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Coefficient estimates and Fekete-Szegő inequalities for a new subclass of *m*-fold symmetric bi-univalent functions satisfying subordinate conditions

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Abstract

In this paper, we introduce a new subclass of the class of m-fold symmetric bi-univalent functions and obtain estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|, |a_{2m+1}|$ and Fekete-Szegő functional problem for functions in this new subclass. The results in this paper generalize some of the results of Huo Tang et al. [18], Altınkaya and Yalçın [3].

Keywords: analytic functions, bi-univalent functions, m-fold symmetric functions, subordination, coefficient estimates, Fekete-Szeg ő problem
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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which are of the form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, z \in \mathbb{U}.$$

The class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$ is denoted by S. Every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \qquad \big(z \in \mathbb{U}\big)$$

and

$$f(f^{-1}(w)) = w \qquad \big(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\big),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. The class of bi-univalent functions in $\mathbb{U}$ is denoted by Σ.

A function is said to be m-fold symmetric (see [14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (m \in \mathbb{N},\ z \in \mathbb{U}). \tag{1.1}$$

The class of m-fold symmetric univalent functions, which are normalized by the above series expansion (1.1), is denoted by S_m. The functions in the class S are one fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, is defined the concept of m-fold symmetric bi-univalent functions. Each function f in the class Σ generates an m-fold symmetric bi-univalent function for each positive integer m. The normalized form of f is given in (1.1) and f^{-1} is given in the followings.

$$\begin{aligned} g(w) =& w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots , \end{aligned} \tag{1.2}$$

where $f^{-1} = g$. The class of m-fold symmetric bi-univalent functions is denoted by Σ_m.

Recently, many authors investigated estimates of coefficients and Fekete-Szegő functional problem for subclasses of m-fold symmetric bi-univalent functions ([1], [2], [4]-[10], [12], [13], [15]-[24]).

Forwards, we need the notion of subordination.

Definition 1.1. [11, p. 4] Let f, F analytic functions in the open unit disk $\mathbb{U}$. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists an analytic function w in the open unit disk $\mathbb{U}$, with $w(0) = 0$ and $\left|w(z)\right| < 1, z \in U$, such that $f(z) = F\left[w(z)\right], z \in \mathbb{U}$.

H. Tang *et al.* [18] introduced the following subclasses of m-fold symmetric bi-univalent functions.

Definition 1.2. [18, Definition 1, p.1066] A function $f(z)$, given by (1.1), is said to be in the class $\mathcal{H}_{\Sigma,m}(\phi)$, if the following conditions are satisfied:

$$f \in \Sigma_m, \qquad f'(z) \prec \phi(z) \qquad \text{and} \qquad g'(w) \prec \phi(w),$$

where the function $g(w)$ is defined by (1.2).

Definition 1.3. [18, Definition 3, p. 1078] A function $f(z)$, given by (1.1), is said to be in the class $\mathcal{M}_{\Sigma,m}(\lambda, \phi)$ if the following conditions are satisfied:

$$f \in \Sigma_m, \qquad (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z)$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \phi(w),$$

where the function $g(w)$ is defined by (1.2).

Ş. Altınkaya and S. Yalçın [3] introduced the following subclass of bi-univalent functions.

Definition 1.4. [3] A function $f \in \Sigma$ is said to be in $S_\Sigma(\lambda, \phi), 0 \le \lambda \le 1$, if the following subordinations hold

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4\left(\lambda - \lambda^2\right) z + \left(2\lambda^2 - \lambda\right) zf'(z) + \left(2\lambda^2 - 3\lambda + 1\right) f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4\left(\lambda - \lambda^2\right) w + \left(2\lambda^2 - \lambda\right) wg'(w) + \left(2\lambda^2 - 3\lambda + 1\right) g(w)} \prec \phi(w),$$

where $g = f^{-1}$.

Motivated by the definition of the above subclass of bi-univalent functions, we introduce below a new subclass of m-fold symmetric bi-univalent functions in a similar manner.

Definition 1.5. A function $f \in \Sigma_m$ said to be in the class $S_{\Sigma_m}(\lambda, \phi)$, $0 \leq \lambda \leq 1$, if the following subordination conditions holds

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4(\lambda - \lambda^2) z + (2\lambda^2 - \lambda) zf'(z) + (2\lambda^2 - 3\lambda + 1) f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4(\lambda - \lambda^2) w + (2\lambda^2 - \lambda) wg'(w) + (2\lambda^2 - 3\lambda + 1) g(w)} \prec \phi(w)$$

where $g = f^{-1}$.

Remark 1.6.

$$S_{\Sigma_m}(0, \phi) = \mathcal{M}_{\Sigma,m}(0, \phi)$$
$$S_{\Sigma_m}\left(\frac{1}{2}, \phi\right) = \mathcal{H}_{\Sigma,m}(\phi)$$
$$S_{\Sigma_m}(1, \phi) = \mathcal{M}_{\Sigma,m}(1, \phi)$$
$$S_{\Sigma_1}(\lambda, \phi) = S_{\Sigma}(\lambda, \phi)$$

In the followings, we introduce a function ϕ used in [18].

ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$ such that

$$\phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0$$

and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. This function has a series expansion of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with

$$u(0) = v(0) = 0, \qquad \max\{|u(z)|, |v(z)|\} < 1,$$

and

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots,$$
$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \cdots.$$

We have the following inequalities

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2, |c_m| \leq 1 \text{ and } |c_{2m}| \leq 1 - |c_m|^2. \tag{1.3}$$

By simple computations, are obtained the followings

$$\phi(u(z)) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \cdots \qquad (|z| < 1) \tag{1.4}$$

and

$$\phi(v(w)) = 1 + B_1 c_m w^m + (B_1 c_{2m} + B_2 c_m^2) w^{2m} + \cdots \qquad (|w| < 1). \tag{1.5}$$

2 Main results

We begin this section by finding the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the class $S_{\Sigma_m}(\lambda, \phi)$.

Theorem 2.1. Let the function $f(z)$, given by (1.1), be in the class $S_{\Sigma_m}(\lambda,\phi)$. Then

$$|a_{m+1}| \le \frac{B_1\sqrt{2B_1}}{\sqrt{\left|\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\right|+2B_1\alpha^2}} \tag{2.1}$$

and

$$|a_{2m+1}| \le \begin{cases} \dfrac{\left(|\beta(m+1)-\alpha\gamma|+|\alpha\gamma|\right)B_1}{\left|\beta(\beta(m+1)-2\alpha\gamma)\right|}, \\ \quad \text{if } |\beta|(m+1)|B_2| \le \left(|\beta(m+1)-\alpha\gamma|+|\alpha\gamma|\right)B_1 \\ \dfrac{\big(|\beta(m+1)-\alpha\gamma|+|\alpha\gamma|\big)\left|(\beta(m+1)-2\alpha\gamma)B_1^2-2\alpha^2B_2\right|B_1+2\alpha^2|\beta|(m+1)|B_2|B_1}{\left|\beta(\beta(m+1)-2\alpha\gamma)\right|\big(\left|(\beta(m+1)-2\alpha\gamma)B_1^2-2\alpha^2B_2\right|+2B_1\alpha^2\big)}, \\ \quad \text{if } |\beta|(m+1)|B_2| > \left(|\beta(m+1)-\alpha\gamma|+|\alpha\gamma|\right)B_1 \end{cases}, \tag{2.2}$$

where

$$\alpha = m+2\lambda^2m^2-\lambda m^2-4\lambda^2+4\lambda,$$
$$\beta = 2(m+4\lambda^2m^2-2\lambda m^2-2\lambda^2+2\lambda),$$
$$\gamma = \big(2\lambda-1\big)\big((m+2)\lambda-1\big).$$

Proof . Let $f\in S_{\Sigma_m}(\lambda,\phi)$. Then there are analytic functions $u:\mathbb{U}\to\mathbb{U}$ and $v:\mathbb{U}\to\mathbb{U}$, with

$$u(0)=v(0)=0,$$

satisfying the following conditions:

$$\frac{zf'(z)+\big(2\lambda^2-\lambda\big)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} = \phi(u(z)) \tag{2.3}$$

and

$$\frac{wg'(w)+\big(2\lambda^2-\lambda\big)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} = \phi(v(w)). \tag{2.4}$$

Since

$$\frac{zf'(z)+\big(2\lambda^2-\lambda\big)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} =$$
$$1+\alpha a_{m+1}z^m+\big(\beta a_{2m+1}-\alpha\gamma a_{m+1}^2\big)z^{2m}+\cdots$$

and

$$\frac{wg'(w)+\big(2\lambda^2-\lambda\big)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} =$$
$$1-\alpha a_{m+1}w^m+\Big(\big(\beta(m+1)-\alpha\gamma\big)a_{m+1}^2-\beta a_{2m+1}\Big)w^{2m}-\cdots,$$

from (1.4), (1.5), (2.3) and (2.4), we find that

$$\alpha a_{m+1} = B_1b_m, \tag{2.5}$$
$$\beta a_{2m+1}-\alpha\gamma a_{m+1}^2 = B_1b_{2m}+B_2b_m^2, \tag{2.6}$$
$$-\alpha a_{m+1} = B_1c_m \tag{2.7}$$

and

$$\big(\beta(m+1)-\alpha\gamma\big)a_{m+1}^2-\beta a_{2m+1} = B_1c_{2m}+B_2c_m^2. \tag{2.8}$$

From (2.5) and (2.7), we get

$$c_m = -b_m. \tag{2.9}$$

By adding the equations (2.6) and (2.8) and, upon some computations using (2.5) and (2.9), we obtain

$$\Big(\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\Big)a_{m+1}^2=B_1^3\big(b_{2m}+c_{2m}\big). \tag{2.10}$$

Further, the equations (2.9), (2.10), together with the equation (1.3), yield

$$\left|\Big(\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\Big)a_{m+1}^2\right|\le 2B_1^3\big(1-|b_m|^2\big). \tag{2.11}$$

Now, from the equations (2.5) and (2.11), we get

$$\big|a_{m+1}\big|\le\frac{B_1\sqrt{2B_1}}{\sqrt{\Big|\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\Big|+2B_1\alpha^2}},$$

as asserted in (2.1).

By simple calculations from (2.6) and (2.8) and using the equation (2.9), we find that

$$\beta\big(\beta(m+1)-2\alpha\gamma\big)a_{2m+1}=\big(\beta(m+1)-\alpha\gamma\big)B_1b_{2m}+\alpha\gamma B_1c_{2m}+\beta(m+1)B_2b_m^2. \tag{2.12}$$

Thus, by using the equations (1.3) and (2.9) in (2.12), we get

$$\begin{gathered}\big|\beta\big(\beta(m+1)-2\alpha\gamma\big)\big|\big|a_{2m+1}\big|\le\\ \big(\big|\beta(m+1)-\alpha\gamma\big|+|\alpha\gamma|\big)B_1-\big(\big|\beta(m+1)-\alpha\gamma\big|+|\alpha\gamma|\big)B_1|b_m|^2+|\beta|(m+1)|B_2||b_m|^2.\end{gathered} \tag{2.13}$$

Since

$$\big|b_m\big|^2\le\frac{2\alpha^2B_1}{\big|\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\big|+2B_1\alpha^2}, \tag{2.14}$$

upon substituting from (2.14) into (2.13), we are led easily to the assertion (2.2) of Theorem 2.1. This evidently completes the proof of Theorem 2.1. □

Taking $m=1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. Let the function $f(z)$, given by (1.1), be in the class $S_\Sigma(\lambda,\phi)$. Then

$$\big|a_2\big|\le\frac{B_1\sqrt{B_1}}{\sqrt{\Big|\big(\beta-\alpha\gamma\big)B_1^2-\alpha^2B_2\Big|+B_1\alpha^2}}$$

and

$$\big|a_3\big|\le\begin{cases}\dfrac{\left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1}{2\big|\beta(\beta-\alpha\gamma)\big|}, & \\ \quad\text{if } 2|\beta||B_2|\le\left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1 & \\[2ex] \dfrac{\big(|2\beta-\alpha\gamma|+|\alpha\gamma|\big)\big|(\beta-\alpha\gamma)B_1^2-\alpha^2B_2\big|B_1+2\alpha^2|\beta||B_2|B_1}{2\big|\beta(\beta-\alpha\gamma)\big|\big(\big|(\beta-\alpha\gamma)B_1^2-\alpha^2B_2\big|+B_1\alpha^2\big)}, & \\ \quad\text{if } 2|\beta||B_2|>\left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1\end{cases},$$

where

$$\alpha=1+3\lambda-2\lambda^2,$$
$$\beta=2(1+2\lambda^2),$$
$$\gamma=(2\lambda-1)(3\lambda-1).$$

Remark 2.3. The estimate for $\big|a_2\big|$ asserted by Corollary 2.2 is obtained in Theorem 1 in [3].

Taking $\lambda=0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. Let the function $f(z)$, given by (1.1), be in the class $\mathcal{M}_{\Sigma,m}(0,\phi)$. Then

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{\left|B_1^2-B_2\right|+B_1}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{m+1}{2m^2}B_1, & \text{if } |B_2|\leq B_1 \\ \dfrac{(m+1)B_1\left(|B_1^2-B_2|+|B_2|\right)}{2m^2\left(|B_1^2-B_2|+B_1\right)}, & \text{if } |B_2|>B_1 \end{cases}.$$

Remark 2.5. The results of Corollary 2.4 are obtained taking $\lambda = 0$ in Theorem 5 in [18].

Taking $\lambda = \frac{1}{2}$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.6. Let the function $f(z)$, given by (1.1), be in the class $\mathcal{H}_{\Sigma,m}(\phi)$. Then

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{(m+1)\left(2(m+1)B_1+\left|(2m+1)B_1^2-2(m+1)B_2\right|\right)}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{B_1}{2m+1}, & \text{if } |B_2|\leq B_1 \\ \dfrac{2(m+1)^3|B_2|B_1(2m+1)+\left|(2m+1)B_1^2-2(m+1)B_2\right|B_1}{(2m+1)\left(|(2m+1)B_1^2-2(m+1)B_2|+2B_1(m+1)\right)}, & \text{if } |B_2|>B_1 \end{cases}.$$

Remark 2.7. The estimate for $\left|a_{m+1}\right|$ asserted by Corollary 2.6 is obtained in Theorem 1 in [18].

Taking $\lambda = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.8. Let the function $f(z)$, given by (1.1), be in the class $\mathcal{M}_{\Sigma,m}(1,\phi)$. Then

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{(m+1)\left(\left|B_1^2-(m+1)B_2\right|+B_1(m+1)\right)}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{B_1}{2m^2}, & \text{if } |B_2|\leq B_1 \\ \dfrac{\left|B_1^2-(m+1)B_2\right|B_1+(m+1)|B_2|B_1}{2m^2\left(\left|B_1^2-(m+1)B_2\right|+B_1(m+1)\right)}, & \text{if } |B_2|>B_1 \end{cases}.$$

Remark 2.9. The results of Corollary 2.8 are obtained taking $\lambda = 1$ in Theorem 5 in [18].

Next we shall solve the Fekete-Szegő problem for functions in the class $S_{\Sigma_m}(\lambda,\phi)$.

Theorem 2.10. Let the function $f(z)$, given by (1.1), be in the class $S_{\Sigma_m}(\lambda,\phi)$. Also let $\delta\in\mathbb{R}$. Then

$$\left|a_{2m+1}-\delta a_{m+1}^2\right| \leq \begin{cases} \dfrac{B_1}{|\beta|}, & \text{for } 0\leq\left|h(\delta)\right|<\dfrac{1}{2|\beta|} \\ 2B_1\left|h(\delta)\right|, & \text{for } \left|h(\delta)\right|\geq\dfrac{1}{2|\beta|} \end{cases}, \tag{2.15}$$

where

$$h(\delta)=\frac{B_1^2(m+1-2\delta)}{2\big[\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\big]},$$

$$\alpha=m+2\lambda^2m^2-\lambda m^2-4\lambda^2+4\lambda,$$

$$\beta=2(m+4\lambda^2m^2-2\lambda m^2-2\lambda^2+2\lambda)\neq 0,$$

$$\gamma=\big(2\lambda-1\big)\big((m+2)\lambda-1\big).$$

Proof . From the equation (2.10), we get

$$a_{m+1}^2=\frac{B_1^3\big(b_{2m}+c_{2m}\big)}{\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2}. \tag{2.16}$$

Subtracting (2.6) from the (2.8), we obtain

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2+\frac{B_1\big(b_{2m}-c_{2m}\big)}{2\beta}. \tag{2.17}$$

From the equations (2.16) and (2.17), it follows that

$$a_{2m+1}-\delta a_{m+1}^2=B_1\left[\left(h(\delta)+\frac{1}{2\beta}\right)b_{2m}+\left(h(\delta)-\frac{1}{2\beta}\right)c_{2m}\right],$$

where

$$h(\delta)=\frac{B_1^2(m+1-2\delta)}{2\big[\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\big]}.$$

We know that all B_i are real and $B_1>0$, which implies the inequality (2.15). This completes the proof of Theorem 2.10. □

Taking $m=1$ in Theorem 2.10, we obtain the following corollary.

Corollary 2.11. Let the function $f(z)$, given by (1.1), be in the class $S_\Sigma(\lambda,\phi)$. Also let $\delta\in\mathbb{R}$. Then

$$\big|a_3-\delta a_2^2\big|\leq\begin{cases}\dfrac{B_1}{2(2\lambda^2+1)}, & \text{for } 0\leq\big|h(\delta)\big|<\dfrac{1}{4(2\lambda^2+1)}\\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big|\geq\dfrac{1}{4(2\lambda^2+1)}\end{cases},$$

where

$$h(\delta)=\frac{B_1^2(1-\delta)}{2\big[(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2\big]}.$$

Taking $\delta=1$ and $\delta=0$ in Theorem 2.10, we have the following corollaries.

Corollary 2.12. Let the function $f(z)$, given by (1.1), be in the class $S_{\Sigma_m}(\lambda,\phi)$. Then

$$\big|a_{2m+1}-a_{m+1}^2\big|\leq\begin{cases}\dfrac{B_1}{|\beta|}, & \text{for } 0\leq\big|h(1)\big|<\dfrac{1}{2|\beta|}\\ 2B_1\big|h(1)\big|, & \text{for } \big|h(1)\big|\geq\dfrac{1}{2|\beta|}\end{cases}$$

where

$$h(1)=\frac{B_1^2(m-1)}{2\big[\big(\beta(m+1)-2\alpha\gamma\big)B_1^2-2\alpha^2B_2\big]},$$

$$\alpha=m+2\lambda^2m^2-\lambda m^2-4\lambda^2+4\lambda,$$

$$\beta=2(m+4\lambda^2m^2-2\lambda m^2-2\lambda^2+2\lambda)\neq 0,$$

$$\gamma=\big(2\lambda-1\big)\big((m+2)\lambda-1\big).$$

Corollary 2.13. Let the function $f(z)$, given by (1.1), be in the class $S_{\Sigma_m}(\lambda,\phi)$. Then

$$|a_{2m+1}| \le \begin{cases} \dfrac{B_1}{|\beta|}, \\ \quad \text{for } \dfrac{B_2}{B_1^2} \in \Big(-\infty; -\dfrac{(m+1)(|\beta|-\beta)+2\alpha\gamma}{2\alpha^2}\Big) \cup \Big(\dfrac{(m+1)(|\beta|+\beta)-2\alpha\gamma}{2\alpha^2}; +\infty\Big) \\ \dfrac{B_1^3(m+1)}{\big|(\beta(m+1)-2\alpha\gamma)B_1^2-2\alpha^2B_2\big|}, \\ \quad \text{for } \dfrac{B_2}{B_1^2} \in \Big(-\dfrac{(m+1)(|\beta|-\beta)+2\alpha\gamma}{2\alpha^2}; \dfrac{\beta(m+1)-2\alpha\gamma}{2\alpha^2}\Big) \cup \\ \quad \Big(\dfrac{\beta(m+1)-2\alpha\gamma}{2\alpha^2}; \dfrac{(m+1)(|\beta|+\beta)-2\alpha\gamma}{2\alpha^2}\Big) \end{cases},$$

where

$$\alpha = m + 2\lambda^2 m^2 - \lambda m^2 - 4\lambda^2 + 4\lambda,$$
$$\beta = 2(m + 4\lambda^2 m^2 - 2\lambda m^2 - 2\lambda^2 + 2\lambda) \neq 0,$$
$$\gamma = \big(2\lambda - 1\big)\big((m+2)\lambda - 1\big).$$

Taking $\lambda = 0$ in Theorem 2.10, we obtain the following corollary.

Corollary 2.14. Let the function $f(z)$, given by (1.1), be in the class $\mathcal{M}_{\Sigma,m}(0,\phi)$. Also let $\delta \in \mathbb{R}$. Then

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \le \begin{cases} \dfrac{B_1}{2m}, & \text{for } 0 \le \big|h(\delta)\big| < \dfrac{1}{4m} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \ge \dfrac{1}{4m} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2\big(B_1^2 - B_2\big)}.$$

Remark 2.15. The result of Corollary 2.14 is obtained taking $\lambda = 0$ in Theorem 6 in [18].

Taking $\lambda = \frac{1}{2}$, Theorem 2.10 reduces to the corresponding result of H. Tang *et al.* [18].

Corollary 2.16. [18, Th. 2, p. 1070] Let the function $f(z)$, given by (1.1), be in the class $\mathcal{H}_{\Sigma,m}(\phi)$. Also let $\delta \in \mathbb{R}$. Then

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \le \begin{cases} \dfrac{B_1}{2m+1}, & \text{for } 0 \le \big|h(\delta)\big| < \dfrac{1}{2(2m+1)} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \ge \dfrac{1}{2(2m+1)} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{2(m+1)\big[(2m+1)B_1^2 - 2(m+1)B_2\big]}.$$

Taking $\lambda = 1$ in Theorem 2.10, we obtain the following corollary.

Corollary 2.17. Let the function $f(z)$, given by (1.1), be in the class $\mathcal{M}_{\Sigma,m}(1,\phi)$. Also let $\delta \in \mathbb{R}$. Then

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \le \begin{cases} \dfrac{B_1}{2m(2m+1)}, & \text{for } 0 \le \big|h(\delta)\big| < \dfrac{1}{4m(2m+1)} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \ge \dfrac{1}{4m(2m+1)} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2(m+1)\big[B_1^2 - (m+1)B_2\big]}.$$

Remark 2.18. The result of Corollary 2.17 is obtained taking $\lambda = 1$ in Theorem 6 in [18].

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