \\ v = V(z), & z = t + \int h(x)dx. \end{array} \tag{2.1}$$

The solutions which are of the form (2.1) are called generalized traveling-wave solutions. The functions $f(x)$ and $h(x)$ may be given or have to be otherwise determined. The goal in a particular situation determines whether they are given or not. Substitution of (2.1) into (1.4) yields a system of functional-differential equations

$$\begin{aligned} &a_1 f^2 U''_{yy} + \{a_1 f' + b_1 f - r_1(x)\} U'_y + c_1(x) G_1(U, \bar{U}, V, \bar{V}) = 0, \\ &a_2 h^2 V''_{zz} + \{a_2 h' + b_2 h - r_2(x)\} V'_z + c_2(x) G_2(U, \bar{U}, V, \bar{V}) = 0, \end{aligned} \tag{2.2}$$

with $f = f(x)$ and $h = h(x)$.

Let the coefficients of the system of functional-differential equations (2.2) satisfy the relations

$$\begin{aligned} c_1(x) &= a_1 p_1 f^2, \\ c_2(x) &= a_2 p_2 h^2, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} a_1 f' + b_1 f &= -q_1 a_1 f^2 + r_1(x), \\ a_2 h' + b_2 h &= -q_2 a_2 h^2 + r_2(x), \end{aligned} \tag{2.4}$$

where p_i and q_i are constants ($p_i \not\equiv 0$) and $i = 1, 2$. Then (2.2) reduces to the coupled ODEs

$$\begin{aligned} U''_{yy} - q_1 U'_y + p_1 G_1(U, \bar{U}, V, \bar{V}) = 0, \\ V''_{zz} - q_2 V'_z + p_2 G_2(U, \bar{U}, V, \bar{V}) = 0, \end{aligned} \tag{2.5}$$

with $\bar{U} = U(Y - \tau)$ and $\bar{V} = V(Z - \tau)$.

Remark 2.1. For $V \equiv 0$ and $G_1(U, \bar{U}) = Ug(\bar{U}/U)$, an exact solution for (2.5) takes the form $U = Ke^{\zeta y}$, where the arbitrary constant is K, while ζ is decided by the abstract equation (See e.g, [23])

$$\zeta^2 - q_1 \zeta + p_1 g\left(e^{-\tau\zeta}\right) = 0. \tag{2.6}$$

Different roots of (2.6) will produce different roots for the delay ODE.

Remark 2.2. For $V \equiv 0, q_1 \equiv 0$ and $G_1(U, \bar{U}) = g(U)$, the general solution of (2.5) in the implicit form for any function $g(U)$ is given by

$$\int \left[K_1 - 2\int g(U)dU\right]^{-1/2} dU = K_2 \pm y,$$

where K_1 and K_2 are arbitrary constants and $p_1 \equiv 1$ (See e.g, [26]). The special case $G_1(U, \bar{U}) = g(U)$ describes an instantaneous system and $q_i \equiv 0$ corresponds to the linear form of (2.4), where $i = 1, 2$.

The relation between the coefficients (real constants and functions) which appear in (1.4) and the functions f and h which appear in (2.1), are given by (2.3) and (2.4). Notice that (2.4) is a system of differential relations with respect to f and h, while it forms algebraic relations as regards to $a_i, b_i, c_i(x)$, and $r_i(x)$, where $i = 1, 2$.

2.1 Constructing exact solutions by finding f and h

A system of Riccati equations is formed in (2.4) for f and h, if the functions a_i, b_i and $r_i(x)$ are assumed given, with $q_i \neq 0$ and $i = 1, 2$. In standard form, the system of Riccati equations for f and h is given by

$$\begin{aligned} a_1 f' + q_1 a_1 f^2 + b_1 f - r_1(x) = 0, \\ a_2 h' + q_2 a_1 h^2 + b_2 h - r_2(x) = 0. \end{aligned} \tag{2.7}$$

Interested readers in the exact solutions for single equations of the form (2.7) and for various forms of arbitrary real constants a, b and arbitrary real functions $c(x)$ and $r(x)$ are referred to [26, 27]. The system of equations (2.7) will be considered under two cases, which are degenerate and nondegenerate.

Degenerate case. For $q_1 = q_2 = 0$, the system of equations in (2.7) take the vector-matrix form

$$F'(x) = AF(x) - r(x), \tag{2.8}$$

where

$$F(x) = \begin{bmatrix} f(x) \\ h(x) \end{bmatrix}, \quad r(x) = \begin{bmatrix} r_1(x)/a_1 \\ r_2(x)/a_2 \end{bmatrix} \text{ and}$$

$$A = \begin{bmatrix} b_1/a_1 & 0 \\ 0 & b_2/a_2 \end{bmatrix}. \tag{2.9}$$

Observe that the matrix A in (2.9) is symmetric and, in particular, it is a diagonal matrix. The fundamental matrix for a homogeneous system of equations which are of the form

$$F'(x) = A(x)F(x), \tag{2.10}$$

is given as

$$\Phi(x) = e^{\int_{x_0}^{x} A(\nu)d\nu}. \tag{2.11}$$

$\Phi(x)$ is called the fundamental system of solutions and its columns are formed by linearly independent solutions of the homogeneous system (See e.g, [32]). Note that the fundamental matrix $\Phi(x)$ is nonsingular. The general solution for the homogeneous system (2.10) is expressed in terms of the fundamental matrix by

$$F_0(x) = \Phi(x)K, \tag{2.12}$$

where $K = (K_1, K_2)^T$, is a vector consisting of arbitrary constants. To solve the nonhomogeneous system (2.8), the constant vectors in (2.12) will be replaced by $K(x)$, which is known as the method of variation of constants (or Lagrange method). $K(x)$ are continuously differentiable functions with respect to independent variable x. The general solution for the nonhomogeneous system (2.8) can be given in the form

$$F(x) = \Phi(x)K(x). \tag{2.13}$$

To find the unknown vector $K(x)$, substitute (2.13) into the nonhomogeneous system (2.8). This gives

$$\begin{aligned} & \cancel{\Phi'(x)}K(x) + \Phi(x)K_x'(x) = \cancel{A\Phi(x)K(x)} - r(x) \\ \Rightarrow \quad & \Phi(x)K_x'(x) = -r(x). \end{aligned} \tag{2.14}$$

$\Phi(x)$ is known to have an inverse since it is nonsingular. Multiplying (2.14) on the left by $\Phi^{-1}(x)$ gives

$$\begin{aligned} & \Phi^{-1}(x)\Phi(x)K_x'(x) = -\Phi^{-1}(x)r(x) \\ \Rightarrow \quad & K_x'(x) = -\Phi^{-1}(x)r(x) \\ \Rightarrow \quad & K(x) = K_0 - \int \Phi^{-1}(x)r(x)dx, \end{aligned} \tag{2.15}$$

where K_0 denotes an arbitrary constant vector. Substituting (2.15) into (2.13) gives the general solution for the nonhomogeneous system as

$$\begin{aligned} F(x) &= \Phi(x)K(x) \\ &= \Phi(x)\left(K_0 - \int \Phi^{-1}(x)r(x)dx\right). \end{aligned} \tag{2.16}$$

Then, c_i, where $i = 1, 2$ are determined by (2.3) and the exact solution is obtained by using (2.1).

Example 2.3. Consider the case where $a_1 = 2, a_2 = 3, b_1 = -2, b_2 = 6, r_1 = 4\sin\ x$, and $r_2 = 12x$. Using (2.9) and (2.11) respectively,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

and the fundamental matrix is

$$\Phi(x) = \begin{bmatrix} e^{-x} & 0 \\ 0 & e^{2x} \end{bmatrix},$$

with

$$\Phi^{-1}(x) = \begin{bmatrix} e^{x} & 0 \\ 0 & e^{-2x} \end{bmatrix}.$$

Using (2.16) gives

$$F(x) = \begin{bmatrix} f(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} e^{x}\left(K_1 + \sin\ x - \cos\ x\right) \\ e^{-2x}\left(K_2 - 2x - 1\right) \end{bmatrix}. \tag{2.17}$$

Setting $K_1 = K_2 = 0$ in (2.17) leads to

$$\begin{aligned} f(x) &= e^{x}\left(\sin\ x - \cos\ x\right), \\ h(x) &= -e^{-2x}\left(2x + 1\right). \end{aligned} \tag{2.18}$$

Consequently, it can be deduced from (2.1) that

$$\begin{aligned} u &= U(y), \quad y = t - \frac{1}{2}e^{x}\cos\ x, \\ v &= V(z), \quad z = t + (x+1)e^{-2x}, \end{aligned} \tag{2.19}$$

and is obtained from (2.3) that

$$\begin{aligned} c_1(x) &= 2e^{2x}\left(\sin\ x - \cos\ x\right)^2, \\ c_2(x) &= 3e^{-4x}\left(2x + 1\right)^2, \end{aligned}$$

where without loss of generality, $p_1 = p_2 \equiv 1$. Thus, the system of nonlinear reaction-diffusion equations

$$\begin{aligned} 4sin\ x\ u_t &= 2u_{xx} - 2u_x + 2e^{2x}\left(\sin\ x - \cos\ x\right)^2 G_1(u, \bar{u}, v, \bar{v}), \\ 12x\ v_t &= 3v_{xx} + 6u_x + 3e^{-4x}\left(2x + 1\right)^2 G_2(u, \bar{u}, v, \bar{v}), \end{aligned}$$

for arbitrary functions $G_1(u, \bar{u}, v, \bar{v})$ and $G_2(u, \bar{u}, v, \bar{v})$, admit the generalized traveling-wave solutions (2.19), where $U(z)$ and $V(z)$ are decided by the system of ODEs

$$\begin{aligned} U''_{yy} + G_1(U, \bar{U}, V, \bar{V}) = 0,\ \bar{U} = U(Y - \tau), \\ V''_{zz} + G_2(U, \bar{U}, V, \bar{V}) = 0,\ \bar{V} = V(Z - \tau). \end{aligned} \tag{2.20}$$

Nondegenerate case. For $q_1 \neq 0$ and $q_2 \neq 0$, substitute

$$\begin{aligned} f &= \frac{1}{q_1}\frac{\Omega'}{\Omega}, \\ h &= \frac{1}{q_2}\frac{\Psi'}{\Psi} \end{aligned} \tag{2.21}$$

into (2.7) to get

$$\begin{aligned} &\frac{a_1}{q_1}\left(\frac{\Omega''}{\Omega} - \frac{(\Omega')^2}{\Omega^2}\right) + q_1 a_1\left(\frac{1}{q_1}\frac{\Omega'_x}{\Omega}\right)^2 + \frac{b_1}{q_1}\frac{\Omega'}{\Omega} - r_1(x) = 0, \\ &\frac{a_2}{q_2}\left(\frac{\Psi''}{\Psi} - \frac{(\Psi')^2}{\Psi^2}\right) + q_2 a_2\left(\frac{1}{q_2}\frac{\Psi'_x}{\Psi}\right)^2 + \frac{b_2}{q_2}\frac{\Psi'}{\Psi} - r_2(x) = 0. \end{aligned} \tag{2.22}$$

Using vector-matrix notation, simplification of (2.22) yields

$$\Gamma''(x) + M\Gamma'(x) - m(x)\Gamma(x) = 0, \tag{2.23}$$

where

$$\Gamma(x) = \begin{bmatrix} \Omega(x) \\ \Psi(x) \end{bmatrix}, M = \begin{bmatrix} b_1/a_1 & 0 \\ 0 & b_2/a_2 \end{bmatrix} \ \&\ m(x) = \begin{bmatrix} q_1 r_1(x)/a_1 \\ q_2 r_2(x)/a_2 \end{bmatrix}.$$

Interested readers in the exact solutions of nonlinear second-order ODEs, which are of the form (2.23) for various values of a, b and $r(x)$ are referred to [26, 27].

Example 2.4. In (2.23), the r_i are taken to be constants such that $r_i = a_i = 1$, where $i = 1, 2$. Let $T = \begin{bmatrix} \Omega' \\ \Psi' \\ \Omega \\ \Psi \end{bmatrix}$, then $T' = \begin{bmatrix} \Omega'' \\ \Psi'' \\ \Omega' \\ \Psi' \end{bmatrix} = \begin{bmatrix} -b_1\Omega' + q_1\Omega \\ -b_2\Psi' + q_2\Psi \\ \Omega' \\ \Psi' \end{bmatrix}$. Thus,

$$T'(x) = \begin{bmatrix} -b_1 & 0 & q_1 & 0 \\ 0 & -b_2 & 0 & q_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} T(x). \tag{2.24}$$

The system of second ODEs (2.23), has been transformed to a homogeneous system of first ODEs (2.24). According to (2.12), the general solution for (2.24) takes the form

$$T(x) = \Phi(x)K, \tag{2.25}$$

where $\Phi(x)$ is the fundamental matrix and $K = [K_1, K_2, K_3, K_4]^T$. To find the fundamental matrix in (2.25), the eigenvalues and the corresponding eigenvectors of the matrix of coefficients in (2.24) need to be determined. Let $P = \begin{bmatrix} -b_1 & 0 & q_1 & 0 \\ 0 & -b_2 & 0 & q_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then

$$|\lambda I - P| = \begin{vmatrix} \lambda + b_1 & 0 & -q_1 & 0 \\ 0 & \lambda + b_2 & 0 & -q_2 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} = [\lambda(\lambda + b_1) - q_1]\,[\lambda(\lambda + b_2) - q_2]\,. \tag{2.26}$$

Solving (2.26) gives

$$\lambda = \frac{-b_i \pm D_i}{2}, \tag{2.27}$$

where

$$D_i = \sqrt{b_i^2 + 4q_i}, \quad i = 1, 2. \tag{2.28}$$

Two cases will be considered for D_i in (2.28).

Case 1: $D_i \geq 0$ e.g take $b_1 = 1, b_2 = 4, q_1 = 6$ and $q_2 = -4$. From (2.27), $\lambda = -3, -2$ and 2. The value -2 has the multiplicity of two. The corresponding eigenvector for $\lambda = -3$ is $[-3, 0, 1, 0]^T$, for $\lambda = 2$ is $[2, 0, 1, 0]^T$, while the corresponding eigenvectors for $\lambda = -2$ are $[0, -2, 0, 1]^T$ and $[0, 1, 0, -1]^T$. These yield the fundamental matrix which is given as

$$\Phi(x) = \begin{bmatrix} -3e^{-3x} & 2e^{2x} & 0 & 0 \\ 0 & 0 & -2e^{-2x} & (-2x+1)e^{-2x} \\ e^{-3x} & e^{2x} & 0 & 0 \\ 0 & 0 & e^{-2x} & (x-1)e^{-2x} \end{bmatrix}.$$

This gives the system of solutions for (2.23) as

$$\begin{aligned} \Omega(x) &= -3K_1e^{-3x} + 2K_2e^{2x}, \\ \Psi(x) &= -2K_3e^{-2x} + K_4(-2x+1)e^{-2x}. \end{aligned} \tag{2.29}$$

Setting $K_1 = K_3 = 0$ in (2.29) leads to

$$\begin{aligned} f(x) &= \frac{1}{3}, \\ h(x) &= \frac{1}{2}\left(1 - \frac{1}{2x-1}\right). \end{aligned} \tag{2.30}$$

Substitute (2.30) into (2.3) to obtain

$$c_1(x) = \frac{1}{9},$$
$$c_2(x) = \frac{1}{4}\left(1 - \frac{1}{2x-1}\right)^2,$$

and into (2.1) to obtain

$$\begin{aligned} u &= U(y), \quad y = t + \frac{1}{3}x, \\ v &= V(z), \quad z = t + \frac{1}{2}x - \frac{1}{4}\ln(2x-1), \end{aligned} \tag{2.31}$$

where without loss of generality, $p_1 = p_2 \equiv 1$. Hence the system of nonlinear RD equations

$$u_t = u_{xx} + u_x + \frac{1}{9}G_1(u, \bar{u}, v, \bar{v}),$$
$$v_t = v_{xx} + 4u_x + \frac{1}{4}\left(1 - \frac{1}{2x-1}\right)^2 G_2(u, \bar{u}, v, \bar{v}),$$

for arbitrary functions $G_1(u, \bar{u}, v, \bar{v})$ and $G_2(u, \bar{u}, v, \bar{v})$, admit the generalized traveling-wave solutions (2.31), where $U(z)$ and $V(z)$ are decided by the system of ODEs (2.5).

Case 2: $D_i < 0$ e.g take $b_1 = 2, b_2 = -2, q_1 = -2$ and $q_2 = -5$. From (2.27), $\lambda = -1 \pm i, 1 \pm 2i$. The corresponding eigenvectors are $\begin{bmatrix} -1+i \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1+2i \\ 0 \\ 1 \end{bmatrix}$, respectively. The fundamental matrix is given by

$$\Phi(x) = \begin{bmatrix} e^{-x}(-\cos x - \sin x) & e^{-x}(\cos x - \sin x) & 0 & 0 \\ 0 & 0 & e^{x}(\cos 2x - 2\sin 2x) & e^{x}(2\cos 2x + \sin 2x) \\ e^{-x}\ \cos x & e^{-x}\ \sin x & 0 & 0 \\ 0 & 0 & e^{x}\ \cos 2x & e^{x}\ \sin 2x \end{bmatrix}.$$

This gives the system of solutions for (2.23) as

$$\begin{aligned} \Omega(x) &= K_1 e^{-x}(-\cos x - \sin x) + K_2 e^{-x}(\cos x - \sin x), \\ \Psi(x) &= K_3 e^{x}(\cos 2x - 2\sin 2x) + K_4 e^{x}(2\cos 2x + \sin 2x). \end{aligned} \tag{2.32}$$

Setting $K_2 = K_4 = 0$ in (2.32) leads to

$$\begin{aligned} f(x) &= \frac{1}{2}\left(1 - \frac{\sin\ x - \cos\ x}{-\cos\ x - \sin\ x}\right), \\ h(x) &= -\frac{1}{5}\left(1 + \frac{-2\sin\ 2x - 4\cos\ 2x}{\cos\ 2x - 2\sin\ 2x}\right). \end{aligned} \tag{2.33}$$

Substitute (2.33) into (2.1) to obtain

$$\begin{aligned} u &= U(y), \quad y = t + \frac{1}{2}\left(x - \ln(\cos\ x + \sin\ x)\right), \\ v &= V(z), \quad z = t - \frac{1}{5}\left(x + \ln(\cos\ 2x - 2\sin\ 2x)\right), \end{aligned} \tag{2.34}$$

and into (2.3) to obtain

$$c_1(x) = \frac{1}{4}\left(1 - \frac{\sin\ x - \cos\ x}{-\cos\ x - \sin\ x}\right)^2,$$
$$c_2(x) = \frac{1}{25}\left(1 + \frac{-2\sin\ 2x - 4\cos\ 2x}{\cos\ 2x - 2\sin\ 2x}\right)^2,$$

where without loss of generality, $p_i \equiv 1, i = 1, 2$. Hence the system of nonlinear RD equations

$$u_t = u_{xx} + 2u_x + \frac{1}{4}\left(1 - \frac{\sin\ x - \cos\ x}{-\cos\ x - \sin\ x}\right)^2 G_1(u, \bar{u}, v, \bar{v}),$$
$$v_t = v_{xx} - 2v_x + \frac{1}{25}\left(1 + \frac{-2\sin\ 2x - 4\cos\ 2x}{\cos\ 2x - 2\sin\ 2x}\right)^2 G_2(u, \bar{u}, v, \bar{v}),$$

for arbitrary functions $G_1(u, \bar{u}, v, \bar{v})$ and $G_2(u, \bar{u}, v, \bar{v})$, admit the generalized traveling-wave solutions (2.34), where $U(z)$ and $V(z)$ are decided by the system of ODEs (2.5).

2.2 Constructing exact solutions when f and h are given

Constructing exact solutions of the system of nonlinear RD equations which are of the form (1.4) requires solving the pair of algebraic equations (2.7) simultaneously, for the pair a_i, b_i and $r_i(x)$, where $i = 1, 2$. The functions f and h, including a pair of any two, among the three functions (or constants) a_i, b_i, and $r_i(x)$ are assumed to be given. The remaining unknown pair of functions (or constants) will be derived by using (2.4) and (2.7). The pair of functions $c_i(x)$ in the nonlinear RD system of the form (1.4) are then determined by using (2.3) and the corresponding exact solutions are deduced by using (2.1).

Example 2.5.

Let $f(x) = e^{-x}, h(x) = 2/x, a_1 = 2, a_2 = 1/2, b_1 = 5, b_2 = 1$ be given. The task is to find $r_i(x)$, where q_i are arbitrary constants and $i = 1, 2$.

(I) Degenerate case: $q_i = 0, i = 1, 2$.

It can be obtained from (2.7) that

$$r_1(x) = 3e^{-x},$$
$$r_2(x) = (2x - 1)/x^2,$$

and from (2.3) that

$$c_1(x) = 2e^{-2x}, \quad c_2(x) = 2/x^2, \tag{2.35}$$

where without loss of generality, $p_1 = p_2 \equiv 1$. Thus, the system of RD equations

$$3e^{-x}u_t = 2u_{xx} + 5u_x + 2e^{-2x}G_1(u, \bar{u}, v, \bar{v}),$$
$$\frac{(2x-1)}{x^2}v_t = \frac{1}{2}v_{xx} + v_x + \frac{2}{x^2}G_2(u, \bar{u}, v, \bar{v}),$$

for arbitrary functions $G_1(u, \bar{u}, v, \bar{v})$ and $G_2(u, \bar{u}, v, \bar{v})$, admit the generalized traveling-wave solutions

$$u = U(y), \quad y = t - e^{-x}, \qquad v = V(z), \quad z = t + ln\ x^2, \tag{2.36}$$

where $U(z)$ and $V(z)$ are decided by the system of ODEs (2.20).

(II) Nondegenerate case: $q_i \neq 0,\ i = 1, 2$.

It can be obtained from (2.7) that

$$r_1(x) = e^{-2x}\left(3e^x + 2q_1\right), \quad r_2(x) = (2x + 2q_2 - 1)/x^2. \tag{2.37}$$

and from (1.4) that

$$e^{-2x}\left(3e^x + 2q_1\right)u_t = 2u_{xx} + 5u_x + 2e^{-2x}G_1(u, \bar{u}, v, \bar{v}), \quad \frac{2x + 2q_2 - 1}{x^2}v_t = \frac{1}{2}v_{xx} + v_x + \frac{2}{x^2}G_2(u, \bar{u}, v, \bar{v}). \tag{2.38}$$

Hence, for arbitrary functions $G_1(u, \bar{u}, v, \bar{v})$ and $G_2(u, \bar{u}, v, \bar{v})$, the system of RD equations (2.38) are solved by (2.36), where $U(z)$ and $V(z)$ are decided by the system of ODEs (2.5).

3 Transformations and extension to more complex multidimensional RD systems

The exact solutions for more complex multidimensional RD systems can be easily derived by applying the results which have been presented above.

3.1 Exact solutions of other nonlinear PDEs with delay

Let τ denote the delay time and $u = u(x,t), v = v(x,t),\ \bar{u} = u(x,t-\tau), \bar{v} = v(x,t-\tau)$. Consider nonlinear delay RD systems which have the form

$$\begin{aligned} r_1(x)u_t &= a_1 u_{xx} + b_1 u_x + c_1(x) f(x) G_1\left(f, u, \bar{u}, v, \bar{v}, u_x/f\right), \\ r_2(x)v_t &= a_2 v_{xx} + b_2 v_x + c_2(x) h(x) G_2\left(h, u, \bar{u}, v, \bar{v}, v_x/h\right), \end{aligned} \tag{3.1}$$

where $G_1\left(f, u, \bar{u}, v, \bar{v}, \theta_1\right),\ G_2\left(h, u, \bar{u}, v, \bar{v}, \theta_2\right)$ are arbitrary functions which take six arguments. The system (3.1) is solved by

$$\begin{aligned} u &= U(y), \qquad y = t + \int f(x)dx, \\ v &= V(z), \qquad z = t + \int h(x)dx. \end{aligned} \tag{3.2}$$

Here, $U(z)$ and $V(z)$ are decided by the system

$$\begin{aligned} U''_{yy} - q_1 U'_y + p_1 G_1(f, U, \bar{U}, V, \bar{V}, U'_y) = 0, \\ V''_{zz} - q_2 V'_z + p_2 G_2(h, U, \bar{U}, V, \bar{V}, V'_z) = 0, \end{aligned}$$

with $\bar{U} = U(Y-\tau)$ and $\bar{V} = V(Z-\tau)$. This can be easily verified by substituting (3.2) into (3.1), while taking into account the relations (2.3) and (2.4).

3.2 Exact solutions of other nonlinear PDEs without delay

A special case $G\left(u, \bar{u}, v, \bar{v}\right) = G\left(u, v\right)$ describes the processes of say, heat and mass transfer which take place in the media with local equilibrium which are without reference to inertial properties. This governs the system which are without the delay time τ and which react instantaneously to action at the given point time t.

Consider nonlinear RD systems which are of the form

$$\begin{aligned} r_1(x)u_t &= a_1 u_{xx} + b_1 u_x + c_1(x) G_1\left(u, v\right), \\ r_2(x)v_t &= a_2 v_{xx} + b_2 v_x + c_2(x) G_2\left(u, v\right), \end{aligned} \tag{3.3}$$

with $u = u(x,t),\ v = v(x,t)$, and $G_1\left(u, v\right), G_2\left(u, v\right)$ are arbitrary functions. The system of equations (3.3) are solved by

$$\begin{aligned} u &= U(y), \qquad y = t + \int f(x)dx, \\ v &= V(z), \qquad z = t + \int h(x)dx. \end{aligned} \tag{3.4}$$

Here, $U(z)$ and $V(z)$ are decided by the system

$$\begin{aligned} U''_{yy} - q_1 U'_y + p_1 G_1(U, V) = 0, \\ V''_{zz} - q_2 V'_z + p_2 G_2(U, V) = 0, \end{aligned}$$

with $U = U(Y)$ and $V = V(Z)$. This can be easily verified by substituting (3.4) into (3.3), while taking into account the relations (2.3) and (2.4).

Conclusion: There is an ever-increasing demand on how to obtain the exact solutions of nonlinear PDEs. Mathematical models that describe several physical phenomena as well as dynamical processes in chemistry, biology, geology, physics and ecology are often expressed in the form of nonlinear PDEs. Many generalizations are available in the literature on how to construct the exact solutions of one-component RD equations. Two-component RD systems of equations allow for the study of a wider range of physical phenomena than their counterpart one-component RD equations. There are certain models for which using two-component RD system is the most suitable and best way to study them. However, the results on two-component RD systems of equations are scarce (see e.g, [5]). Also, in the presence of delay time, nonlinear PDEs are more difficult to study. The precepts for obtaining the exact solutions of a generalized form of two-component RD systems of equations with delay have been presented in this study. How to apply the results of this study to obtain the exact solutions of miscellaneous multidimensional RD systems has also been elucidated. The efficiency of numerical methods which are being used for solving nonlinear delay PDEs can be verified by the results which have been presented in this paper.

List of Abbreviations
ODEs: Ordinary Differential Equations
PDEs: Partial Differential Equations
RD: Reaction-Diffusion

Acknowledgments

The first author acknowledges with thanks the postdoctoral fellowship and financial support from the DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

[1] M.O. Aibinu, S.C. Thakur and S. Moyo, *Exact solutions of nonlinear delay reaction-diffusion equations with variable coefficients*, Partial Differ. Equ.n Appl. Math. **4** (2021), 100170.

[2] A.A. Alderremy, T.M. Elzaki and M. Chamekh, *Modified Adomian decomposition method to solve generalized Emden-Fowler systems for singular IVP*, Math. Probl. Eng. **2019** (2019), 6097095.

[3] M. Al-Shaeer, *Solutions to nonlinear partial differential equations by Tan-Cot method*, IOSR J. Math. **5** (2013), no. 3, 6–11.

[4] I. Aziz, I. Siraj-ul-Islam and M. Asif, *Haar wavelet collocation method for three-dimensional elliptic partial differential equations*, Comput. Math. Appl. **73** (2017), 2023–2034.

[5] R. Bakhshandeh-Chamazkoti, *Abelian Lie symmetry algebras of two-dimensional quasilinear evolution equations*, Math. Meth. Appl. Sci. In Press, doi:10.1002/mma.8552.

[6] R. Bakhshandeh-Chamazkoti and M. Alipour, *Lie symmetries reduction and spectral methods on the fractional two-dimensional heat equation*, Math. Comput. Simul. **200** (2022), 97–107.

[7] N. Bilidik, *General convergence analysis for the perturbation iteration technique*, Turk. J. Math. Comput. Sci. **6** (2017), 1–9.

[8] R. Cherniha and O. Pliukhin, *New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations*, J. Phys. A: Math. Theory **40** (2007), no. 33, 10049–10070.

[9] R. Cherniha and O. Pliukhin, *New conditional symmetries and exact solutions of reaction diffusion systems with power diffusivities*, J. Phys. A: Math. Theo. **41** (2008), 185208.

[10] T.M. Elzaki and M. Chamekh M. *Solving nonlinear fractional differential equations using a new decomposition method*, Univ. J. Appl Math. Comput. **6** (2018), 27–35.

[11] L. Gai, M. Li and B. Sudao, *Approximate analytical and numerical solutions of a nonlinear boundary value problem in fluid mechanics based on symmetry reduction*, Partial Differ. Equ. Appl. Math. **2** (2020), 100011.

[12] M.B. Hossen and M.Z. Ali, *Dynamical structures of exact soliton solutions to Burgers equation via the bilinear approach*, Partial Differ. Equ. Appl. Math. **3** (2021), 100035.

[13] Z. Jackiewicza and B. Zubik-Kowal, *Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations*, Appl. Numerical Math. **56** (2006), 433–443.

[14] R.M. Jena and S. Chakraverty, *A new iterative method based solution for fractional Black-Scholes option pricing equations (BSOPE)*, SN Appl. Sci. **1** (2019), no. 1, 1–11.

[15] S. Kumbinarasaiah, *Numerical solution of partial differential equations using Laguerre wavelets collocation method*, Int. J. Manag. Technol. Eng. **9** (2019), no. 1, 3635–3639.

[16] M. Marzougue and I. Sane I, *Homogenization of periodic elliptic degenerate PDEs with non-linear Neumann boundary condition*, Partial Differ. Equ. Appl. Math. **3** (2021), 100021.

[17] S.V. Meleshko and S. Moyo, *On the complete group classification of the reaction-diffusion equation with a delay*, J. Math. Anal. Appl. **338** (2008), 448–466.

[18] A.G. Nikitin, *Group classification of systems of non-linear reaction-diffusion equations with general diffusion matrix. II. Generalized turing systems*, J. Math. Anal. Appl. **332** (2007), 666–690.

[19] P. Pandey, S. Kumar, J.F. Gómez-Aguilar and D. Baleanu, *An efficient technique for solving the space-time fractional reaction-diffusion equation in porous media*, Chinese J,f Phys. **68** (2020), 483–492.

[20] A.D. Polyanin, *Exact solutions of nonlinear sets of equations of the theory of heat and mass transfer in reactive media and mathematical biology*, Theor. Found. Chem. Engin. **38** (2004), no. 6, 622–635.

[21] A.D. Polyanin, *Functional separable solutions of nonlinear convection-diffusion equations with variable coefficients*, Comm. Nonlinear Sci. Numer. Simul. **73** (2019), 379–390.

[22] A.D. Polyanin, *Functional separable solutions of nonlinear reaction-diffusion equations with variable coefficients*, Appl. Math. Comput. **347** (2019), 282–292.

[23] A.D. Polyanin, *Generalized traveling-wave solutions of nonlinear reaction-diffusion equations with delay and variable coefficients*, Appl. Math. Lett. **90** (2019), 49–53.

[24] A.D. Polyanin and V.G. Sorokin, *New exact solutions of nonlinear wave type PDEs with delay*, Appl. Math. Lett. **108** (2020), 106512.

[25] A.D. Polyanin and V.G. Sorokin, *A method for constructing exact solutions of nonlinear delay PDEs*, J. Math. Anal. Appl. **494** (2021), 124619.

[26] A.D. Polyanin and V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations*, 2nd Edition, Boca Raton, Chapman & Hall/CRC Press, 2003.

[27] A.D. Polyanin and V.F. Zaitsev, *Handbook of ordinary differential equations: exact solutions, methods, and problems*, Boca Raton, CRC Press, 2018.

[28] A.D. Polyanin and A.I. Zhurov, *The generating equations method: Constructing exact solutions to delay reaction-diffusion systems and other non-linear coupled delay PDEs*, Int. J. Non-Linear Mech. **71** (2015), 104–115.

[29] A.D. Polyanin and A.I. Zhurov, *Non-linear instability and exact solutions to some delay reaction-diffusion systems*, Int. J. Nonlinear Mech. **62** (2014), 33–40.

[30] J. Rajarama and S. Chakraverty, *Residual power series method for solving time-fractional model of vibration equation of large membranes*, J. Appl. Comput. Mech. **5** (2018), 603–615.

[31] V.G. Sorokin, A. Vyazmin, A.I. Zhurov, V. Reznik and A.D. Polyanin, *The heat and mass transfer modeling with time delay*, Chem. Engin. Trans. **57** (2017), 1465–1470.

[32] W.F. Trench, *Elementary differential equations*, Faculty Authored and Edited Books & CDs. 8, 2013. https://digitalcommons.trinity.edu/mono/8.

[33] M.S. Ullah, M.Z. Ali and N.F.M. Noor, *Novel dynamics of wave solutions for Cahn-Allen and diffusive predator-prey models using MSE scheme*, Partial Differ. Equ. Appl. Math. **3** (2021), 100017.

[34] B.W. Williams, *Exact traveling wave solutions of fast reaction-diffusion-convection equations based on the Lambert function W*, Partial Differ. Equ. Appl. Math. **2** (2020), 100013.

[35] T. Zhang and Y. Jin, *Traveling waves for a reaction-diffusion-advection predator-prey model*, Nonlinear Anal.: Real World Appl. **36** (2017), 203–232.