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On CLS-modules and the S-closure of a submodule

Bahram Vakili

Department of Mathematics, Shabestar Branch, Islamic Azad University, Shabestar, Iran

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Abstract

A module M is called a CLS-module if every S-closed submodule of M is a direct summand of M [9]. We give a characterization for CLS-modules and obtain a sufficient condition for CLS-submodules of a CLS-module. Also, we characterize the splitting property in terms of UT-modules and the S-closure of submodules.

Keywords: S-closure, CLS-module, splitting property
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1 Introduction

In what follows, all rings R have identities and all modules are unitary right R-modules, unless otherwise stated. Let us give some basic notations and terminologies. If M is a module, then the notations $A \leqq M$, $A \leqq_e M$ mean A is a submodule of M, A is an essential submodule of M, respectively. The singular submodule of M is $Z(M) = \{m \in M \mid ann_r(m) \leqq_e R_R\}$. M is called a singular module if $Z(M) = M$; and M is nonsingular if $Z(M) = 0$. The singular submodule of R_R is denoted by $Z_r(R)$.

Recall from [3] that a submodule A of a module M will be called S-closed if M/A is nonsingular (In [10], Tercan and Yücel call S-closed by "z-closed"). We use $L^*(M)$ to denote the collection of all S-closed submodules of M. A submodule K of M is called closed (in M) if K has no proper essential extension in M. In general, closed submodules need not be S-closed. For example, 0 is a closed submodule of any module M, but 0 is S -closed in M only if M is nonsingular. It is well known that, every S-closed submodule of a module M is a closed submodule, and every closed submodule of a nonsingular module is S-closed in M. (For example, see [7, Lemma 2.3] or [3, Proposition 2.4].)

Let M be an R-module and $A \leqq M$. The purpose of this paper is to study the S-closure of A in M. In section 2, we show that if M is nonsingular and K is the S-closure of a submodule A in M, then K is the only essential closure of A (i.e. maximal essential extention (see [8])) in M; in particular, K is the only S-closed submodule of M for which $A \leqq_e K$. This generalizes [3, Proposition 2.3(c)] without the condition $Z_r(R) = 0$.

A module M is said to be a CLS-module if every S-closed submodule of M is a direct summand of M [9]. We give a characterization for CLS-modules, and we show that if M is a CLS-module and L is a submodule of M with the property that $X \leqq^{\oplus} M$ implies $X \cap L \leqq^{\oplus} L$, then L is a CLS-module. As a consequence, every fully invariant submodule and every distributive submodule of a CLS-module is a CLS-module.

*Corresponding author
Email address: bvakil2011@gmail.com, bvakil@iaushab.ac.ir, bvakil2004@yahoo.com (Bahram Vakili)

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An important question for a module over a commutative ring is: when does it split, in the sense that its singular submodule is a direct summand? As in [1], a ring R has the splitting property (SP) if every R-module splits. We characterize the splitting property in terms of UT-modules and the S-closure of submodules.

2 The results

Let M be an R-module and $A \leqq M$. Recall from [3] that the S-closure of A in M is the intersection of all S-closed submodules of M containing A.

Proposition 2.1. Let M be a nonsingular R-module, $A \leqq M$ and K be the S-closure of A in M. Then K is the only essential closure of A in M.

Proof . We know that the essential closure always exists. Now let K' be an essential closure of A in M. Hence $A \leqq_e K'$ and K' is closed in M. But since M is nonsingular so K' is S-closed in M, i.e. $K' \in L^*(M)$, and so $K = \bigcap_{A \subseteq N \in L^*(M)} N \subseteq K'$. Now, $A \leqq K \leqq K'$ and $A \leqq_e K'$, hence $K \leqq_e K'$ and since K is closed in M so $K = K'$. Therefore, K is the only essential closure of A in M. □

Recall from [8] that a module M is called UC-module if every submodule has a unique essential closure in M. An immediate consequence of 2.1 is the following corollary which is a Johnson's result [4, Theorem 6.4].

Corollary 2.2. Every nonsingular module is a UC-module.

The next result is a generalization of [3, Proposition 2.3(c)] without the condition $Z_r(R) = 0$.

Proposition 2.3. Let M be a nonsingular R-module, $A \leqq M$ and K be the S-closure of A in M. Then K is the only S-closed submodule of M for which $A \leqq_e K$.

Proof . We know that K is closed in M. Also by 2.1, $A \leqq_e K$. Now let $K' \in L^*(M)$ with $A \leqq_e K'$. We have $A \leqq K \leqq K'$ and $A \leqq_e K'$. Hence $K \leqq_e K'$ and since K is closed in M so $K = K'$, as required. □

Let M be an R-module. The S-closure of any submodule of M is S-closed in M, and the S-closure of any S-closed submodule A of M is A itself. Therefore, M is a CLS-module if and only if the S-closure of any submodule of M is a direct summand of M. The following proposition gives a characterization for CLS-modules in the case that the ring is nonsingular.

Proposition 2.4. Let M be an R-module. If for every $A \leqq M$ there exists an S-closed submodule K of M such that K is a direct summand of M with $K \supseteq A$ and K/A is singular, then M is a CLS-module. The converse is true if R is nonsingular.

Proof . Suppose that M has the stated property. Let A be an S-closed submodule of M. By hypothesis, there exists an S-closed submodule K of M such that K is a direct summand of M with $K \supseteq A$ and K/A is singular. Since A is S-closed in M, $K/A \leqq M/A$ implies that K/A is nonsingular. Hence K/A is both singular and nonsingular, which implies that $K/A = 0$ and so $K = A$. Therefore, A is a direct summand of M. It follows that M is a CLS-module.

Conversely, let R be nonsingular and M be a CLS-module. Let $A \leqq M$. Suppose that K is the S-closure of A in M. Hence $K \supseteq A$ and K is S-closed in M. Also since $Z_r(R) = 0$, it follows from [3, Proposition 2.3(a)] that $K/A = Z(M/A)$. Hence K/A is singular. Also since M is a CLS-module and K is S-closed in M, so K is a direct summand of M. □

The following proposition gives a sufficent condition for CLS-submodules of a CLS-module.

Proposition 2.5. Let M be a CLS-module. Let L be a submodule of M with the property that $X \leqq^{\oplus} M$ implies $X \cap L \leqq^{\oplus} L$. Then L is a CLS-module.

Proof . Let $A \leqq L$. Then $A \leqq M$ and since M is a CLS-module, it follows from Proposition 2.4 that there exists an S-closed submodule K of M such that K is a direct summand of M with $K \supseteq A$ and K/A is singular. Now, by hypothesis, $K \leqq^{\oplus} M$ implies that $K \cap L \leqq^{\oplus} L$. Now, $K \cap L \supseteq A$ and $(K \cap L)/A$ is a submodule of the singular module K/A, so $(K \cap L)/A$ is sigular. On the other hand, $L/(K \cap L) \cong (K + L)/K \leqq M/K$ implies that $K \cap L$ is an S-closed submodule of L. Therefore, by Proposition 2.4, L is a CLS-module. □

Given any map $f : M \to N$ in Mod-R, we have $f(Z(M)) \subseteq Z(N)$. In particular, for any module M we have $f(Z(M)) \subseteq Z(M)$ for all $f \in End_R(M)$, so that $Z(M)$ is a fully invariant submodule of M. (Recall that a submodule A of M is called a fully invariant submodule if $f(A) \subseteq A$ for all $f \in End_R(M)$.) Also, since $Hom_R(Z(M), Z(M)/N) = 0$ for all $N \in L^*(Z(M))$, so $L^*(Z(M)) = \{Z(M)\}$, i.e. $Z(M)$ is the only S-closed submodule of $Z(M)$. It follows that $Z(M)$ is a CLS-module for any module M. In general we have the following corollary:

Corollary 2.6. Let M be a CLS-module. Then every fully invariant submodule of M is a CLS-module.

Proof . Let L be a fully invariant submodule of M. If $M = X \oplus X'$ for some $X,\ X' \leqq M$, then $L = (L \cap X) \oplus (L \cap X')$. It follows from Proposition 2.5 that L is a CLS-module.. □

A submodule A of an R-module M is called a distributive submodule if $A \cap (X + Y) = (A \cap X) + (A \cap Y)$, for all submodules X, Y of M. Clearly, if L is a distributive submodule of M then $X \leqq^{\oplus} M$ implies $X \cap L \leqq^{\oplus} L$. Hence we have the following corollary by using Proposition 2.5.

Corollary 2.7. Let M be a CLS-module. Then every distributive submodule of M is a CLS-module.

In the rest of this section, R will denote a commutative ring. Recall from [5] and [2] that an R-module N is UF if N is a nonsingular module and $\mathrm{Ext}_1^{\mathrm{R}}(\mathrm{N}, \mathrm{S}) = 0$ for all singular modules S. Motivated by [5] and [2], we say that an R-module S is UT if S is a singular module and $\mathrm{Ext}_1^{\mathrm{R}}(\mathrm{N}, \mathrm{S}) = 0$ for all nonsingular modules N. An R-module M is called split if $Z(M)$ is a direct summand of M. As in [1], a ring R has spliting property (SP) if every R-module splits. An immediate consequence of Cateforis and Sandomierski [1, Proposition 1.12] is the following proposition.

Proposition 2.8. For any ring R, the following statements are equivalent:

1. R has SP;
2. Z(R) = 0, and every nonsingular R-module is UF;
3. Z(R) = 0, and every singular R-module is UT.

To prove the main results of this part, we first bring the following proposition.

Proposition 2.9. Let R be a nonsigular ring. Then the following statements are equivalent:

1. Every singular R-module is UT;
2. For every R-module M, K/A is a UT-module for all $A \leqq M$, where K is the S-closure of A in M.

Proof . $(i) \Rightarrow (ii)$: Let M be an R-module and $A \leqq M$. Suppose that K is the S-closure of A in M. Since R is nonsigular , it follows from [3, Proposition 2.3(a)] that $K/A = Z(M/A)$. Hence K/A is a singular R-module and so K/A is a UT-module by (i).

$(ii) \Rightarrow (i)$: Let S be a singular R-module. Then there exist R-modules $A \leqq_e B$ such that $M \cong B/A$ by [6, Example 7.6(3)]. Suppose that K is the S-closure of A in B. By (ii), K/A is a UT-module. Now, $A \leqq K \leqq B$ and $A \leqq_e B$ implies that $K \leqq_e B$. But since K is S-closed in B so K is closed in B and it follows that $K = B$. Hence $B/A = K/A$ is a UT-module. Therefore, M is a UT-module. □

Combining Propositions 2.8 and 2.9, we are now ready to state the most important result of this paper, which is a characterization of rings with SP in terms of UF- modules, UT-modules and the S-closure of submodules.

Corollary 2.10. For any ring R, the following statements are equivalent:

1. R has SP;
2. Z(R) = 0, and every nonsingular R-module is UF;
3. Z(R) = 0, and every singular R-module is UT.
4. Z(R) = 0, and For every R-module M, K/A is a UT-module for all $A \leqq M$, where K is the S-closure of A in M.

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