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An inertial-based hybrid and shrinking projection methods for solving split common fixed point problems in real reflexive spaces

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Abstract

We introduce and study an inertial-based iterative algorithm for solving the split common fixed point problem involving a finite family of Bregman quasi-strictly pseudocontractive mappings in real reflexive Banach spaces. Strong convergence of the proposed algorithm is obtained under mild assumptions.

Keywords: Bregman distance, quasi-Bregman strictly pseudocontractive mapping, fixed point problem, Banach space

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1 Introduction

Throughout this paper, C will denote a nonempty, closed and convex subset of a real Banach space E while E^* denotes the dual space of E. Strong and weak convergence will be denoted by $\to$ and $\rightharpoonup$ respectively. A point x^* is called a fixed of an operator T if $Tx^* = x^*$. The set of fixed points of an operator $T : C \to C$ is denoted by $F(T) = \{x \in C : Tx = x\} \neq \emptyset$.

Many problems in applied mathematics can be reduced to finding a fixed point of an operator. The problems in signal processing and image recovery are iterative in nature. The pioneering work of Censor and Elfving [13] on split feasibility problems (SFP) to model inverse problems arising in medical imaging and signal processing triggered vigorous research interest by many authors [11, 10] and the references therein. Numerous research papers on iterative algorithms for solving SFP have been published after the work of Censor and Elfving [13], for instance, see [10, 12, 28]. A lot of modifications and generalizations have been made to accommodate other problems arising in applied sciences. For example, generalizations have been made from SFP to multi-sets SFP (MSSFP) [14, 20], the split common fixed point problem (SCFPP) [16, 21] for details, and split common null point problem (SCNPP) [12] for details.

Many of the papers mentioned above were done in Euclidean and Hilbert spaces. Several attempts have been made by different researchers to extend the results to a more general space, the Banach space; for details see [25, 31, 32]

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and the references therein. The major drawback in the proposed algorithms is that the daulity mapping of a Banach space which is a nonlinear mapping has to be used for the computations and it can be reasonably difficult to achieve. Based on the foregoing, in 1967, Bregman [7] introduced an effective technique to overcome this challenge with the concept that is popularly known today as the Bregman distance. This tool has been widely used to design and analyze problems in feasibility and optimization algorithms as well as fixed point problems for nonlinear operators in general (see for instance [9, 27]).

It is pertinent to note, however that many of these algorithms developed are modifications of Mann iterative scheme, which in general, is known to have slow rate of convergence. Polyak [23] was the first to propose an acceleration process called inertial-type algorithm for solving a smooth convex minimization problem. It has been observed that incorporating inertial terms into algorithms increases the rate of convergence of such algorithms.

Consequently, researchers have in different ways constructed fast iterative algorithms by means of inertial-term techniques, (see for instance [3, 5, 18, 19]).

Recently, Bashir et al. [3] studied an inertial algorithm for quasi-Bregman strictly pseudocontractive mappings in real reflexive spaces. They established a strong convergence of the sequence generated by their algorithm.

In 2020, Reich and Tuyen [28] constructed the following hybrid and shrinking projection algorithms for solving split common fixed point problem in real reflexive Banach spaces. For definition of terms used in the Theorems of Reich and Tuyen, please see preliminary section, section 2 of this manuscript.

Theorem 1.1. (Reich and Tuyen [28]) Let E and F be two real reflexive Banach spaces. Let $f : E \to E$ and $g : F \to F$ be two Legendre functions, which are bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E and F, respectively. Let $T_i : E \to E, i = 1, 2, ..., N$, and $S_j : F \to F, j = 1, 2, ..., M$, be Bregman relatively nonexpansive operators. Let $\Theta : E \to F$ be a bounded linear operator. Suppose that $\Omega = \cap_{i=1}^{N} F(T_i) \cap \Theta^{-\prime}(\cap_{j=1}^{M} F(S_j)) \neq \emptyset$.

$$(1.1)\quad \begin{cases} x_0 \in E, \\ y_{i,n} = T_i(x_n), i = 1, 2, ..., N, \\ z_{j,n} = S_j(\Theta x_n), j = 1, 2, 3, ..., M, \\ \text{let } \ d_{1,n} := \max_{1\leqslant i\leqslant N}\{D_f(y_{i,n}, x_n)\}, d_{2,n} : \max_{1\leqslant j\leqslant M}\{D_f(z_{j,n}, \Theta(x_n))\}, \\ K_n: \ = \{i \in \{1, 2, ..., N\} : D_f(y_{i,n}, x_n) = d_{1,n}\}, \\ L_n := \{j \in \{1, 2, ..., M\} : D_g(z_{i,n}, x_n) = d_{2,n}\}, \\ \text{if } \ d_{1,n} \geq d_{2,n}, \text{choose } \ i_n \in K_n \ \text{ and let } t_n := y_{i_n,n}, \Phi = I^E, h = f, \\ \text{if } d_{1,n} < d_{2,n}, \text{choose } j_n \in L_n \ \text{ and let } t_n := z_{j_n,n}, \Phi = \Theta, h = g, \\ C_n := \{z \in E : D_h(\Phi z, t_n) \leqslant D_h(\Phi z, \Phi(x_n))\}, \\ Q_n := \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}, \\ x_{n+1} = proj^f_{C_n\cap Q_n}(x_0), \quad n \geq 0. \end{cases}$$

$$(1.2)\quad \begin{cases} x_0 \in E, \\ y_{i,n} = T_i(x_n), i = 1, 2, ..., N, \\ z_{j,n} = S_j(\Theta x_n), j = 1, 2, 3, ..., M, \\ \text{let } \ d_{1,n} := \max_{1\leqslant i\leqslant N}\{D_f(y_{i,n}, x_n)\}, d_{2,n} : \max_{1\leqslant j\leqslant M}\{D_f(z_{j,n}, \Theta(x_n))\}, \\ K_n: \ = \{i \in \{1, 2, ..., N\} : D_f(y_{i,n}, x_n) = d_{1,n}\}, \\ L_n := \{j \in \{1, 2, ..., M\} : D_g(z_{i,n}, x_n) = d_{2,n}\}, \\ \text{if } \ d_{1,n} \geq d_{2,n}, \text{choose } \ i_n \in K_n \ \text{ and let } t_n := y_{i_n,n}, \Phi = I^E, h = f, \\ \text{if } d_{1,n} < d_{2,n}, \text{choose } j_n \in L_n \ \text{ and let } t_n := z_{j_n,n}, \Phi = \Theta, h = g, \\ C_{n+1} := \{z \in C_n : D_h(\Phi(z), t_n) \leqslant D_h(\Phi(z), \Phi(x_n))\}, \\ x_{n+1} = proj^f_{C_{n+1}} x_0, \quad n \geq 0. \end{cases}$$

Then, the sequences $\{x_n\}$ generated by the Algorithms 1.1 and 1.2 converge strongly to $x^* = proj^f_{\Omega} x_0$ as $n \to \infty, x^* \in \Omega$. Inspired and motivated by the above mentioned results, we propose in this article, an inertial-based hybrid and shrinking projection algorithm for solving split common fixed point problem in real reflexive spaces for finite families of quasi-Bregman strictly pseudocontractive mappings.

Remark 1.2. Our proposed algorithm has the following features.

a) The class of Bregman quasi-strictly pseudocontractive operators we studied is more general than the class of Bregman quasi nonexpansive studied by Reich and Tuyen (2020).

b) If we set the operators T_i and S_j to be Bregman quasi nonexpansive in our result (see Theorem 3.1 below)and $\alpha \equiv 0$, we recover the result of Reich and Tuyen [27].

c) The problem studied in this manuscript includes the problem studied by Bashir et al, [3] as a special case.

d) Our algorithms contain inertial term which is known to increase the rate of convergence.

2 Preliminaries

The normalized duality map $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle , \rangle$ is the daulity pairing between E and E^*. We denote by $f : E \to (-\infty, +\infty]$ a proper, lower semi continous convex function. The domain of f is defined by $domf : \{x \in E : f(x) < +\infty\}$. A function f defined on a Banach space E is called coercive [17] if the sublevel set of f is bounded; that is,

$$\lim_{\|x\|\to\infty} f(x) = +\infty,$$

where a sublevel set of f is defined by

$$lev^f \leqslant (r) := \{x \in E : f(x) \leqslant r\},$$

for some $r \in \mathbb{R}$. The function f defined on E is strongly coercive [38] if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\| x \|} = +\infty.$$

For any $x \in int(domf)$ and $y \in E$ we denote by $f^{'}(x, y)$ the right-hand derivative of f at x in the direction of y which is defined by

$$f^{'}(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called Gateaux differentiable at x if $\lim_{t\to 0^+} f(x + ty) - f(x)/t$ exists for each y in E. In this case, $f^{'}(x, y) = (\nabla f)(x)$, the value of the gradient ∇f of f at x and $\nabla f(x) : E \to (-\infty, +\infty]$. The function f is said to be Frechet differentiable at x if this limit exists uniformly $\forall y \in E$ such that $\| y \|= 1$. Lastly, f is said to be uniformly Frechet differentiable on a subset C of E, if the limit is attainable uniformly for all $x \in C$ and $y \in E$ with $\| y \|= 1$. It has been proved (see [2, 6]) that if f is Gateaux differentiable(respectively, Frechet differentiable) on the $int(domf)$, then f is continuous and its Gateaux derivative ∇f is norm-to-$weak^*$ continuous (respectively, norm-to-norm continuous) on the $int(domf)$.

The subdifferential of f at x is a convex set defined by

$$\partial f(x) = \{x^* \in E : \langle x^*, y - x\rangle \leqslant f(y) - f(x), \forall y \in E\},$$

while the Frechet conjugate of f is a function $f^* : E^* \to (-\infty, +\infty]$ which is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

Let $f : E \to (-\infty, +\infty]$ be a Gateaux differentiable function. The modulus of total convexity of f at $x \in int(domf)$ is a function $v_f : int(domf) \times [0, \infty[\to [0, \infty[$, defined by

$$v_f(x, t) :\ = \inf\{D_f(y, x) : y \in domf, \|y - x\| = t\}.$$

v_f is positive whenever $t > 0$. A function f is called totally convex if it is totally convex at every $x \in int(domf)$; it is totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$ and it is said to be totally convex on a bounded set B if $v_f(B, t) > 0$

for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(B, t) \colon = \inf\{v_f(x, t) \colon x \in B \cap domf\}.$$

If E be a reflexive Banach space, a function $f : E \to (-\infty, +\infty]$ is said to be Legendre, if it satisfies the following conditions:

(C1). The $int(domf)$ is nonempty, f is Gateaux differentiable on the $int(domf)$, and $dom\nabla f = int(domf)$;

(C2). The $int(domf^*)$ is nonempty, f^* is Gateaux differentiable on the $int(domf^*)$, and $dom\nabla f^* = int(domf^*)$, (see [4], Theorem 5.10).

Remark 2.1. Since E is reflexive, it is well known that $(\partial f)^{-1} = \partial f^*$ (see [6], p. 83). Combining (C1) and (C2) with the fact that $(\partial f)^{-1} = \partial f^*$, we obtain the following results:

$$\nabla f = (\nabla f^*)^{-1},\ ran\ \nabla f = dom\nabla f^* = int(domf^*)$$

and

$$ran\nabla f^* = dom\ \nabla f = int(domf).$$

The range of ∇f is denoted by $ran\ \nabla f$. It is also observed that conditions (C1) and (C2) further imply that f and f^* are strictly convex on the interior of their respective domains (see [4] for details). When the subdifferential ∂f is single-valued, then $\partial f = \nabla f$ (see for details [9]).

The function f is said to be, (see [4]):

i). Essentially smooth, if ∂f is both localy bounded and single-valued on its domain;

ii). Essentially strictly convex, if $(\partial f)^{-\prime}$ is locally bounded and f is strictly convex on every subset of $domf$;

iii). Legendre, if it is both essentially smooth and essentially strictly convex.

If E be a smooth and strictly convex Banach space, then an important Legendre function is

$$f(x) = \frac{1}{p}\|x\|^p, 1 \leq p < \infty.$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function $D_f : domf \times int(domf) \to [0, \infty[$, defined by

(2.1) $D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle$

is called the Bregman distance with respect to f (see [7]). The following identities are well known with Bregman distance:

(1) the two points identity, for any $x \in domf$ and any $y \in int(domf)$

(2.2) $D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(x), x - y\rangle$,

(2) three points identity, for any $y, \omega \in domf$ and $x, y \in int(domf)$,

(2.3) $D_f(y, x) - D_f(y, z) - D_f(\omega, x) + D_f(\omega, z) = \langle \nabla f(z) - \nabla f(y), y - \omega\rangle$.

It is well known that the Bregman projection of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset domf$ is a unique vector $P_C(x) \in C$ satisfying the condition below:

$$D_f(P_K(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 2.2. a) If E be a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then $\nabla f(x) = 2Jx$, for all $x \in E$ where, J is the normalized duality mapping. Hence, $D_f(x, y)$ becomes $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$.
b) If $E = H$ be a Hilbert space, then it is known that J is an identity mapping. Thus, the Bregman projection $proj^f_C(x)$ is the metric projection of H onto C.

A function f is called sequential consistent (see Butnariu and Resmerita [9]), if for any two sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in $int(domf)$ and $domf$, respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, it follows that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.
Let C be a convex subset of $int(domf)$ and let T be a self mapping of C. A point p in the closure of C is said to be asymptotical fixed point of T, if there is a sequence $\{x_n\} \subset C$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\overline{F}(T)$.
The operator T is called:
i). quasi-Bregman nonexpansive, if $F(T) \neq \varnothing$ and

(2.4) $D_f(p, T(x)) \leqslant D_f(p, x), \forall x \in C, p \in F(T);$

ii). Bregman relatively nonexpansive, if $\overline{F}(T) = F(T) \neq \varnothing$ and

(2.5) $D_f(p, T(x)) \leq D_f(p, x), \forall x \in C, p \in F(T).$

iii). quasi-Bregman strictly pseudocontractive if there exists a constant $\lambda \in [0, 1)$ with $F(T) \neq \varnothing$ such that

(2.6) $D_f(p, Tx) \leq D_f(p, x) + \lambda D_f(x, Tx), \forall x \in C, p \in F(T);$

Remark 2.3. If $\lambda = 0$ in definition (iii), we observe that every Bregman relatively nonexpanxive mapping with nonempty fixed point set is quasi Bregman nonexpanxive and hence is a quasi-Bregman strictly pseudocontractive mapping. Hence, quasi-Bregman strictly pseudocontractive mapping is more general than the other two class of mappings.

The following established results will be useful in our result:

Lemma 2.4 ([26], Proposition 1). If $f\colon E \to \mathbb{R}$ be a uniformly Frechet differentiable and bounded on a bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from strong topology of E to that of E^* respectively.

Lemma 2.5 ([1], Theorem 1.8). If $f\colon E \to \mathbb{R}$ be a uniformly Frechet differentiable, then f is uniformly continuous on E.

Lemma 2.6 ([9], Corollary 4.4). Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f\colon E \to R$ be a Gateaux differentiable and totally convex function and let $x \in C$. Then, the following conditions are equivalent:
i). $z = P_C(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leqslant 0, \forall y \in C;$
ii). $D_f(y, P_C(x)) + D_f(P_C(x), x) \leqslant D_f(y, x), \forall x \in E, y \in C;$
iii). z is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geqslant 0, \forall y \in C.$$

Lemma 2.7 ([29], Theorem 1.8). If $x \in int(domf)$, then the following statements are equivalent:
(a) the function f is totally convex at x;
(b) for any sequence $\{y_n\} \subset domf$, the condition

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x\| = 0.$$

Lemma 2.8 ([22], Lemma 2.4). Let E be a Banach space and $f\colon E \to \mathbb{R}$ be a Gateaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be bounded sequences in E. Then $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.9 ([8], Lemma 2.1.2). The function f is totally convex on bounded sets, if and only if it is sequentially consistent.

Lemma 2.10 ([25],Proposition 3). Let $f\colon E \to \mathbb{R}$ be Gateaux differentiable and totally convex function. If $\{x_0\} \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.

Lemma 2.11 ([22], Lemma 2.5). Let E be a reflexive Banach space, let $f\colon E \to \mathbb{R}$ be a strongly coercive Bregman function and let V be a function defined by

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), x \in E, x^* \in E^*.$$

Then, the following results are true:
(1) $D_f(x, \nabla f^*(x^*)) = V(x, x^*), \forall x \in E, x^* \in E^*$.
(2) $V(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leqslant V(x, x^* + y^*), \forall x \in E, \forall x^*, y^* \in E^*$.

Lemma 2.12 ([37], Lemma 2.1). Let $f\colon E \to \mathbb{R}$ be a Legendre function which is uniformly Frechet differentiable and bounded on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and let $T\colon C \to C$ be a closed quasi-Bregman strictly pseudocontractive mapping with respect to f.
Then for any $u \in C,\ p \in F(T),$ and $\lambda \in [0, 1)$, the following results hold:
(a) $D_f(u, Tu) \leqslant \frac{\lambda}{1-\lambda}\langle \nabla f(u) - \nabla f(Tu), u - p\rangle$.
(b) $F(T)$ is closed and convex.
Proof: a) Let $u \in C,\ p \in F(t)$ and $\lambda \in [0, 1)$. Then by the definition of T we have,

$$D_f(p, Tu) \leqslant D_f(p, u) + \lambda D_f(u, Tu). \tag{2.7}$$

Applying the two points identity of the Bregman distance (2.7), we obtain $D_f(u, Tu) \leqslant \frac{\lambda}{1-\lambda}\langle \nabla f(u) - \nabla f(Tu), u - p\rangle$ as asserted.

b) For the closedness: let $\{x_n\}$ be in $F(T)$ such that $x_n \to z$ as $n \to \infty$. We show that $z \in F(T)$. From Lemma 2.9(a), we obtain that

$$D_f(z, Tz) \leq \frac{\lambda}{1-\lambda}\langle \nabla f(z) - \nabla f(Tz), z - x_n\rangle. \tag{2.8}$$

It follows from (2.8) that $D_f(z, Tz) \leq 0$ and consequently, from [4], Lemma 7.3, it follows that $F(T)$ is closed.
For convexity: let $z_1, z_2 \in F(T)$, for any $\alpha \in (0, 1)$, set $z = \alpha z_1 + (1 - \alpha)z_2$. We show that $z \in F(T)$. It follows from Lemma 2.9(a), that

$$D_f(z, Tz) \leq \frac{\lambda}{1-\lambda}\langle \nabla f(z) - \nabla f(Tz), z - z_1\rangle \tag{2.9}$$

and

$$D_f(z, Tz) \leq \frac{\lambda}{1-\lambda}\langle \nabla f(z) - \nabla f(Tz), z - z_2\rangle \tag{2.10}$$

respectively.
If we multiply inequalities (2.9) by t and (2.10) by $(1 - t)$ and sum the result, we have

$$D_f(z, Tz) \leq \frac{\lambda}{1-\lambda}\langle \nabla f(z) - \nabla f(Tz), z - z\rangle = 0. \tag{2.11}$$

It follows from (2.11) that $D_f(z, Tz) \leq 0$. From [4], Lemma 7.3 we conclude that $F(T)$ is convex. Therefore, $F(T)$ is closed and convex. This completes the proof.

3 Main Results

In this section, we state and prove our main result.

Theorem 3.1. Let E and F be two real reflexive Banach spaces and $f\colon E\to\mathbb{R}$ and $g\colon F\to\mathbb{R}$ be two Legendre functions which are bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E and F respectively. Let $T_i\colon E\to E, i=\overline{1,N}$ and $S_j\colon F\to F, j=\overline{1,M}$ be finite families of λ_i and β_j closed quasi-Bregman strictly pseudocontractive mappings respectively with λ_i and β_j in (0,1). Let $\Psi\colon E\to F$ be a bounded linear operator. Suppose that $\Omega\colon=\bigcap_{i=1}^N F(T_i)\bigcap\Psi^{-\prime}(\bigcap_{j=1}^M F(S_j))\neq\emptyset$ and assume $(I-T_i), i=1,2,...,N$ is demiclosed at the origin, that is, if $x_n\rightharpoonup x\in C$ and $x_n-T_ix_n\to 0$, then $x=T_ix$. For $\lambda=\min_{1\leqslant i\leqslant N}\{\lambda_i\}, \beta=\min_{1\leqslant j\leqslant N}\{\beta_j\}$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$(3.1)\qquad \begin{cases} x_0, x_{-1}\in E,\\ u_n=x_n+\alpha_n(x_n-x_{n-1}),\\ v_n=\nabla f^*(\sigma\nabla f(u_n)+(1-\sigma)\nabla f(T_iu_n)),\\ y_{i,n}=T_i(v_n), i=1,2,...,N,\\ z_{j,n}=S_j(\Psi y_{i,n}), j=1,2,3,...,M,\\ \text{let } \ d_{1,n}:=\max_{1\leqslant i\leqslant N}\{D_f(y_{i,n},u_n)\}, d_{2,n}:\max_{1\leqslant j\leqslant M}\{D_f(z_{j,n},\Psi u_n)\},\\ K_n\colon=\{i\in\{1,2,...,N\}\colon D_f(y_{i,n},u_n)=d_{1,n}\},\\ L_n:=\{j\in\{1,2,...,M\}:D_g(z_{i,n},\Psi u_n)=d_{2,n}\},\\ \text{if } \ d_{1,n}\geq d_{2,n}, \text{choose } \ i_n\in K_n \text{ and let } t_n:=y_{i_n,n}, \Phi=I^E, h=f,\\ \text{if } d_{1,n}<d_{2,n}, \text{choose } j_n\in L_n \text{ and let } t_n:=z_{j_n,n}, \Phi=\Theta, h=g,\\ C_n:=\{z\in E:D_h(\Phi z,z_{j,n})\leqslant D_f(\Phi z,y_{j,n})\leqslant D_f(\Phi z,v_n)\leqslant D_f(\Phi z,u_n)+\\ \frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\},\\ Q_n:=\{z\in E:\langle\nabla f(x_0)-\nabla f(x_n),z-x_n\rangle\leqslant 0\},\\ x_{n+1}=proj^f_{C_n\cap Q_n}(x_0), n\geq 0, \end{cases}$$

where σ is fixed in (0,1) and $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\lim x_n=0$.
Then, the sequence $\{x_n\}$ converges strongly to $proj^f_\Omega(x_0)$, where $proj^f_\Omega(x_0)$ is the Bregman projection of x_0 onto Ω.

Proof . The proof is divided into six steps.

Step 1: We show that $\Omega\colon=\bigcap_{i=1}^N F(T_i)\bigcap\Psi^{-\prime}(\bigcap_{j=1}^M F(S_j))$ is closed and convex. From Lemma 2.9(b), $F(T_i)$ is closed and convex for each $i=1,2,...,N$. So, $F(S_j)$ is closed and convex for each $j=1,2,...,M$ and consequently, it follows that $\bigcap_{i=1}^N F(T_i)$ and $\bigcap_{j=1}^M F(S_j)$ are closed and convex for each $i=1,2,...,N$ and $j=1,2,...,M$, respectively. Next, we show that $\Psi^{-\prime}(\bigcap_{j=1}^M F(S_j))$ is closed and convex. But Ψ is a bounded linear operator, it follows that $\Psi^{-\prime}(\bigcap_{j=1}^M F(S_j))$ is a closed and convex subset of E. Consequently, $\Omega\colon\bigcap_{i=1}^N F(T_i)\bigcap\Psi^{-\prime}(\bigcap_{j=1}^M F(S_j))$ is closed and convex.

Step II: We show that C_n is closed and convex. Let $D_n\colon=\{z\in E\colon D_h(\Phi z,v_n)\leqslant D_h(\Phi z,u_n)+\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}$, $A_n\colon\{z\in E\colon D_h(\Phi z,z_{j,n})\leqslant D_h(\Phi z,y_{i,n})\}$ and $B_n\colon=\{z\in E\colon D_h(\Phi z,y_{i,n})\leqslant D_h(\Phi z,v_n)\}$. So,

$$C_n\colon=D_n\cap A_n\cap B_n.$$

To show that C_n is closed and convex, it suffices to show that each of D_n, A_n and B_n is closed and convex. First, we show that D_n is closed and convex. To see this, observe that

$$\begin{aligned} D_n\colon&=\{z\in E\colon D_h(\Phi z,v_n)\leqslant D_h(\Phi z,u_n)+\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}\\ &=\{z\in E\colon D_h(\Phi z,v_n)-D_h(\Phi z,u_n)\leqslant\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}\\ &=\{z\in E\colon h(\Phi z)-h(v_n)-\langle\nabla h(v_n),\Phi z-v_n\rangle-(h(\Phi z)-h(u_n)-\langle\nabla h(u_n),\Phi z-u_n\rangle)\\ &\qquad\leqslant\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}\\ &=\{z\in E\colon h(u_n)-h(v_n)-\langle\nabla h(v_n),\Phi z-v_n\rangle+\langle\nabla h(u_n),\Phi z-u_n\rangle\\ &\qquad\leqslant\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}\\ &=\{z\in E\colon\langle\nabla h(u_n),\Phi z-u_n\rangle-\langle\nabla h(v_n),\Phi z-v_n\rangle\leqslant h(v_n)-h(u_n) \end{aligned}$$

$$+\frac{\lambda}{1-\lambda}\langle\nabla f(u_n)-\nabla f(T_iu_n),u_n-\Phi z\rangle\}$$
$$=\{z\in E\colon \langle\nabla h(u_n),\Phi z\rangle-\langle\nabla h(u_n),u_n\rangle-\langle\nabla h(v_n),\Phi z\rangle+\langle\nabla h(v_n),v_n\rangle\leqslant h(v_n)-h(u_n)+\frac{\lambda}{1-\lambda}\langle\nabla f(u_n),u_n\rangle$$
$$-\frac{\lambda}{1-\lambda}\langle\nabla f(u_n),\Phi z\rangle-\frac{\lambda}{1-\lambda}\langle\nabla f(T_iu_n),u_n\rangle+\frac{\lambda}{1-\lambda}\langle\nabla f(T_iu_n),\Phi z\rangle\}$$
$$=\{z\in E\colon \langle\nabla h(u_n)-\nabla h(v_n)+\frac{\lambda}{1-\lambda}\nabla f(u_n)-\frac{\lambda}{1-\lambda}\nabla f(T_iu_n),\Phi z\rangle\leqslant h(v_n)-h(u_n)+\langle\nabla h(u_n)+\frac{\lambda}{1-\lambda}\nabla f(u_n)$$
$$-\frac{\lambda}{1-\lambda}\nabla f(T_iu_n),u_n\rangle-\langle\nabla h(v_n),v_n\rangle\}.$$

Taking either $\Phi=I^E$ or $\Phi=\Psi, d_{1,n}\geq d_{2,n}$, and choosing $i_n\in K_n$ and $t_n\colon=y_{i_n},\Phi=I^E,h=g$, this shows that D_n is closed and convex for all $n\geq 1$. Next, we show that A_n is closed and convex. To see this, observe that

$$A_n\colon\{z\in E\colon D_h(\Phi z,z_{j,n})\leqslant D_h(\Phi z,y_{i,n})\}$$

can be rewritten for each $n\geq 1$ as,

$$\begin{aligned}A_n&=\{z\in E\colon h(\Phi z)-h(z_{j,n})-\langle\nabla f(z_{j,n}),\Phi z-y_{i,n}\rangle\leqslant h(\Phi z)-h(y_{i,n})-\langle h(y_{i,n}),\Phi z-y_{i,n}\rangle\}\\&=\{z\in E\colon -\langle\nabla h(z_{j,n}),\Phi z-z_{j,n}\rangle\leqslant h(z_{j,n})-h(y_{i,n})-\langle\nabla h(y_{i,n}),\Phi z-y_{i,n}\rangle\}\\&=\{z\in E\colon\langle\nabla h(z_{j,n}),z_{j,n}\rangle-\langle\nabla h(z_{j,n}),\Phi z\rangle\leqslant h(z_{j,n})-h(y_{i,n})+\langle\nabla h(y_{i,n}),y_{i,n}\rangle-\langle\nabla h(y_{i,n}),\Phi z\rangle\}\\&=\{z\in\colon\langle\nabla h(y_{i,n}),\Phi z\rangle-\langle\nabla h(z_{j,n}),\Phi z\rangle\leqslant h(z_{j,n})-h(y_{i,n})+\langle\nabla h(y_{i,n}),y_{i,n}\rangle-\langle\nabla h(z_{j,n}),z_{j,n}\rangle\}\\&=\langle\nabla h(y_{i,n})-\nabla h(z_{j,n}),\Phi z\rangle\leqslant h(z_{j,n})-h(y_{i,n})+\langle\nabla h(y_{i,n}),y_{i,n}\rangle-\langle\nabla h(z_{j,n}),z_{j,n}\rangle.\end{aligned}$$

Applying the conditions that either $\Phi=I^E$ or $\Phi=\Psi$ (the bounded linear operator), we obtain that A_n is closed and convex for all $n\geq 1$. Next, we show that B_n is closed and convex.

$$\begin{aligned}B_n\colon&=\{z\in E\colon D_h(\Phi z,y_{i,n})\leqslant D_h(\Phi z,v_n)\}\\&=\{z\in E\colon h(\Phi z)-h(y_{i,n})-\langle\nabla f(y_{i,n}),\Phi z-y_{i,n}\rangle\leqslant h(\Phi z)-h(v_n)-\langle\nabla h(v_n),\Phi z-v_n\rangle\}\\&=\{z\in E\colon\langle\nabla h(y_{i,n}),y_{i,n}\rangle-\langle\nabla h(y_{i,n}),\Phi z\rangle\leqslant h(y_{i,n})-h(v_n)+\langle\nabla h(v_n),v_n\rangle-\langle\nabla h(v_n),\Phi z\rangle\}\\&=\{z\in E\colon\langle\nabla h(v_n),\Phi z\rangle-\langle\nabla h(y_{i,n}),\Phi z\rangle\leqslant h(y_{i,n})-h(v_n)+\langle\nabla h(v_n),v_n\rangle-\langle\nabla h(y_{i,n}),y_{i,n}\rangle\}\\&=\{z\in E\colon\langle\nabla h(v_n)-\nabla h(y_{i,n}),\Phi z\rangle\leqslant h(y_{i,n})-h(v_n)+\langle\nabla h(v_n),v_n\rangle-\langle\nabla h(y_{i,n}),y_{i,n}\rangle\}.\end{aligned}$$

Taking either $\Phi=I^E$ or $\Phi=\Psi$, we obtain that B_n is closed and bounded $\forall n\geq 1$.

Step III: We show that $\Omega\subset C_n\cap Q_n$, for all $n\geq 0$. We prove first that $\Omega\subset C_n$. Let $p\in\Omega$ be arbitrary. Then, $T_i(p)=p$ and $S_j(\Psi p)=\Psi p$ for each $i=1,2,...,N$ and $j=1,2,...,M$ respectively. From the fact that T_i and S_j are quasi-Bregman strictly pseudocontractive mappings for each $i=1,2,...,N$ and $j=1,2,...,M$ and Lemma 2.9(b), we have

$$\begin{aligned}D_h(\Phi p,t_n)&=\begin{cases}D_f(\Phi p,T_{i,n}(v_n)),\text{if } & d_{i,n}\geq d_{2,n},\\ D_g(\Psi p,S_{j,n}(\Psi y_{i,n}),\text{if } & d_{1,n}<d_{2,n}\end{cases}\\&\leq\begin{cases}\frac{\lambda}{1-\lambda}\langle\nabla f(v_n)-\nabla f(T_i(v_i)),v_n-p\rangle,\text{ if } & d_{1,n}\geq d_{2,n}\\ \frac{\lambda}{1-\lambda}\langle\nabla g(\Psi y_{i,n})-\nabla g(S_j\Psi y_{i,n}),\Psi y_{i,n}-\Phi p\rangle,\text{if } & d_{1,n}\leq d_{2,n}.\end{cases}\end{aligned}$$

It follows from this last inequality and by the definition of C_n that $\Omega\subset C_n$, for all $n\geq 0$. Next, we prove that $\Omega\subset Q_n$, for all $n\geq 0$. From (3.1), we know that

$$Q_n\colon=\{z\in E\colon\langle\nabla f(x_0)-\nabla f(x_n),z-x_n\rangle\leqslant 0\}.$$

For $n=0$, we obtain that $Q_0=E$, which further implies that $\Omega\subset Q_0$. Now for some $n\geq 0$, we assume that $\Omega\subset Q_{n-1}$. It then follows that $\Omega\subset C_{n-1}$, for some $n\geq 0$. So that $\Omega\subset C_{n-1}\cap Q_{n-1}$. From the definition of Q_n we get $Q_n=\langle\nabla f(x_0)-\nabla f(x_n),z-x_n\rangle\leqslant 0$, for all $z\in E$. Then

$$\langle\nabla f(x_0)-\nabla f(x_n),z\rangle\leqslant\langle\nabla f(x_0)-\nabla f(x_n),x_n\rangle\leqslant 0.$$

From Lemma 2.3 (iii), we get $\langle \nabla f(x_0) - \nabla f(x_n), y - x_n \rangle \leqslant 0$, for all $y \in C_{n-1} \cap Q_{n-1}$. Following the fact that $p \in \Omega \subset C_{n-1} \cap Q_{n-1}$, we obtain $\langle \nabla f(x_0) - \nabla f(x_n), p - x_n \rangle \leqslant 0$. Then $p \in Q_n$. From mathematical induction, we have that $\Omega \subset Q_n$, for all $n \geq 0$. Hence, $p \in \Omega$ and so $p \in C_n \cap Q_n$ and $C_n \cap Q_n$ is nonempty, closed and convex.

Step IV: We prove that $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_{i,n}\}$ and $\{z_{j,n}\}$ are bunded sequences. We show that $\{x_n\}$ is bounded. Following the definition of Q_n and from Lemma 2.3(i), we know that $x_n = proj^f_{Q_n}(x_0)$. Also, from Lemma 2.3(ii) for each $p \in \Omega$, we have

$$D_f(x_n, x_0) = D_f(proj^f_{Q_n}(x_0), x_0) \leqslant D_f(p, x_0) - D_f(p, proj^f_{Q_n}(x_0)) \leqslant D_f(p, x_0).$$

This shows that $\{D_f(x_n, x_0)\}$ is bounded for all $n \geq 0$. Since $\{D_f(x_n, x_0)\}$ is bounded, from Lemma 2.7, it follows that $\{x_n\}$ is bounded too. Again, applying Lemma 2.3 (ii), for $m \geq n$,

$$D_f(x_m, x_n) = D_f(x_m, prof^f_{Q_n}(x_0)) \leq D_f(x_m, x_0) - D_f(x_0, x_n) \leq D_f(x_m, x_0) - D_f(p, x_0).$$

Since f is totally convex, sequentially consistent (see [9]) and $\{x_n\}$ is bounded too, it follows that

$$\lim_{n\to\infty} D_f(x_m, x_n) = 0.$$

From Lemma 2.5, we obtain that since $\lim_{n\to\infty} D_f(x_m, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|x_m - x_n\| = 0$. Since $\{x_n\}$ is a Cauchy sequence, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| \to 0. \tag{3.2}$$

From equation (3.1), we know that $u_n = x_n + \alpha_n(x_n - x_{n-1})$. So,

$$\begin{aligned} \|x_n - u_n\| &= \|x_n - (x_n + \alpha_n(x_n - x_{n-1}))\| = \|x_n - x_n - \alpha_n(x_n - x_{n-1})\| \\ &= \alpha_n \|x_n - x_{n-1}\| \to 0, \end{aligned}$$

since $\alpha_n \to 0$ and $\{x_n\}$ is bounded. This shows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.3}$$

Consequently, $\{u_n\}$ is bounded as required. Furthermore, we have

$$\|x_{n+1} - u_n\| = \|x_{n+1} - x_n + x_n - u_n\| \leqslant \|x_{n+1} - x_n\| + \|x_n - u_n\|.$$

Combining (3.2) and (3.3), we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

From Lemma 2.5, we get $\lim_{n\to\infty} D_f(x_{n+1} - u_n) = 0$. Next, we prove that $\{v_n\}$ is bounded. From (3.1), we know that

$$v_n = \nabla f^*(\beta \nabla f(u_n) + (1-\beta)\nabla f(T_i u_n)).$$

Then from Lemma 2.8 and for all $p \in \Omega$, we get

$$\begin{aligned} D_f(p, v_n) &= D_f(p, \nabla f^*(\beta \nabla f(u_n) + (1-\beta)\nabla f(T_i u_n)) \\ &= V_f(p, \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(T_i u_n)) \\ &= f(p) - \langle p, \beta \nabla f(u_n) + (1-\beta)\nabla f(T_i u_n)\rangle + f^*(\beta \nabla f(u_n) + (1-\beta)\nabla f(T_i u_n)) \\ &\leqslant \beta f(p) + (1-\beta) f(p) - \beta\langle p, \nabla f(u_n)\rangle - (1-\beta)\langle p, \nabla f(T_i u_n)\rangle + \beta f^*(\nabla f(u_n)) + (1-\beta) f^*(\nabla f(T_i n_n)) \\ &= \beta V_f(p, \nabla f(u_n)) + (1-\beta) V_f(p, \nabla f(T_i u_n)) \\ &= \beta D_f(p, u_n) + (1-\beta) D_f(q, T_i u_n) \\ &\leqslant \beta D_f(p, u_n) + (1-\beta)(D_f(p, u_n) + \lambda D_f(u_n, T_i u_n)) \\ &\leqslant D_f(p, u_n) + \lambda D_f(u_n, T_i u_n) \\ &\leqslant D_f(p, u_n) + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - p\rangle. \qquad (3.4) \end{aligned}$$

So, $D_f(p,v_n) \leqslant D_f(p,u_n) + \dfrac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - p\rangle$. Following the boundedness of $\{u_n\}$, we have from (3.4) that $\lim\limits_{n\to\infty} D_f(p,v_n) = 0$. which further implies from Lemma (2.4)(b) that

$$\lim_{n\to\infty} \|p - v_n\| = 0. \tag{3.5}$$

From Lemma 2.5, we have that $\{D_f(p,v_n)\}$ is bounded. Using Lemma 2.7 we have that $\{v_n\}$ is bounded. Next we show that $\{y_{i,n}\}$ and $\{z_{j,n}\}$ are bounded sequences. Since $x_n = prof^f_{Q_n}(x_0), x_{n+1} \in Q_n$ we obtain from Lemma 2.3 (ii) that,

$$D_f(x_{n+1}, proj^f_{Q_n}(x_0)) + D_f(proj^f_{Q_n}(x_n), x_0) \leqslant D_f(x_{n+1}, x_0).$$

This implies that $D_f(x_{n+1},x_n)+D_f(x_n,x_0) \leqslant D_f(x_{n+1},x_0)$. Since $D_f(x_{n+1},x_n)$ is non-negative and $D_f(x_{n+1},x_n) \geq D_f(x_n,x_0)$, for all $n \geq 1$, and from the fact that $\{D_f(x_n,x_0)\}$ is bounded, we conclude that $\lim\limits_{n\to\infty}\{D_f(x_n,x_0)\}$ exists. Again from Lemma (2.6), we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

It follows from the definition of $\Phi = I^E$ that

$$\lim_{n\to\infty} \|\Phi x_{n+1} - \Phi x_n\| = 0. \tag{3.6}$$

We know from Lemma 2.2 that h is uniformly continuous on E. It follows from (3.6) that

$$\lim_{n\to\infty} \|h(\Phi x_{n+1}) - h(\Phi x_n)\| = 0. \tag{3.7}$$

From the assumptions on f and g and their convexity property, we get that ∇f and ∇g are bounded on bounded subsets of E and F respectively. Thus, there exists a number $K \geq 0$ such that

$$\|\nabla h(x_n)\| \leqslant K, \forall n \geq 0.$$

Also from the boundedness of $\{\Phi x_n\}$, f, and g, there exists a positive number K' such that

$$\|h(\Phi x_n)\| \leqslant K', \forall n \geq 0.$$

But

$$D_h(\Phi p, \Phi x_n) = h(\Phi p) - h(\Phi x_n) - \langle \nabla h(\Phi x_n), \Phi p - \Phi x_n\rangle. \tag{3.8}$$

We know also that $-h(\Phi x_n) \leqslant h(\Phi x_n) \leqslant \|h(\Phi x_n)\|$. It follows from this fact and (3.8) that

$$\begin{aligned} D_h(\Phi p, \Phi x_n) &\leqslant \|h(\Phi p)\| + K' + \|\nabla h(\Phi x_n)\|\|\Phi p - \Phi x_n\| \\ &\leqslant \|h(\Phi p)\| + K' + \|\nabla h(\Phi x_n)\|(\|\Phi p\| + \|\Phi x_n\|) \\ &\leqslant \|h(\Phi p\| + K' + K(L + \|h(\Phi p)\|. \end{aligned}$$

Let $J\colon = \|h(\Phi p\| + K' + K(L + \|h(\Phi p)\|)$ with $L\colon = \sup_n\{\|h(\Phi x_n)\} < \infty$. So, $D_h(\Phi p, \Phi x_n) \leqslant J$. Again from (3.8) and Lemma (2.8), we get

$$h(\Phi p) - \langle \nabla h(\Phi x_n), \Phi p\rangle + h^* \nabla h(\Phi x_n) = V_h(\Phi p, \nabla h(\Phi x_n)) = D_h(\Phi p, \Phi x_n) \leqslant J.$$

We observe that $\{\nabla h(\Phi x_n)\}$ is contained in the level set $lev^{\omega}_{\leqslant}(J - h(\Phi p))$, where $\omega = h^* - \langle ., \Phi p\rangle$. Since h is lower semicontinuous, then h^* is. Since ω is coercive from ([30], Theorem 7A), it follows that $\{\nabla h(\Phi x_n)\}$ is bounded. Using (3.6) and (3.7), we get

$$D_h(\Phi x_{n+1}, \Phi x_n) = h(\Phi x_{n+1}) - h(\Phi x_n) - \langle \nabla h(\Phi x_n), \Phi x_{n+1} - \Phi x_n\rangle \to 0.$$

That is, $D_h(\Phi x_{n+1}, \Phi x_n) \to 0$. From the definition of C_n, and $x_{n+1} \in C_n$, we see that

$$D_h(\Phi x_{n+1}, t_n) \leqslant D_f(\Phi x_{n+1}, \Phi x_n) \to 0,$$

which further implies that $\lim_{n\to\infty} D_f(\Phi x_{n+1}, t_n) = 0$. We conclude from Lemma (2.5) and Lemma (2.6) that

$$\lim_{n\to\infty} \|\Phi x_{n+1} - t_n\| = 0. \tag{3.9}$$

Following (3.6) and (3.9), we get that

$$\|\Phi x_n - t_n\| = \|\Phi x_n - \Phi x_{n+1} + \Phi x_{n+1} - t_n\| \leqslant \|\Phi x_n - \Phi x_{n+1}\| + \|\Phi x_n - \Phi x_{n+1}\|,$$

which implies that

$$\lim_{n\to\infty} \|\Phi x_n - t_n\| = 0. \tag{3.10}$$

Now if $d_{1,n} \geq d_{2,n}$, choose $i_n \in K_n$ and let $t_n\colon = y_{i_n}$, $\Phi = I^E$ and $h = f$, then, for all $i = 1, 2, ..., N$, we get

$$\lim_{n\to\infty} \|\Phi x_n - t_n\| = \lim_{n\to\infty} \| x_n - y_{i,n}\| = \lim_{n\to\infty} \| x_n - T_i(v_n)\| = 0. \tag{3.11}$$

Again following the same argument as above, if $d_{1,n} < d_{2,n}$, then choose $j_n \in L_n$ and let $t_n\colon = z_{j_n}, \Phi = \Psi, h = g$. Then for all $j = 1, 2, ..., M$, we obtain

$$\lim_{n\to\infty} \|\Phi x_n - t_n\| = \lim_{n\to\infty} \| x_n - z_{i,n}\| = \lim_{n\to\infty} \| x_n - S_j(\Psi y_{i,n})\| = 0. \tag{3.12}$$

Therefore, $\{y_{i,n}\}$ and $\{z_{j,n}\}$ are bounded sequences.

Step V: We show that every weak accumulation point of $\{x_n\}$ belongs to Ω. Let q be an arbitrary weak cluster point of Ω. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ as $k \to \infty$. From the fact that Ψ is a linear and bounded map, we know that Ψ is continuous. Hence, $\Psi x_{n_k} \rightharpoonup \Psi q$. Therefore, it follows from (3.11) and (3.12) that $q \in F(T_i)$ and $\Psi q \in F(S_j)$ for all $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$. Therefore, $q \in \Omega = \cap_{i=1}^{N} F(T_i) \cap \Psi^{-\prime}(\cap_{j=1}^{M} F(S_j))$ as required.

Step VI: We show that $x_n \to x^* = proj_{\Omega}^{f}(x_0)$ as $n \to \infty$. Suppose $x^* = proj_{\Omega}^{f}(x_0)$. Since $x_{n+1} = proj_{C_n \cap Q_n}^{f}(x_0)$ and $\Omega \subset C_n \cap Q_n$, we get

$$\begin{aligned} D_f(proj_{\Omega}^{f}(x_0), x_0) =& D_f(x_{n+1}, x_0) \leqslant D_f(x^*, x_0) - D_f(x^*, x_{n+1}) \\ \leqslant& D_f(x^*, x_0), \end{aligned}$$

which implies that

$$D_f(x_{n+1}, x_0) \leqslant D_f(x^*, x_0). \tag{3.13}$$

But by the three points identity, we know that

$$D_f(x_n, x_0) + D_f(x_0, x^*) - D_f(x_n, x^*) = \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle.$$

Then

$$D_f(x_n, x^*) = D_f(x_n, x_0) + D_f(x_0, x^*) - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle. \tag{3.14}$$

It follows from (3.13) and (3.14), that
$D_f(x_n, x^*) \leqslant D_f(x^*, x_0) + D_f(x_0, x^*) - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle$
$= -\langle \nabla f(x_0), x^* - x_0\rangle - \langle \nabla f(x^*), x_0 - x^*\rangle - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle$
$= \langle -\nabla f(x_0), x_0 - x^*\rangle - \langle \nabla f(x^*), x_0 - x^*\rangle - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle$
$= -\langle \nabla f(x^*) - f(x_0), x_0 - x^*\rangle - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0\rangle$
$= -\langle \nabla f(x^*) - \nabla f(x_0), x_0\rangle + \langle \nabla f(x^*) - \nabla f(x_0), x^*\rangle - \langle \nabla f(x^*) - \nabla f(x_0), x_n\rangle + \langle \nabla f(x^*) - \nabla f(x_0), x_0\rangle$
$= \langle \nabla f(x^*) - \nabla f(x_0), x^*\rangle - \langle \nabla f(x^*) - \nabla f(x_0), x_n\rangle$
$= \langle \nabla f(x^*) - \nabla f(x_0), x^* - x_n\rangle$
which implies that

$$D_f(x_n, x^*) \leqslant \langle \nabla f(x^*) - \nabla f(x_0), x^* - x_n\rangle. \tag{3.15}$$

Because $\{x_n\}$ is a bounded sequence in E and E is a real reflexive Banach space, we know from this fact that $\{x_n\}$ has a subsequence that converges weakly in E. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ that converge weakly to some point $x^\dagger \in E$. It follows from the same argument in step (IV) that $x^\dagger \in \Omega$. From Lemma 2.3(ii), (3.15) and the fact that x^* is a subsequential limit of $\{x_{n_k}\}$, we have

$$\begin{aligned}\liminf_{k\to\infty} D_f(x_{n_k}, x^*) &\leqslant \limsup_{k\to\infty} D_f(x_{n_k}, x^*)\\ &\leqslant \limsup_{k\to\infty}\langle \nabla f(x^*) - \nabla f(x_0), x^\dagger - x_{n_k}\rangle\\ &\langle \nabla f(x^*) - \nabla f(x_0), x^* - x^*\rangle = 0.\end{aligned}$$

Hence,

$$\lim_{k\to\infty} D_f(x_{n_k}, x^*) = 0. \tag{3.16}$$

□

Theorem 3.2. Let E and F be two real reflexive Banach spaces and $f\colon E \to \mathbb{R}$ and $g\colon F \to \mathbb{R}$ be two Legendre functions which are bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E and F respectively. Let $T_i\colon E \to E, i = \overline{1, N}$ and $S_j\colon F \to F, j = \overline{1, M}$ be finite families of λ_i and β_j closed quasi-Bregman strictly pseudocontractive mappings respectively with λ_i and β_j in (0,1). Let $\Psi\colon E \to F$ be a bounded linear operator. Suppose that $\Omega\colon = \bigcap_{i=1}^N F(T_i) \bigcap \Psi^{-\prime}(\bigcap_{j=1}^M F(S_j)) \neq \emptyset$. Assume $(I - T_i), i = 1, 2, ..., N$ is demiclosed at the origin. For $\lambda = \min_{1\leqslant i\leqslant N}\{\lambda_i\}, \beta = \min_{1\leqslant j\leqslant N}\{\beta_j\}$, let $\{x_n\}$ be the sequence generated by the following algorithm;

$$\begin{cases} x_0, x_{-1} \in E, C_0 = E,\\ u_n = x_n + \alpha_n(x_n - x_{n-1}),\\ v_n = \nabla f^*(\sigma\nabla f(u_n) + (1-\sigma)\nabla f(T_i n_n)),\\ y_{i,n} = T_i(v_n), i = 1, 2, ..., N,\\ z_{j,n} = S_j(\Psi y_{i,n}), j = 1, 2, 3, ..., M,\\ \text{let } \ d_{1,n} := \max_{1\leqslant i\leqslant N}\{D_f(y_{i,n}, u_n)\}, d_{2,n} : \max_{1\leqslant j\leqslant M}\{D_f(z_{j,n}, \Psi u_n)\},\\ K_n\colon = \{i \in \{1, 2, ..., N\}\colon D_f(y_{i,n}, u_n) = d_{1,n}\},\\ L_n := \{j \in \{1, 2, ..., M\} : D_g(z_{i,n}, \Psi u_n) = d_{2,n}\},\\ \text{if } \ d_{1,n} \geq d_{2,n}, \text{choose } \ i_n \in K_n\\ \text{and let } t_n := y_{i_n}, \Phi = I^E, h = f,\\ \text{if } d_{1,n} < d_{2,n}, \text{choose } j_n \in L_n\\ \text{and let } t_n := z_{j_n}, \Phi = \Theta, h = g,\\ C_{n+1} := \{z \in E : D_h(\Phi z, z_{j,n}) \leqslant D_f(\Phi z, y_{j,n}) \leqslant D_f(\Phi z, v_n) \leqslant D_f(\Phi z, u_n) +\\ \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - \Phi z\rangle\},\\ x_{n+1} = proj^f_{C_{n+1}}(x_0), n \geq 0,\end{cases} \tag{3.17}$$

where σ is fixed in (0,1) and $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\lim x_n = 0$. Then $\{x_n\}$ converges strongly to $proj^f_\Omega(x_0)$, where $proj^f_\Omega(x_0)$ is the Bregman projection of x_0 onto Ω.

Proof . We divide the proof into five steps.

Step 1: We first establish that C_n is a closed and convex subset of E. Because $C_0 = E$, then C_0 is closed and convex. Assume that C_n is a closed and convex subset of E for some $n \geq 0$. Let

$$C_{n+1} = C_n \cap \left\{ z \in E\colon D_h(\Phi z, z_{j,n}) - D_h(\Phi z, y_{i,n}) \leqslant D_h(\Phi z, u_n) - D_h(\Phi z, v_n) + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - \Phi z\rangle \right\} \tag{3.18}$$

To see that C_{n+1} is closed and convex for all $n \geq 0$, we can rewrite C_{n+1} in the form below;

(3.19)

$$\begin{aligned}
C_{n+1} =& C_n \cap \Big\{ z \in E\colon D_h(\Phi z, z_{j,n}) - D_h(\Phi z, y_{i,n}) \leqslant D_h(\Phi z, u_n) - D_h(\Phi z, v_n) + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - \Phi z\rangle \Big\}\\
=& C_n \cap \{ z \in E : h(\Phi z) - h(z_{j,n}) - \langle \nabla h(z_{j,n}), \Phi z - z_{j,n}\rangle - (h(\Phi z) - h(y_{i,n}) - \langle \nabla h(y_{i,n}), \Phi z - y_{i,n}\rangle) \leqslant h(\Phi z)\\
& -h(u_n) - \langle \nabla h(u_n), \Phi z - u_n\rangle - (h(\Phi z) - h(v_n) - \langle \nabla h(v_n), \Phi z - v_n\rangle) + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - \Phi z\rangle \Big\}\\
=& C_n \cap \{ z \in E\colon \langle \nabla h(y_{i,n}), \Phi z - y_{i,n}\rangle - \langle \nabla h(z_{j,n}), \Phi z - z_{j,n}\rangle + h(y_{i,n}) - h(z_{j,n}) \leqslant h(v_n) - h(u_n)\\
& + \langle \nabla h(v_n), \Phi z - v_n\rangle - \langle \nabla h(u_n), \Phi z - u_n\rangle + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n) - \nabla f(T_i u_n), u_n - \Phi z\rangle \Big\}\\
=& C_n \cap \Big\{ z \in E\colon \langle \nabla h(y_{i,n}), \Phi z\rangle - \langle \nabla h(y_{i,n}), y_{i,n}\rangle - \langle \nabla h(z_{j,n}), \Phi z\rangle + \langle \nabla h(z_{j,n}), z_{j,n}\rangle + h(y_{i,n}) - h(z_{j,n}) \leqslant h(v_n)\\
& - h(u_n) + \langle \nabla h(v_n), \Phi z\rangle - \langle \nabla h(v_n), v_n\rangle - \langle \nabla h(u_n), \Phi z\rangle + \langle \nabla h(u_n), u_n\rangle + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n), u_n\rangle\\
& - \frac{\lambda}{1-\lambda}\langle \nabla f(u_n), \Phi z\rangle - \frac{\lambda}{1-\lambda}\langle \nabla f(T_i u_n), u_n\rangle + \frac{\lambda}{1-\lambda}\langle \nabla f(T_i u_n), \Phi z\rangle \Big\}\\
=& C_n \cap \Big\{ z \in E\colon \langle \nabla h(y_{i,n}), \Phi z\rangle - \langle \nabla h(z_{j,n}), \Phi z\rangle + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n), \Phi z\rangle + \langle \nabla h(v_n), \Phi z\rangle - \frac{\lambda}{1-\lambda}\langle \nabla f(T_i u_n), \Phi z\rangle\\
& - \langle \nabla h(u_n), \Phi z\rangle \leqslant h(z_{j,n}) - h(y_{i,n}) + h(v_n) - h(u_n) - \langle \nabla h(z_{j,n}), z_{j,n}\rangle - \langle \nabla h(v_n), v_n\rangle + \langle \nabla h(u_n), u_n\rangle\\
& + \frac{\lambda}{1-\lambda}\langle \nabla f(u_n), u_n\rangle - \frac{\lambda}{1-\lambda}\langle \nabla f(T_i u_n), u_n\rangle \Big\}\\
=& C_n \cap \Big\{ z \in E\colon \langle \nabla h(y_{i,n}) - \nabla h(z_{j,n}) + \frac{\lambda}{1-\lambda}\nabla f(u_n) + \nabla h(v_n) - \frac{\lambda}{1-\lambda}\nabla f(T_i u_n) - \nabla h(u_n), \Phi z\rangle \leqslant h(z_{j,n})\\
& - h(y_{i,n}) + h(v_n) - h(u_n) - \langle \nabla h(z_{j,n}), z_{j,n}\rangle - \langle \nabla h(v_n), v_n\rangle + \langle \nabla h(u_n) + \frac{\lambda}{1-\lambda}\nabla f(u_n) - \frac{\lambda}{1-\lambda}\nabla f(T_i u_n), u_n\rangle \Big\}
\end{aligned}$$

Taking either $\Phi = I^E$ or $\Phi = \Psi$, we see that C_{n+1} is closed and convex.

Step II: We prove that $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_{i,n}\}$ and $\{z_{j,n}\}$ are bunded sequences. The boundedness of these sequences is obtained from the same argument as in the Step IV of Theorem 3.1 above.

Step III: We show that $\{x_n\}$ converges to some point $p \in \Omega$. Since $C_{n+1} \subset C_n, x_n = proj^f_{C_n}(x_0)$ and from Lemma 2.3 (ii), we have

$$D_f((x_{n+1}), proj^f_{C_n}(x_0)) + D_f(proj^f_{C_n}(x_0), x_0) \leqslant D_f(x_{n+1}, x_0)$$

that is,

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \leqslant D_f(x_{n+1}, x_0) \tag{3.20}$$

which implies that

$$D_f(x_n, x_0) \leqslant D_f(x_{n+1}, x_0).$$

Thus, we get $\{D_f(x_n, x_0)\}$ is non-decreasing. Hence,

$$D_f(x_n, x_0) \leqslant D_f(p, x_0) - D_f(p, x_n), \forall p \in \Omega, \forall n \geq 0,$$

which shows that $\{D_f(x_n, x_0)\}$ is bounded and consequently, from Lemma 2.7, we obtain that $\{x_n\}$ is bounded too. Therefore, $\lim_{n\to\infty}\{D_f(x_n, x_0)\}$ exists. For $m \geq n$, it follows from the definition that $C_m \subset C_n$. Hence, $x_m \in C_n$. From Lemma 2.3 and the fact that $x_n = proj^f_{C_n}(x_0)$, we obtain that

$$D_f(x_m, x_n) = D_f(x_m, proj^f_{C_n}(x_0)) \leqslant D_f(x_m, x_0) - D_f(x_n, x_0) \to 0$$

which implies that $D_f(x_m, x_n) \to 0$. We conclude from Lemmas 2.5 and 2.6 that

$$\lim_{n\to\infty} \|x_m - x_n\| = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence and it converges strongly to some point $p \in \Omega$.

Next, we show that $x_n \to p \in \Omega$ as $n \to \infty$. But, $\{x_n\}$ is a Cauchy sequence. It follows that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Following conclusions (3.10), (3.11) and (3.12) we get

$$\lim_{n\to\infty} \|\Phi x_n - t_n\| = 0. \tag{3.21}$$

From (3.21) we conclude that

$$\lim_{n\to\infty} \|\Phi x_n - T_i(x_n)\| = 0 \tag{3.22}$$

and

$$\lim_{n\to\infty} \|\Psi x_n - S_j(\Psi x_n)\| = 0, \tag{3.23}$$

for any $i = 1, 2, ..., N$ and $j = 1, 2, \ldots, M$. Since Ψ is a linear and bounded operator, it follows that $\Psi x_n \to \Psi p$. From (3.22) and (3.23), we have $p \in F(T_i)$ and $\Psi p \in F(S_j)$ for each $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$. That is, $p \in \Omega = \cap_{i=1}^{N} F(T_i) \cap \Psi^{-\prime}(\cap_{j=1}^{M} F(S_j))$ as required.

Step IV: We prove that $p = x^* = proj_{\Omega}^{f}(x_0)$. Let $x^* = proj_{\Omega}^{f}(x_0)$. Since $x_n = proj_{\Omega}^{f}(x_0)$ and $\Omega \subset C_n$, we obtain

$$D_f(proj_{\Omega}^{f}(x_0), x_0) = D_f(x_n, x_0) \leqslant D_f(x^*, x_0) - D_f(x^*), x_n \leqslant D_f(x^*, x_0),$$

which implies that

$$D_f(x_n, x_0) \leqslant D_f(x^*, x_0). \tag{3.24}$$

Using the three points identity, we have

$$D_f(x_n, x^*) = D_f(x^*, x_0) + D_f(x_0, x^\dagger) - \langle \nabla f(x^*) - \nabla f(x_0), x_n - x_0 \rangle$$

which we have from (3.15) that

$$D_f(x_n, x^*) \leqslant \langle \nabla f(x^*) - \nabla f(x_0), x^* - x_n \rangle. \tag{3.25}$$

We combine the facts that $p \in \Omega$, Lemma 2.3 (iii) and (3.24), to get

$$\lim_{n\to\infty} \sup D_f(x_n, x^*) \leqslant \lim_{n\to\infty} \sup \langle \nabla f(x^*) - \nabla f(x_0), x^* - x_n \rangle.$$

So,

$$\lim_{n\to\infty} D_f(x_n, x^*) \leqslant \langle \nabla f(x^*) - \nabla f(x_0), x^* - p \rangle \leqslant 0.$$

That is, $\lim_{n\to\infty} D_f(x_n, x^*) = 0$. From Lemma 2.4, we get that $\lim_{n\to\infty} D_f(x_n, x^*) = 0$. Then $\lim_{n\to\infty} \|x_n - x^*\| = 0$. That is, $\{x_n\}$ converges strongly to x^* as $n \to \infty$. Hence, $x^* = p$. This completes the proof.

Consequently from Lemma 2.4(b), it follows from (3.16) that

$$\lim_{k\to\infty} \|x_{n_k} - x^*\| = 0. \tag{3.26}$$

That is, $\{x_{n_k}\}$ converges strongly to x^* as $k \to \infty$. But $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$. It follows that $\{x_n\}$ converges strongly to x^* as $n \to \infty$. This completes the proof. □

4 Conclusion

We studied two independent algorithms, the hybrid and the shrinking projection algorithms for solving the split common fixed point problem for finite families of Bregman quasi-strictly pseudocontractive mappings in reflexive spaces. The Bregman distance techniques were employed in both methods. Strong convergence of the iterative sequences were otanined.

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