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Nonlinear implicit Caputo-Fabrizio fractional hybrid differential equations

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Abstract

The present paper is mainly concerned with the existence, uniqueness, estimates on solutions and discuss the continuous dependence on the initial data of the nonlinear implicit Caputo-Fabrizio fractional hybrid differential equations with an initial condition. The results are obtained by using fractional calculus, contraction principle theorem and the Gronwall's inequality to show the estimate of the solutions. An example is given to illustrate the effectiveness of our main results.

Keywords: Implicit hybrid fractional differential equations, Caputo-Fabrizio fractional derivative, fixed point theorems, Gronwall's inequality, continuous dependence
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1 Introduction

Fractional calculus (integrals and derivatives) are the generalizations of integer-order differential and integral operators. It is well known that the motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. For details, see ([3], [8], [11], [16], [18], [21]-[24]).

There are different definitions of fractional derivatives. The popular derivatives of fractional order we mention Riemann-Liouville, Grunwald-Letnikov, Hadamard, Caputo and Hilfer. For example, the Caputo fractional derivative of order $0 < \alpha < 1$ of a function $g \in L^1[0,T]$ $(T > 0)$ is given by

$$^{C}D_0^{\alpha} g(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{-\alpha} g'(s) ds.$$

In 2015, Caputo and Fabrizio developed and proposed a new version of fractional derivative by changing the Kernel $(t-s)^{-\alpha}$ by the function $(t,s) \longmapsto \exp\left(\frac{-\alpha(t-s)}{\Gamma(1-\alpha)}\right)$ and $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{M(\alpha)}{1-\alpha}$.

In this work we will consider that the normalized function M is given by $M(\alpha) = 1$, for every $\alpha \in [0,1]$. For more details, see [20]. There are many works concerning with the applications and some fractional differential equations of Caputo-Fabrizio derivative. For details, see ([1]-[3], [8]-[10], [21], [24], [26], [28]).

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Fractional differential hybrid equations involving Riemann-Liouville, Caputo, Hadamard and Hilfer type fractional derivatives have extensively been studied by several researchers, see in [[4], [5], [6], [7], [17], [19], [25], [27]]. However, the literature on Caputo-Fabrizio type fractional differential hybrid equations is not enriched yet.

Recently, Ardjouni, Lachouri and Djoudi [7] studied the existence, interval of existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations with the nonlocal conditions:

$$\begin{cases} {}^{C}D_{1}^{\alpha}\Big(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\Big) = h\Bigg(t, x(t), {}^{C}D_{1}^{\alpha}\Big(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\Big)\Bigg), \qquad t\in[1,T] \\ \\ x(1) = \theta g(1,x(1)) + f(1,x(1)), \quad \theta\in\mathbb{R}, \end{cases}$$

where $f : [1,T]\times\mathbb{R} \longrightarrow \mathbb{R}$, $g : [1,T]\times\mathbb{R} \longrightarrow \mathbb{R}/\{0\}$ and $h : [1,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are nonlinear continuous functions and ${}^{C}D_{1}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order $0<\alpha<1$.

In [13], Eiman, K. Shah, M. Sarwar and D. Baleanu established existence theory of solutions to a class of implicit fractional differential equations involving Caputo-Fabrizio derivative:

$$\begin{cases} {}^{CF}_{0}D_{x}^{\theta}u(x) = f\Big(t, x(t), {}^{CF}_{0}D_{x}^{\theta}u(x)\Big), \qquad t\in\mathcal{J}=[0,T] \\ \\ u(0) = u_0, \quad u_0\in\mathbb{R}, \end{cases}$$

where ${}^{CF}_{0}D_{x}^{\theta}$ is the Caputo-Fabrizio fractional derivative of order $0<\theta\leq 1$, $f : \mathcal{J}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$.

In [15], R. Gul, M. Sarwar, K. Shah and T. Abdeljawad concerned with the existence and uniqueness of solutions of implicit fractional differential equations involving a Caputo-Fabrizio fractional derivative under the following Dirichlet boundary conditions:

$$\begin{cases} {}^{CF}_{c}D_{\omega}^{\mu}z(\omega) = F\Big(\omega, z(\omega), {}^{CF}_{c}D_{\omega}^{\mu}z(\omega)\Big), \qquad 1<\mu\leq 2,\ \omega\in[c,d] \\ \\ z(c)=0,\ z(d)=0 \ and\ c,\ d\in\mathbb{R}, \end{cases}$$

where ${}^{CF}_{c}D_{\omega}^{\mu}$ is used for CFFD and $I=[c,d]$, $f : I\times\mathbb{R}\times\mathbb{R}$ is a continuous function.

In this paper, we are interested in the existence, uniqueness and continuous dependence of solutions for a nonlinear implicit Caputo-Fabrizio fractional hybrid differential equation:

$$\begin{cases} {}^{CF}D^{\theta}[x(t)-f(t,x(t))] = g\Big(t, x(t), {}^{CF}D^{\theta}[x(t)-f(t,x(t))]\Big), \qquad t\in J=[0,T] \\ \\ x(0)-f(0,x(0)) = \eta, \quad \eta\in\mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^{CF}D^{\theta}$ is the Caputo-Fabrizio fractional derivative of order $0<\alpha\leq 1$, $g : J\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $f : J\times\mathbb{R} \longrightarrow \mathbb{R}$ are given functions and satisfy some assumptions.

To show the existence, uniqueness of solutions of (1.1) we transform (1.1) into an integral equation and then use the contraction mapping principle. Further, we discuss the continuous dependence on the data and uniqueness of solution for (1.1). The rest of this paper is organized as follows. In Section 2, we introduce all the background material used in this paper such as definition of Caputo-Fabrizio derivatives of fractional order and some properties of generalized Banach spaces and fixed point theory. In Section 3, we will prove the existence and uniqueness of solutions for (1.1). In section 4, we obtain the estimate of solutions of (1.1) by the Gronwall's inequality. In Section 5, we discuss the continuous dependence on the data and uniqueness of solution for (1.1). Finally, an example is given in Section 6 to illustrate the usefulness of our main results.

2 Preliminaries

In this section we present some basic definitions, notations and preliminaries. Basics of new fractional Caputo-Fabrizio derivative that are needed in this work. Let $J=[0,T]$ be an interval in $\mathbb{R}$, where $T>0$. We denote by $C(J,\mathbb{R})$ the space of continuous functions $x : J \longrightarrow \mathbb{R}$. The space $C(J,\mathbb{R})$ is a Banach space with the supremum norm $\|\cdot\|$ defined by

$$\|x\|_{\infty} = \sup\{|x(t)| : \quad t\in J\}.$$

By $L^1(J,\mathbb{R})$ we denote the Banach space of measurable functions $u : J \longrightarrow \mathbb{R}$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_0^T |x(t)|dt.$$

Definition 2.1. (See [20]) The Caputo-Fabrizio fractional derivative for a function $h \in C^1(J)$ of order $0 < \alpha < 1$, is defined by

$$^{CF}D^{\theta}h(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(\frac{-\alpha}{1-\alpha}(t-x))h'(t)dt, \ t \in J.$$

Note that $^{CF}D^{\theta}h(t) = 0$ if and only if h is a constant function.

Definition 2.2. (See [20]) The Caputo-Fabrizio fractional integral of order $0 < \alpha < 1$ for a function $h \in L^1(J)$ is defined by

$$^{CF}D^{\theta}h(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t h(x)dx, \ t \geq 0,$$

where $M(\alpha)$ is the normalization constant depending on α.

Proposition 2.3. (See [20]) Let $0 < \alpha < 1$. Then the unique solution of the following initial value problem

$$\begin{cases} {}^{C}D^{\alpha}f(t) = \sigma(t), & t \in [1,T] \\ f(0) = f_0 \in \mathbb{R}, \end{cases}$$

is given by

$$f = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha I^1\sigma(t), \ t \geq 0,$$

where $I^1\sigma(t)$ denotes a primitive of $\sigma(t)$, $a_\alpha = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$ and $b_\alpha = \dfrac{2\alpha}{(2-\alpha)M(\alpha)}$.

Theorem 2.4. (Gronwall's inequality Theorem [12]). Let x, Ψ and χ be real continuous functions defined in $[a,b]$, $\chi(t) \geq 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)x(s)ds.$$

Then

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)\Psi(s)\exp[\chi(u)du]ds,$$

in $[a,b]$.

Corollary 2.5. (Corollary 3, [12]) If Ψ is constant, then from the fact that

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)x(s)ds,$$

it follows that

$$x(t) \leq \Psi(t)\exp(\int_a^t \chi(u)du).$$

Theorem 2.6. (Banach's fixed point Theorem [14]). Let Ω be a non-empty closed convex subset of a Banach space $(S, \|\cdot\|)$, then any contraction mapping P of Ω into itself has a unique fixed point.

3 Existence and Uniqueness Results

Let us define what we mean by a solution of problem (1.1).

Definition 3.1. A function $x \in C(J, \mathbb{R})$ is said to be a solution of the problem (1.1) if x satisfies the equation ${}^{CF}D^{\alpha}[x(t) - f(t, x(t))] = f\Big(t, x(t), {}^{CF}D^{\alpha}[x(t) - f(t, x(t))]\Big)$ and the condition $x(0) - f(0, x(0)) = \eta$ on J.

To prove the existence of solutions to the problem (1.1), we need the following auxiliary lemma.

Lemma 3.2. Let $0 < \alpha \leq 1$. If $h \in L^1(J, \mathbb{R})$ and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, then x is solution of the (FHDE):

$$\begin{cases} {}^{CF}D^{\alpha}[x(t) - f(t, x(t))] = h(t), \qquad t \in J = [0, T] \\ x(0) - f(0, x(0)) = \eta, \end{cases} \tag{3.1}$$

if and only if x satisfies the following hybrid integral equation (HIE):

$$x(t) = C + f(t, x(t)) + a_{\alpha} h(t) + b_{\alpha} \int_0^t h(s) ds, \ t \in J, \tag{3.2}$$

where $a_{\alpha} = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$, $b_{\alpha} = \dfrac{2\alpha}{(2-\alpha)M(\alpha)}$ and $C = \eta - a_{\alpha} h(0)$.

Proof . Let x be a solution of the problem (FHDE).From Proposition 2.3, the equation ${}^{CF}D^{\alpha}[x(t) - f(t, x(t))] = h(t)$ implies that

$$[x(t) - f(t, x(t))] - [x(0) - f(0, x(0))] = a_{\alpha}(h(t) - h(0)) + b_{\alpha} \int_0^t h(s) ds.$$

Thus

$$x(t) = x(0) - f(0, x(0)) + f(t, x(t)) + a_{\alpha} h(t) - a_{\alpha} h(0) + b_{\alpha} \int_0^t h(s) ds,$$

so

$$\begin{aligned} x(t) &= \eta + f(t, x(t)) + a_{\alpha} h(t) - a_{\alpha} h(0) + b_{\alpha} \int_0^t h(s) ds \\ &= C + f(t, x(t)) + a_{\alpha} h(t) + b_{\alpha} \int_0^t h(s) ds, \end{aligned}$$

we get (3.2). □

Our first result is based on Banach fixed point Theorem. We assume the following conditions to prove the existence and uniqueness of a solution of problem (1.1):

(H1) The function $g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exist constants $L_1 > 0$ and $0 < L_2 < 1$ such that

$$|g(t, u, v) - g(t, \overline{u}, \overline{v})| \leq L_1 |u - \overline{u}| + L_2 |v - \overline{v}|,$$

for any u, v, $\overline{u}$ and $\overline{v} \in \mathbb{R}$, for a.e., $t \in J$.

(H3) The function $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H4) There exists constant $0 < L_3 < 1$ such that

$$|f(t, u) - f(t, \overline{u})| \leq L_3 |u - \overline{u}|,$$

for any u, $\overline{u} \in \mathbb{R}$, for a.e., $t \in J$.

Theorem 3.3. If the hypotheses (H1)-(H4) are satisfied and if

$$\rho = L_3 + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2} < 1, \tag{3.3}$$

then the problem (1.1) has a unique solution $x \in C(J, \mathbb{R})$.

Proof . Transform the problem (1.1) into a fixed point problem. In fact, consider the operator $\mathcal{P} : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ defined by

$$(\mathcal{P}x)(t) = C + f(t, x(t)) + a_\alpha \sigma_x(t) + b_\alpha \int_0^t \sigma_x(s) ds,$$

$a_\alpha = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$, $b_\alpha = \dfrac{2\alpha}{(2-\alpha)M(\alpha)}$, $C = \eta - a_\alpha h(0)$ and $\sigma_x(t) = g\Big(t, x(t), {}^{CF}D^\theta[x(t) - f(t, x(t))]\Big)$.

It is clear that the fixed points of $\mathcal{P}$ are solutions of problem (1.1). Let $x, y \in C(J, \mathbb{R})$. We have

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &= \left| f(t, x(t)) + a_\alpha \sigma_x(t) + b_\alpha \int_0^t \sigma_x(s) ds - f(t, y(t)) + a_\alpha \sigma_y(t) + b_\alpha \int_0^t \sigma_y(s) ds \right| \\ &\le |f(t, x(t)) - f(t, y(t))| + a_\alpha |\sigma_x(t) - \sigma_y(t)| + b_\alpha \int_0^t |\sigma_x(s) - \sigma_y(s| ds, \end{aligned} \tag{3.4}$$

where $\sigma_x(t), \sigma_y(t) \in C(J, \mathbb{R})$ such that $\sigma_x(t) = g(t, x(t), \sigma_x(t))$ and $\sigma_y(t) = g(t, y(t), \sigma_y(t))$. By (H2), we have

$$\begin{aligned} |\sigma_x(t) - \sigma_y(t)| &\le L_1 |x(t) - y(t)| + L_2 |\sigma_x(t) - \sigma_y(t)| \\ &\le \frac{L_1}{1 - L_2} |x(t) - y(t)|. \end{aligned} \tag{3.5}$$

By (H4) and replacing (3.5) in the inequality (3.4), we also have

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &\le L_2 |x(t) - y(t)| + a_\alpha \frac{L_1}{1 - L_2} |x(t) - y(t)| + b_\alpha \frac{L_1}{1 - L_2} \int_0^t |x(s) - y(s)| ds \\ &\le \Big(L_2 + \frac{a_\alpha L_1}{1 - L_2} + \frac{b_\alpha L_1 T}{1 - L_2} \Big) \|x - y\|. \end{aligned}$$

Thus

$$\|\mathcal{P}(x) - \mathcal{P}(y)\| \le \rho \|x - y\|,$$

for $x, y \in C(J, \mathbb{R})$, where

$$\rho = L_2 + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2}.$$

Since $\rho < 1$, the operator $\mathcal{P}$ is a contraction on $C(J, \mathbb{R})$. Also, $\mathcal{P}$ satisfies the Banach contraction theorem. We deduce that $\mathcal{P}$ has a fixed point which is solution to the problem (1.1). This completes the proof. □

4 Estimates on the solutions

Theorem 4.1. Assume that $g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy (H1)-(H4). Let x be a solution of (1.1). If $\frac{a_\alpha L_1}{(1-L_2)(1-L_3)} < 1$, then

$$\|x\| \le \frac{K}{A} \exp(BT),$$

where $A = \frac{(1-L_2)(1-L_3) - a_\alpha L_1}{1 - L_2}$, $B = \frac{b_\alpha}{(1-L_2)(1-L_3) - a_\alpha L_1}$, $K = |C| + Q_1 + \frac{(a_\alpha + b_\alpha T)Q_2}{1 - L_2}$, $Q_1 = \sup_{t \in [0,T]} |f(t, 0)|$ and $Q_2 = \sup_{t \in [0,T]} |g(t, 0, 0)|$.

Proof . Let ${}^{CF}D^\alpha[x(t) - f(t, x(t))] = \sigma_x(t)$, $x(0) + f(0, x(0)) = \eta$. By Lemma 3.1,

$$x(t) = C + f(t, x(t)) + a_\alpha \sigma_x(t) + b_\alpha \int_0^t \sigma_x(s) ds, \tag{4.1}$$

where $a_\alpha = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$, $b_\alpha = \dfrac{2\alpha}{(2-\alpha)M(\alpha)}$ and $C = \eta - a_\alpha \sigma_x(0)$. Then by (H2) and (H3), for any $t \in [0,T]$ we have

$$\begin{aligned}
|x(t)| &= |C + f(t,x(t)) + a_\alpha \sigma_x(t) + b_\alpha| \int_0^t \sigma_x(s)ds| \\
&\leq |C| + |f(t,x(t)) - f(0,x(0))| + |f(0,x(0))| + a_\alpha |\sigma_x(t)| + b_\alpha| \int_0^t \sigma_x(s)|ds \\
&\leq |C| + L_3|x(t)| + Q_1 + a_\alpha |\sigma_x(t)| + b_\alpha| \int_0^t \sigma_x(s)|ds, \qquad (4.2)
\end{aligned}$$

where $Q_1 = \sup_{t\in[0,T]} |f(t,0)|$. On the other hand, for any $t \in [0,T]$, we get

$$\begin{aligned}
|\sigma_x(t)| &= |g(t,x(t),\sigma_x(t))| \\
&\leq |g(t,x(t),\sigma_x(t)) - g(t,0,0)| + |g(t,0,0)| \\
&\leq L_1|x(t)| + L_2|\sigma_x(t)| + Q_2 \\
&\leq \frac{L_1}{1-L_2}|x(t)| + \frac{Q_2}{1-L_2}, \qquad (4.3)
\end{aligned}$$

where $Q_2 = \sup_{t\in[0,T]} |g(t,0,0)|$. By replacing (4.3) in the inequality (4.2), we also get

$$\begin{aligned}
|x(t)| &\leq |C| + L_3|x(t)| + Q_1 + a_\alpha\left(\frac{L_1}{1-L_2}|x(t)| + \frac{Q_2}{1-L_2}\right) + b_\alpha \int_0^t \left(\frac{L_1}{1-L_2}|x(s)| + \frac{Q_2}{1-L_2}\right)ds \\
&\leq |C| + Q_1 + \left(L_3 + \frac{a_\alpha L_1}{1-L_2}\right)|x(t)| + \frac{a_\alpha Q_2}{1-L_2} + \frac{b_\alpha Q_2 T}{1-L_2} + \frac{b_\alpha L_1}{1-L_2}\int_0^t |x(s)|ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big(1 - (L_3 + \frac{a_\alpha L_1}{1-L_2})\Big)|x(t)| &\leq |C| + Q_1 + \frac{a_\alpha Q_2}{1-L_2} + \frac{b_\alpha Q_2 T}{1-L_2} + \frac{b_\alpha L_1}{1-L_2}\int_0^t |x(s)|ds, \\
&\leq |C| + Q_1 + \frac{a_\alpha Q_2}{1-L_2} + \frac{b_\alpha Q_2 T}{1-L_2} \\
&\quad + \frac{b_\alpha L_1}{(1-L_2)\Big(1-(L_3 + \frac{a_\alpha L_1}{1-L_2})\Big)} \int_0^t \Big(1 - (L_3 + \frac{a_\alpha L_1}{1-L_2})\Big)|x(s)|ds,
\end{aligned}$$

and so

$$\begin{aligned}
\Big(\frac{(1-L_2)(1-L_3) - a_\alpha L_1}{1-L_2}\Big)|x(t)| \leq &\frac{(a_\alpha + b_\alpha T)Q_2}{1-L_2} + \frac{b_\alpha L_1}{(1-L_2)(1-L_3) - a_\alpha L_1}\int_0^t \Big(\frac{(1-L_2)(1-L_3) - a_\alpha L_1}{1-L_2}\Big)|x(s)|ds \\
&+ |C| + Q_1,
\end{aligned}$$

we put $A = \frac{(1-L_2)(1-L_3)-a_\alpha L_1}{1-L_2}$, $B = \frac{b_\alpha L_1}{(1-L_2)(1-L_3)-a_\alpha L_1}$ and $K = |C| + Q_1 + \frac{(a_\alpha + b_\alpha T)Q_2}{1-L_2}$. We can write

$$A|x(t)| \leq K + B\int_0^t |x(s)|ds,$$

from Gronwall's inequality (Corollary 2.5), we have

$$A|x(t)| \leq K\exp(BT),$$

using the condition $(1-L_2)(1-L_3) > a_\alpha L_1$, one has

$$\|x\| \leq \frac{K}{A}\exp(BT).$$

This completes the proof of Theorem 4.1. □

5 Continuous Dependence on the data and uniqueness of solution

Now, we give the continuous dependence of solution of problem (1.1) on the initial condition.

Theorem 5.1. Suppose that (H1)-(H4) hold. Let x and y be two solutions of the hybrid implicit Caputo-Fabrizio fractional differential equation (1.1) with $\eta = \eta_1$ and $\eta = \eta_2$, respectively. If $\rho := L_3 + \dfrac{(a_\alpha + b_\alpha T)L_1}{1 - L_2} < 1$, then

$$\|x - y\| \leq \frac{\left(1 + \dfrac{a_\alpha L_1}{(1 - L_2)(1 - L_3)}\right)}{(1 - \rho)}\|\eta_1 - \eta_2\|. \tag{5.1}$$

Proof . Let x and y be two solutions of problem (1.1). Then from equation (4.1) we have

$$|x(t) - y(t)| \leq |\eta_1 - \eta_2| + a_\alpha|\sigma_x(0) - \sigma_y(0)| + |f(t, x(t)) - f(t, y(t))| + a_\alpha|\sigma_x(t) - \sigma_y(t)| + b_\alpha \int_0^t |\sigma_x(s) - \sigma_y(s)|ds.$$

By (H2), for each $t \in J$, we get

$$|x(t) - y(t)| \leq |\eta_1 - \eta_2| + \frac{a_\alpha L_1}{1 - L_2}|x(0) - y(0)| + L_3|x(t) - y(t)| + \frac{a_\alpha L_1}{1 - L_2}|x(t) - y(t)| + \frac{b_\alpha L_1}{1 - L_2} \int_0^t |x(s) - y(s)|ds. \tag{5.2}$$

On the other hand, by (H4) we have

$$\begin{aligned} |x(0) - y(0)| &= |\eta_1 + f(0, x(0)) - \eta_2 - f(0, y(0))| \\ &\leq |\eta_1 - \eta_2| + |f(0, x(0)) - f(0, y(0))| \\ &\leq |\eta_1 - \eta_2| + L_3|x(0) - y(0)|, \\ &\leq \frac{1}{1 - L_3}|\eta_1 - \eta_2|. \end{aligned} \tag{5.3}$$

Replacing (5.3) in the inequality (5.2), we also have

$$\begin{aligned} |x(t) - y(t)| &\leq |\eta_1 - \eta_2| + \frac{a_\alpha L_1}{(1 - L_2)(1 - L_3)}|\eta_1 - \eta_2| + L_3|x(t) - y(t)| + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2}|x(t) - y(t)| \\ &= \left(1 - \frac{a_\alpha L_1}{(1 - L_2)(1 - L_3)}\right)|\eta_1 - \eta_2| + \left(L_3 + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2}\right)|x(t) - y(t)|. \end{aligned}$$

Thus,

$$\left(1 - \left(L_3 + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2}\right)\right)|x(t) - y(t)| \leq \left(1 - \frac{a_\alpha L_1}{(1 - L_2)(1 - L_3)}\right)|\eta_1 - \eta_2|,$$

and so

$$\|x - y\| \leq \frac{\left(1 - \dfrac{a_\alpha L_1}{(1 - L_2)(1 - L_3)}\right)}{1 - \rho}\|\eta_1 - \eta_2\|,$$

where $\rho := L_3 + \frac{(a_\alpha + b_\alpha T)L_1}{1 - L_2}$. From which we get the inequality (5.1). This completes the proof. □

6 Example

Consider the following nonlinear problem:

$$\begin{cases} {}^{CF}D^{\frac{1}{2}}[x(t) - \frac{1}{5}\sin(x(t))] = \dfrac{t^{\frac{1}{2}}}{2} + \dfrac{t}{20}\cos(|x(t)|) + \dfrac{1}{10 + |{}^{CF}D^{\frac{1}{2}}[x(t) - \frac{1}{5}\sin(x(t))]|}, \qquad t \in J = [0, 1] \\ x(0) - \frac{1}{5}\sin(x(0)) = 1. \end{cases} \tag{6.1}$$

Clearly, from (6.1), $T=1$, $g(t,x(t),{}^{CF}D^{\theta}[x(t)-f(t,x(t))]) = \frac{t^{\frac{1}{2}}}{2}+\frac{t}{20}\cos(|x(t)|)+\frac{1}{10+|{}^{CF}D^{\frac{1}{2}}[x(t)-\frac{1}{5}\sin(x(t))]|}$ and $f(t,x(t))=\frac{1}{5}\sin(x(t))$ are continuous for all $t\in[0,1]$. Further, let u, v, $\overline{u}$, $\overline{v}\in\mathbb{R}$ and $t\in[0,1]$. Then one has

$$g(t,u,v)=\frac{t^{\frac{1}{2}}}{2}+\frac{t}{20}\cos(|u|)+\frac{1}{10+|v|},$$

and

$$f(t,u)=\frac{1}{5}\sin(u).$$

We also have

$$|g(t,u,v)-g(t,\overline{u},\overline{v})|\leq\frac{1}{20}|u-\overline{u}|+\frac{1}{10}|v-\overline{v}|,$$

and

$$|f(t,u)-f(t,\overline{u})|\leq\frac{1}{5}|u-\overline{u}|.$$

Hence, all conditions of Theorem 3.3 are fulfilled and

$$\rho=L_3+\frac{(a_\alpha+b_\alpha T)L_1}{1-L_2}=0,3418<1,$$

with $L_1=\frac{1}{20}$, $L_2=\frac{1}{10}$, $L_3=\frac{1}{5}$, $\alpha=\frac{1}{2}$, $T=1$, $a_\alpha=\frac{2}{3}$, $b_\alpha=\frac{2}{3}$. Thus, the problem (6.1) has a unique solution $x\in C(J,\mathbb{R})$ and from Theorem 5.1 we deduce that the continuous dependence on the initial data and uniqeness of solution for (6.1).

7 Conclusion

In this paper we have discussed the existence, uniqueness, estimates on solutions and discuss the continuous dependence on the initial data of solutions for a class of nonlinear implicit Caputo-Fabrizio fractional hybrid differential equations with an initial condition, we made use of the Banach contraction principle and the Gronwall's inequality to show the estimate of the solutions. The differential operator used is extended by Caputo-Fabrizio, which generalizes the Caputo and the Fabrizio fractional derivatives into a single form.

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