Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 2771–2786
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29160.4076

Hybrid iterative algorithms for finding common solutions of a system of generalized mixed quasi-equilibrium problems and fixed point problems of nonexpansive semigroups

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(Communicated by Ali Farajzadeh)

Abstract

In this paper, we introduced a hybrid iterative method for finding the set of common solutions for a system of generalized mixed quasi-equilibrium problems, the set of common fixed points for nonexpansive semigroup and the set of solutions of quasi-variational inclusion problems with multi-valued maximal monotone mappings and inverse strongly monotone mappings in Hilbert spaces. Under suitable assumptions, we prove some strong convergence theorems for the iteration.

Keywords: Generalized mixed quasi-equilibrium problems, nonexpansive semigroup, viscosity approximation method, generalized quasi-variational inclusions problems, multi-valued maximal monotone mappings, α-inverse strongly monotone mappings
2020 MSC: 49J40, 47H09, 47J20

1 Introduction

Let H be a real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C a nonempty closed convex subset of H and $F(T)$ denotes the set of all fixed points of the mapping $T : C \to C$.

A bounded linear operator $A : H \to H$ is said to be strongly positive if there exists a constant $\bar{\gamma}$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.1}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. The generalized quasi-variational inclusion problem is to find $u \in H$ such that

$$\theta \in Bu + Mu. \tag{1.2}$$

The set of solutions of problem (1.2) is denoted by $VI(H, B, M)$.

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Received: December 2022 *Accepted:* January 2023

Example 1.1.

Let C be a nonempty closed convex subset of a Hilbert space H and $\delta_C : H \to [0,\infty)$ be the indicator function of C, *i.e.*,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

If M is the subdifferential of δ_C, that is, $M = \partial\delta_C$, then the variational inclusion problem (1.2) is equivalent to find $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0, \quad \forall u \in C. \tag{1.3}$$

(1.3) is called the Hartmann-Stampacchia variational inequality problem ([16]) and the solution set of (1.3) is denoted by $VI(B,C)$.

A mapping $B : H \to H$ is called α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Bx - By, x-y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in H.$$

A multi-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx$ and $v \in My$ implies that

$$\langle u-v, x-y\rangle \geq 0,$$

and $M : H \to 2^H$ is called maximal monotone if it is monotone and for any $(x,u) \in H \times H$ such that

$$\langle u-v, x-y\rangle \geq 0, \forall\ (y,v) \in G(M)$$

implies that $u \in Mx$, where $G(M)$ is the graph of mapping M.

We can easily prove the following proposition from the definition.

Proposition 1.2. Let $B : H \to H$ be an α-inverse strongly monotone mapping. Then,

(i) B is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;

(ii) if λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.

Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction and $\varphi : C \to \mathbb{R}$ a real valued function. We consider the following generalized mixed equilibrium problem ([7, 26]) for finding $z \in C$ such that

$$\Theta(z,y) + \langle Az, y-z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C, \tag{1.4}$$

where $A : C \to C$ is single-valued mapping and the solution set of (1.4) is denoted by $GMEP(\Theta)$, *i.e.*,

$$GMEP(\Theta) = \{z \in C : \Theta(z,y) + \langle Az, y-z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\}.$$

Recently Ceng and Yao ([4]) introduced the following mixed equilibrium problem for finding $z \in C$ such that

$$\Theta(z,y) + \varphi(y) - \varphi(z) \geq 0, \forall y \in C, \tag{1.5}$$

and the set of solutions (1.5) is denoted by $MEP(\Theta)$, *i.e.*

$$MEP(\Theta) = \{z \in C : \Theta(z,y) + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\}.$$

In particular, if $\varphi = 0$, then (1.5) reduces to the equilibrium problem for finding $z \in C$ such that

$$\Theta(z,y) \geq 0, \forall y \in C, \tag{1.6}$$

and the set of solutions of (1.6) is $EP(\Theta)$.

On the other hand Li *et al.* ([15]) introduced two iterative procedures for the approximation of common fixed points of a one parameter nonexpansive semigroup $\{T(s) : 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert spaces (see [5, 6, 12, 13, 18]).

Very recently Saeidi ([20, 21]) introduced the following general iterative algorithm for finding a common element of the set of solutions of a system of equilibrium problems $EP(g)$ for a family $g = \{F_i : i = 1, 2, \cdots, M\}$ of bifunctions and of the set of fixed points of a finite family of nonexpansive mappings $\varphi = \{T_i : i = 1, 2, \cdots, N\}$ and a left amenable semigroup $\mathcal{S} = \{T(t) : t \in S\}$ of nonexpansive mapping with respect to W-mappings and left regular sequence $\{\mu_n\}$ of means defined on an approximate space of bounded real valued functions of the semigroup S.

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T(\mu_n)W_n J^{F_M}_{r_M,n} \cdots J^{F_2}_{r_2,n} J^{F_1}_{r_1,n} x_n.$$

Recall that a family $\Im = \{T(s) : 0 \le s < \infty\} : C \to C$ of a mapping is called a one parameter nonexpansive semigroup if it is satisfied the following conditions:

(a) $T(s+t) = T(s)T(t), \forall s, t \ge 0$ and $T(0) = I$;

(b) $\|T(s)x - T(s)y\| \le \|x - y\|, \forall x, y \in C$;

(c) the mapping $T(\cdot)x$ is continuous for each $x \in C$.

Motivated and inspired by the recent works [1, 2, 4, 8, 9, 10, 11, 14, 24, 26], we introduced a hybrid iterative scheme for finding a set of common solutions for a system of mixed equilibrium problems, the set of common fixed point for nonexpansive semigroup and the set of solutions of the quasi-variational inclusion problems with multi-valued maximal monotone mapping and inverse strongly monotone mappings in Hilbert spaces. We also prove some strong convergence theorem under suitable conditions.

2 Preliminaries

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Definition 2.1. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the single-valued mapping $J^M_\lambda : H \to H$ defined by

$$J^M_\lambda(u) = (I + \lambda M)^{-1}(u), \forall u \in H,$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Proposition 2.2. ([27]) Let J^M_λ be the resolvent operator associated with M. Then we have:

(i) J^M_λ is single-valued and nonexpansive for all $\lambda > 0$, i.e.,

$$\|J^M_\lambda(x) - J^M_\lambda(y)\| \le \|x - y\|, \forall x, y \in H.$$

(ii) J^M_λ is I-inverse strongly monotone, i.e.,

$$\|J^M_\lambda(x) - J^M_\lambda(y)\|^2 \le \langle x - y, J^M_\lambda(x) - J^M_\lambda(y)\rangle, \forall x, y \in H.$$

Definition 2.3. A single-valued mapping $P : H \to H$ is said to be hemicontinuous if for any $x, y, z \in H$, the function $t \to \langle P(x + ty), z\rangle$ is continuous at 0^+.

Remark 2.4. Every continuous mapping must be hemicontinuous.

Lemma 2.5. ([17]) Let E be a real Banach spaces and E^* be the dual space of E. Let $T : E \to 2^{E^*}$ be a maximal monotone mapping and $P : E \to E^*$ be a hemicontinuous bounded monotone mapping with $D(T) = X$, then the mapping $U := T + P : E \to 2^{E^*}$ is a maximal monotone mapping.

Lemma 2.6. ([22]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$$

for all $n \ge 1$ and $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Lemma 2.7. ([25]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\gamma_n \in (0,1)$ and δ_n are sequences satisfying:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} \mid \delta_n \mid = \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.8. ([3]) Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, we have

$$\|x+y\|^2 \leq \|x\|^2 + \langle y\ , j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Especially if $E = H$ is a real Hilbert space, then

$$\|x+y\|^2 \leq \|x\|^2 + \langle y\ , x+y\rangle, \forall x, y \in H.$$

For solving the equilibrium problems for bifunction $\Theta : C \times C \to \mathbb{R}$, we assume that Θ satisfies the following conditions:

(C1) $\Theta(x,x) = 0, \forall x \in C$;

(C2) Θ is monotone *i.e.,*

$$\Theta(x,y) + \Theta(y,x) \leq 0, \forall x, y \in C;$$

(C3) for any $y \in C$, $x \to \Theta(x,y)$ is concave and weakly upper semi-continuous;

(C4) for each $x \in C$, $y \to \Theta(x,y)$ is convex and lower semi-continuous.

A mapping $\eta : C \times C \to H$ is called Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|\eta(x,y)\| \leq L\|x-y\|, \forall x, y \in C.$$

A differentiable function $K : C \to \mathbb{R}$ on a convex set C is called:

(i) η-convex ([4]) if

$$K(y) - K(x) \geq \langle K'(x), \eta(y,x)\rangle, \forall x, y \in C,$$

where $K'(x)$ is the Frechet derivative of K at x.

(ii) η-strongly convex ([4]) if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y,x)\rangle \geq \frac{\mu}{2}\|x-y\|^2, \forall x, y \in C.$$

A mapping $F : C \to \mathbb{R}$ is called sequentially continuous at x_0 if $F(x_n) \to F(x_0)$ for each sequence $\{x_n\}$ satisfying $x_n \to x_0$. F is called sequentially continuous on C if it is sequentially continuous at each point of C.

Lemma 2.9. ([4]) Suppose that for each fixed $y \in C, \eta(y,\cdot) : C \to H$ be sequentially continuous from the weak topology to the weak topology and that $K' : C \to H$ is sequentially continuous from the weak topology to the strong topology. Then $g_y : C \to \mathbb{R}$ defined as $g_y(x) = \langle K'(x), \eta(y,x)\rangle$ for each fixed $y \in C$ is sequentially continuous in the weak topology.

If an equilibrium bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions $(C1)$-$(C4)$ and $A : C \to C$ a single-valued mapping. Let r be a positive parameter. For a given point $x \in C$, consider the auxiliary problem for $GMEP$ (for short, $GMEP(x,r)$) which consists of finding $y \in C$ such that

$$\Theta(y,z) + \langle Az, y-z\rangle + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \forall z \in C,$$

where $\eta : C \times C \to H$ and $K'(x)$ is the Frechet derivative of a functional $K : C \to \mathbb{R}$ at x. Let $\mathcal{V}_r^{\Theta} : C \to C$ be the mapping such that for each $x \in C, \mathcal{V}_r^{\Theta}(x)$ is the solution set of $GMEP(x,r)$, *i.e.,* for all $z \in C$

$$\mathcal{V}_r^{\Theta}(x) = \Big\{y \in C : \Theta(y,z) + \langle Az, y - z\rangle + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0\Big\}.$$

We can prove the following lemma by using the same method as Lemma 3.1 in [4].

Lemma 2.10. ([4]) Let C be a nonempty closed convex subset of a Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions $(C1)$-$(C4)$. Assume that

(1) $\eta : C \times C \to H$ is Lipschitz continuous with constant $L > 0$ such that
   (a) $\eta(x,y) + \eta(y,x) = 0, \forall x, y \in C$;
   (b) $\eta(\cdot,\cdot)$ is affine in the first variable;
   (c) for each fixed $y \in C, x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;

(2) $K : C \to \mathbb{R}$ is η-strongly convex with constant $\mu > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;

(3) for each $x \in C$ there exists a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$ we have

$$\Theta(y,z_x) + \langle Az_x, y - z_z\rangle + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x,y)\rangle < 0.$$

Then we have:

(i) $\mathcal{V}_r^{\Theta}$ is single-valued;

(ii) $\mathcal{V}_r^{\Theta}$ is nonexpansive if K' is Lipschitz continuous with constant $\upsilon > 0$ such that $\mu \geq L\upsilon$;

(iii) $F(\mathcal{V}_r^{\Theta}) = GMEP(\Theta)$;

(iv) $GMEP(\Theta)$ is closed and convex.

Lemma 2.11. ([23]) Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then for any $h \geq 0$ and $t > 0$,

$$\lim_{t\to\infty} \sup_{x\in C} \Big\|\frac{1}{t}\int_0^t T(s)xds - T(h)\Big(\frac{1}{t}\int_0^t T(s)xds\Big)\Big\| = 0, \quad \text{for all } x \in C.$$

Lemma 2.12. ([15]) Let C be nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties:

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$.

Then $z \in F(\Im) := \bigcap_{s\geq 0} F(T(s))$.

3 Main results

In order to prove the main results, we first give the following lemma.

Lemma 3.1. ([20]) We have the following statements for the solutions of the variational inclusion (1.2):

(i) $u \in H$ is a solution of variational inclusion (1.2) if and only if

$$u = J_\lambda^M(u - \lambda Bu), \forall \lambda > 0,$$

i.e.,

$$VI(H, B, M) = F(J_\lambda^M(I - \lambda B)),\ \forall \lambda > 0.$$

(ii) If $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H, B be an α-inverse strongly monotone mapping from C into H and M be a multi-valued mapping of C. Let $\Im = \{T(s) : 0 \leq s < \infty\}$ be an one parameter nonexpansive semigroup and $\Theta_i : C \times C \to \mathbb{R}$, $(i = 1, 2, \cdots, N)$ be a bifunction which satisfies $(C1)$-$(C4)$ such that

$$\Omega := F(\Im) \cap GMEP(\Theta) \cap VI(H, B, M) \neq \emptyset,$$

where

$$GMEP(\Theta) := \bigcap_{l=1}^{N} GMEP(\Theta_l).$$

Let $\varphi_i : C \to \mathbb{R}, (i = 1, 2, \cdots, N)$ be a lower semi-continuous and convex functional. Let A be a strongly positive bounded linear operator with a coefficient $\bar{\gamma} > 0$ and f be a contraction of H into itself with a contractive constant $h(0 < h < 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{h}$. Let $\{x_n\}, \{\rho_n\}, \{\xi_n\}$ and $\{y_n\}$ be implicit iterative sequences generated by $x_1 \in H$ and

$$\begin{cases} x_n = \alpha_n \gamma f\Big(\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s)x_n ds\Big) + \beta_n x_n + \Big((1 - \beta_n)I - \alpha_n A\Big)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s)\rho_n ds \\ \rho_n = J_\lambda^M(I - \lambda B)\xi_n, \\ \xi_n = J_\lambda^M(I - \lambda B)y_n, \\ y_n = \mathcal{V}_{r_N}^{\Theta_N} \cdots \mathcal{V}_{r_2}^{\Theta_2}\mathcal{V}_{r_1}^{\Theta_1} x_n, \end{cases} \tag{3.1}$$

where $\{r_i\}(i = 1, 2 \cdots N)$ is a finite family of positive parameters, $\lambda \in (0, 2\alpha], \{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$. Assume that the following conditions are hold:

(i) For each $i = 1, 2, \cdots N$, $\eta_i : C \times C \to H$ is a Lipschitz continuous mapping with constant $L_i > 0$ such that

  (a) $\eta_i(x, y) + \eta_i(y, x) = 0, \forall x, y \in C$;

  (b) $\eta_i(\cdot, \cdot)$ is affine in the first variable;

  (c) for each fixed $y \in C, x \to \eta_i(y, x)$ is sequentially continuous from the weak topology to the weak topology;

(ii) For each $i = 1, 2, \cdots N$, $K_i : C \to \mathbb{R}$ is a η_i-strongly convex with constant $\mu_i > 0$ and its derivative $K_i^{'}$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\upsilon_i > 0, \mu_i \geq L_i \upsilon_i$;

(iii) For each $x \in C$ there exists a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C - D_x$

$$F_i(y, z_x) + \langle A_i z_x, y - z_x\rangle + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i^{'}(y) - K_i^{'}(x), \eta_i(z_x, y)\rangle < 0,$$

(iv) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$
and $\lim\limits_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ provided that $\mathcal{V}_{r_i}^{\Theta_i}$ is firmly nonexpansive, and x^* is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - z\rangle \leq 0, \forall z \in \Omega. \tag{3.2}$$

Proof . We observe that from conditions (iv), we can assume without loss of generality that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is a bounded linear self-adjoint operator on H, we have

$$\|A\| = \sup\{|\ \langle Au, u\rangle\ | : u \in H, \|u\| = 1\}.$$

Since

$$\begin{aligned}\langle((1-\beta_n)I-\alpha_nA)u,u\rangle &= 1-\beta_n-\alpha_n\langle Au,u\rangle\\ &\geq 1-\beta_n-\alpha_n\|A\|\\ &\geq 0,\end{aligned}$$

this implies that $(1-\beta_n)I-\alpha_nA$ is positive. Hence we have

$$\begin{aligned}\|(1-\beta_n)I-\alpha_nA\| &= \sup\{|\ \langle((1-\beta_n)I-\alpha_nA)u,u\rangle\ |: u\in H,\|u\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Au,u\rangle : u\in H,\|u\|=1\}\\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned} \tag{3.3}$$

In the sequel, we denote by $\mathcal{V}^l=\mathcal{V}_{r_l}^{\Theta_l}\cdots\mathcal{V}_{r_2}^{\Theta_2}\mathcal{V}_{r_1}^{\Theta_1}$ for $l\in\{1,2,\cdots,N\}$ and $\mathcal{V}^0=I$.

We divide the proof into serval steps:

Step 1. First prove that sequences $\{x_n\},\{\rho_n\},\{\xi_n\}$ and $\{y_n\}$ are bounded.

For each given $n\geq 1$, define the mapping $W_n : C\to C$ as:

$$W_n=\alpha_n\gamma f\frac{1}{t_n}\int_0^{t_n}T(s)ds+\beta_nI+((1-\beta_n)I-\alpha_nA)\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Nds.$$

Then we shall show that the mapping W_n is a contraction. Indeed for any $x,y\in C$, we have

$$\begin{aligned}\|W_n(x)-W_n(y)\| &= \Big\|\alpha_n\gamma f\Big(\frac{1}{t_n}\int_0^{t_n}T(s)xds\Big)+\beta_nx\\ &\quad+\Big((1-\beta_n)I-\alpha_nA\Big)\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Nxds\\ &\quad-\alpha_n\gamma f\Big(\frac{1}{t_n}\int_0^{t_n}T(s)yds\Big)-\beta_ny\\ &\quad+\Big((1-\beta_n)I-\alpha_nA\Big)\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Nyds\Big\|\\ &\leq \alpha_n\gamma\Big\|f\Big(\frac{1}{t_n}\int_0^{t_n}T(s)xds\Big)-f\Big(\frac{1}{t_n}\int_0^{t_n}T(s)yds\Big)\Big\|+\beta_n\|x-y\|\\ &\quad+\Big((1-\beta_n)I-\alpha_n\bar{\gamma}\Big)\frac{1}{t_n}\int_0^{t_n}\|T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Nxds\\ &\quad-T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Ny\|ds\\ &\leq \alpha_n\gamma h\|x-y\|+\beta_n\|x-y\|+(1-\beta_n-\alpha_n\bar{\gamma})\|x-y\|\\ &< \|x-y\|.\end{aligned}$$

Therefore, $W_n : C\to C$ is a contraction. Let $x_n\in C$ be the unique fixed point of W_n. Then

$$\begin{aligned}x_n &= \alpha_n\gamma f\Big(\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\Big)+\beta_nx_n\\ &\quad+\Big((1-\beta_n)I-\alpha_nA\Big)\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^Nx_nds\end{aligned}$$

is well-defined. Let $p\in\Omega$. Since $y_n=\mathcal{V}^Nx_n$, we have

$$\|y_n-p\|=\|\mathcal{V}^Nx_n-p\|\leq\|x_n-p\|. \tag{3.4}$$

Since $p \in VI(H, B, M)$ and $\rho_n = J_\lambda^M(I - \lambda B)\xi_n$, we have $p = J_\lambda^M(I - \lambda B)p$ and so

$$\begin{aligned}
\|\rho_n - p\| &= \|J_\lambda^M(I - \lambda B)\xi_n - J_\lambda^M(I - \lambda B)p\| \\
&\le \|(I - \lambda B)\xi_n - (I - \lambda B)p\| \\
&\le \|\xi_n - p\| \\
&= \|J_\lambda^M(I - \lambda B)y_n - J_\lambda^M(I - \lambda B)p\| \\
&\le \|y_n - p\| \\
&\le \|x_n - p\|. \qquad (3.5)
\end{aligned}$$

Let

$$u_n = \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$$

and

$$q_n = \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds.$$

Then we have

$$\begin{aligned}
\|u_n - p\| &= \Big\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\Big\| \\
&\le \frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - T(s)p\| ds \\
&\le \|x_n - p\|. \qquad (3.6)
\end{aligned}$$

Similarly, we have

$$\|q_n - p\| \le \|\rho_n - p\|. \qquad (3.7)$$

From (3.1)-(3.7), we have

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n\gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)q_n - p\| \\
&= \|\alpha_n\gamma(f(u_n) - f(p)) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(q_n - p) \\
&\quad + \alpha_n(\gamma f(p) - Ap)\| \\
&\le \alpha_n\gamma h\|u_n - p\| + \beta_n\|x_n - p\| + ((1 - \beta_n)I - \alpha_n\bar{\gamma})\|q_n - p\| \\
&\quad + \alpha_n\|\gamma f(p) - Ap\| \\
&\le \alpha_n\gamma h\|x_n - p\| + \beta_n\|x_n - p\| + ((1 - \beta_n)I - \alpha_n\bar{\gamma})\|x_n - p\| \\
&\quad + \alpha_n\|\gamma f(p) - Ap\|.
\end{aligned}$$

And so, we have

$$\|x_n - p\| \le \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|.$$

This implies that $\{x_n\}$ is a bounded sequence in H. Therefore $\{y_n\}, \{\rho_n\}, \{\xi_n\}, \{\gamma f(u_n)\}$ and $\{q_n\}$ are also bounded.

Step 2. Next, we prove that for each $0 \le s < \infty$,

$$\|x_n - T(s)x_n\| \to 0 (n \to \infty). \qquad (3.8)$$

Since $x_n = \alpha_n\gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)q_n$, we have

$$\|x_n - q_n\| \le \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|.$$

Hence

$$\|x_n - q_n\| \le \frac{\alpha_n}{1 - \beta_n}\|\gamma f(u_n) - Aq_n\|.$$

From $\alpha_n \to 0$, we have

$$\|x_n - q_n\| \to 0. \qquad (3.9)$$

Let

$$K = \Big\{w \in C : \|w - p\| \le \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|\Big\}.$$

Then K is a nonempty bounded closed and convex subset of C and $T(s)$-invariant. Since $\{x_n\} \subset K$, there exists $r > 0$ such that $K \subset B_r$. It follows from Lemma 2.11 that

$$\lim_{n\to\infty} \|q_n - T(s)q_n\| \to 0. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned}\|x_n - T(s)x_n\| &= \|x_n - q_n + q_n - T(s)q_n + T(s)q_n - T(s)x_n\| \\ &\le \|x_n - q_n\| + \|q_n - T(s)q_n\| + \|T(s)q_n - T(s)x_n\| \\ &\le \|x_n - q_n\| + \|q_n - T(s)q_n\| + \|q_n - x_n\| \\ &\to 0.\end{aligned}$$

Step 3. Next, we prove that

$$\lim_{n\to\infty} \|\mathcal{V}^{l+1}x_n - \mathcal{V}^l x_n\| = 0,$$

for all $l \in \{0, 1, \cdots, N-1\}$. Especially,

$$\lim_{n\to\infty} \|\mathcal{V}^N x_n - x_n\| = \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.11}$$

In fact, since $\mathcal{V}_{r_{l+1}}^{\Theta_{l+1}}$ is firmly nonexpansive, for any given $p \in \Omega$ and $l \in \{0, 1, \cdots, N-1\}$, we have

$$\begin{aligned}\|\mathcal{V}^{l+1}x_n - p\|^2 &= \|\mathcal{V}_{r_{l+1}}^{\Theta_{l+1}}(\mathcal{V}^l x_n) - \mathcal{V}_{r_{l+1}}^{\Theta_{l+1}}p\|^2 \\ &\le \langle \mathcal{V}_{r_{l+1}}^{\Theta_{l+1}}(\mathcal{V}^l x_n) - p, \mathcal{V}^l x_n - p\rangle \\ &= \langle \mathcal{V}^{l+1}x_n - p, \mathcal{V}^l x_n - p\rangle \\ &= \frac{1}{2}\Big(\|\mathcal{V}^{l+1}x_n - p\|^2 + \|\mathcal{V}^l x_n - p\|^2 - \|\mathcal{V}^l x_n - \mathcal{V}^{l+1}x_n\|^2\Big).\end{aligned}$$

It implies that

$$\|\mathcal{V}^{l+1}x_n - p\|^2 \le \|x_n - p\|^2 - \|\mathcal{V}^l x_n - \mathcal{V}^{l+1}x_n\|^2. \tag{3.12}$$

On the other hane, from (3.1), we have

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n\gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - p\|^2 \\ &= \|\alpha_n(\gamma f(u_n) - Ap) + \beta_n(x_n - q_n) + (I - \alpha_n A)(q_n - p)\|^2 \\ &\le \|(I - \alpha_n A)(q_n - p) + \beta_n(x_n - q_n)\|^2 \\ &\quad + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle \\ &\le [\|(I - \alpha_n A)(q_n - p)\| + \beta_n\|x_n - q_n\|]^2 \\ &\quad + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle \\ &\le [(I - \alpha_n\bar{\gamma})\|\rho_n - p\| + \beta_n\|x_n - q_n\|]^2 \\ &\quad + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle \\ &\le (1 - \alpha_n\bar{\gamma})^2\|\rho_n - p\|^2 + \beta_n^2\|x_n - q_n\|^2 \\ &\quad + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\|x_n - q_n\| \\ &\quad + 2\alpha_n\|\gamma f(u_n) - Ap\|\|x_n - p\|.\end{aligned} \tag{3.13}$$

And note that

$$\|\rho_n - p\| \le \|\xi_n - p\| \le \|\mathcal{V}^N x_n - p\| \le \|\mathcal{V}^{l+1}x_n - p\|, \quad \forall l \in \{0, 1, \cdots, N-1\}.$$

Substituting (3.12) into (3.13), it yields

$$\begin{aligned}\|x_n - p\|^2 &\le (1 - \alpha_n\bar{\gamma})^2\{\|x_n - p\|^2 - \|\mathcal{V}^l x_n - \mathcal{V}^{l+1}x_n\|^2\} + \beta_n^2\|x_n - q_n\|^2 \\ &\quad + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\|x_n - p\| \\ &= (1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2)\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|\mathcal{V}^l x_n - \mathcal{V}^{l+1}x_n\|^2 \\ &\quad + \beta_n^2\|x_n - q_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\|x_n - q_n\| \\ &\quad + 2\alpha_n\|\gamma f(u_n) - Ap\|\|x_n - p\|.\end{aligned}$$

Simplifying the above inequaity, we have

$$\begin{aligned}(1-\alpha_n\bar{\gamma})^2\|\mathcal{V}^l x_n-\mathcal{V}^{l+1}x_n\|^2 \leq& (1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2-\|x_n-p\|^2+\beta_n^2\|x_n-q_n\|^2\\ &+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\|x_n-q_n\|\\ &+2\alpha_n\|\gamma f(u_n)-Ap\|\|x_n-p\|.\end{aligned}$$

Since $\alpha_n\to 0, \|x_n-q_n\|\to 0$ from condition (iv). Hence we have

$$\lim_{n\to\infty}\|\mathcal{V}^{l+1}x_n-\mathcal{V}^l x_n\|=0,$$

for all $l\in\{0,1,\cdots,N-1\}$.

Step 4. Now, we prove that for any given $p\in\Omega$,

$$\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{3.14}$$

In fact, it follows from (3.5) that

$$\begin{aligned}\|\rho_n-p\|^2\leq\|\xi_n-p\|^2&=\|J_\lambda^M(I-\lambda B)y_n-J_\lambda^M(I-\lambda B)p\|^2\\ &\leq\|(I-\lambda B)y_n-(I-\lambda B)p\|^2\\ &\leq\|y_n-p\|^2-2\lambda\langle y_n-p,By_n-Bp\rangle+\lambda^2\|By_n-Bp\|^2\\ &\leq\|y_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2\\ &\leq\|x_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2.\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.13), we obtain

$$\begin{aligned}\|x_n-p\|^2\leq&(1-\alpha_n\bar{\gamma})^2\{\|x_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2\}\\ &+\beta_n^2\|x_n-q_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\|x_n-q_n\|\\ &+2\alpha_n\|\gamma f(u_n)-Ap\|\|x_n-p\|.\end{aligned}$$

Simplifying this, we have

$$\begin{aligned}(1-\alpha_n\bar{\gamma})^2\lambda(2\alpha-\lambda)\|By_n-Bp\|^2\leq&(1+\alpha_n(\bar{\gamma})^2)\|x_n-p\|^2-\|x_n-p\|^2\\ &+\beta_n^2\|x_n-q_n\|^2\\ &+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\|x_n-q_n\|\\ &+2\alpha_n\|\gamma f(u_n)-Ap\|\|x_n-p\|\\ =&\alpha_n(\bar{\gamma})^2\|x_n-p\|^2+\beta_n^2\|x_n-q_n\|^2\\ &+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\|x_n-q_n\|\\ &+2\alpha_n\|\gamma f(u_n)-Ap\|\|x_n-p\|.\end{aligned}$$

Since $\alpha_n\to 0$, $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, $\|x_n-q_n\|\to 0$, and $\{\gamma f(u_n)-Ap\}$, $\{x_n\}$ are bounded, these imply that

$$\lim_{n\to\infty}\|By_n-Bp\|=0.$$

Step 5. Next, we prove that

$$\begin{cases}\lim_{n\to\infty}\|y_n-\rho_n\|=0,\\ \lim_{n\to\infty}\|x_n-\rho_n\|=0.\end{cases} \tag{3.16}$$

In fact, since

$$\|y_n-\rho_n\|\leq\|y_n-\xi_n\|+\|\xi_n-\rho_n\|,$$

it is sufficient to prove $\|y_n - \xi_n\| \to 0$ and $\|\xi_n - \rho_n\| \to 0$. First we have to prove that $\|y_n - \xi_n\| \to 0$. In fact, since

$$\begin{aligned}\|\xi_n - p\|^2 &= \|J_\lambda^M (I - \lambda B)y_n - J_\lambda^M (I - \lambda B)p\|^2 \\ &\le \langle y_n - \lambda B y_n - (p - \lambda B p), \xi_n - p\rangle \\ &= \frac{1}{2}\Big\{\|y_n - \lambda B y_n - (p - \lambda B p)\|^2 + \|\xi_n - p\|^2 \\ &\quad - \|y_n - \lambda B y_n - (p - \lambda B p) - (\xi_n - p)\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n - \lambda(By_n - Bp)\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n\|^2 \\ &\quad + 2\lambda\langle y_n - \xi_n, By_n - Bp\rangle - \lambda^2\|By_n - Bp\|^2\Big\},\end{aligned}$$

we have

$$\|\xi_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, By_n - Bp\rangle - \lambda^2\|By_n - Bp\|^2. \tag{3.17}$$

Substituting (3.17) into (3.13), it yields that

$$\begin{aligned}\|x_n - p\|^2 \le (1 - \alpha_n\bar{\gamma})^2\{&\|y_n - p\|^2 - \|y_n - \xi_n\|^2 \\ &+ 2\lambda\langle y_n - \xi_n, By_n - Bp\rangle - \lambda^2\|By_n - Bp\|^2\} \\ &+ \beta_n^2\|x_n - q_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\|x_n - q_n\| \\ &+ 2\alpha_n\|\gamma f(u_n) - Ap\|\|x_n - p\|.\end{aligned}$$

Simplifying this, we have

$$\begin{aligned}(1 - \alpha\bar{\gamma})^2\|y_n - \xi_n\|^2 \le\ &\alpha_n\bar{\gamma}^2\|x_n - p\|^2 + 2(1 - \alpha_n\bar{\gamma}^2)\lambda\langle y_n - \xi_n, By_n - Bp\rangle \\ &- (1 - \alpha\bar{\gamma})^2\lambda^2\|By_n - Bp\|^2 \\ &+ \beta_n^2\|x_n - q_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\|x_n - q_n\| \\ &+ 2\alpha_n\|\gamma f(u_n) - Ap\|\|x_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, $\|x_n - q_n\| \to 0$, $\|By_n - Bp\| \to 0$ $(n \to \infty)$ and $\{\gamma f(u_n) - Ap\}$, $\{x_n\}$, $\{\rho_n\}$ are bounded, these imply that

$$\|y_n - \xi_n\| \to 0 (n \to \infty).$$

Next we prove that

$$\lim_{n\to\infty} \|\xi_n - \rho_n\| = 0. \tag{3.18}$$

Since

$$\begin{aligned}\|\xi_n - \rho_n\| &= \|J_\lambda^M (I - \lambda B)y_n - J_\lambda^M (I - \lambda B)\xi_n\| \\ &\le \|y_n - \xi_n\| \\ &\to 0,\end{aligned}$$

we have

$$\begin{aligned}\|y_n - \rho_n\| &= \|y_n - \xi_n + \xi_n - \rho_n\| \\ &\le \|y_n - \xi_n\| + \|\xi_n - \rho_n\| \\ &\to 0.\end{aligned}$$

This together with (3.11) shows that $\|x_n - \rho_n\| \to 0$.

Step 6. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. In this case, we will prove that

$$x^* \in \Omega := F(\Im) \cap GMEP(\Theta) \cap VI(H, B, M)$$

and x^* is the unique solution of the variational inequality (3.2).

We first prove that $x^* \in F(\Im)$. From Lemma 2.12 and Step 2, we obtain $x^* \in F(\Im)$.

Next, we prove that

$$x^* \in GMEP(\Theta) := \bigcap_{l=1}^{N} GMEP(\Theta_l).$$

Since $x_{n_k} \rightharpoonup x^*$ and noting Step 3, without loss of generality, we may assume that $\mathcal{V}^l x_{n_k} \rightharpoonup x^*$, for all $l \in \{0, 1, 2, \cdots, N-1\}$. Hence for any $x \in C$, we have

$$\Big\langle \frac{K'_{l+1}(\mathcal{V}^{l+1}x_{n_k}) - K'_{l+1}(\mathcal{V}^l x_{n_k})}{r_{l+1}}, \eta_{l+1}(x, \mathcal{V}^{l+1}x_{n_k})\Big\rangle \geq -\Theta_{l+1}(\mathcal{V}^{l+1}x_{n_k}) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathcal{V}^{l+1}x_{n_k}).$$

By the assumptions and the condition $(C2)$, we know that the function φ_i and the mapping $x \to (-\Theta_{l+1}(x, y))$ are convex and lower semi-continuous. Hence they are weakly lower semi-continuous. These together with

$$\frac{K'_{l+1}(\mathcal{V}^{l+1}x_{n_k}) - K'_{l+1}(\mathcal{V}^l x_{n_k})}{r_{l+1}} \to 0$$

and $\mathcal{V}^{l+1}x_{n_k} \rightharpoonup x^*$, we have

$$\begin{aligned} 0 &= \liminf_{k\to\infty} \Big\langle \frac{K'_{l+1}(\mathcal{V}^{l+1}x_{n_k}) - K'_{l+1}(\mathcal{V}^l x_{n_k})}{r_{l+1}}, \eta_{l+1}(x, \mathcal{V}^{l+1}x_{n_k})\Big\rangle \\ &\geq \liminf_{k\to\infty} \big\{ - \Theta_{l+1}(\mathcal{V}^{l+1}x_{n_k}) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathcal{V}^{l+1}x_{n_k})\big\}. \end{aligned}$$

This implies that for $x \in C$ and $l \in \{0, 1, \cdots, N-1\}$,

$$\Theta_{l+1}(x^*, x) + \varphi_{l+1}(x) - \varphi_{l+1}(x^*) \geq 0.$$

Hence, we have

$$x^* \in \bigcap_{l=1}^{N} GMEP(\Theta_l) = GMEP(\Theta).$$

Now, we prove that $x^* \in VI(H, B, M)$. In fact, since B is α-inverse strongly monotone, it follows from Proposition 1.2 that B is an $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping and $D(B) = H$, (where $D(B)$ is the domain of B). From Lemma 2.5 that $M + B$ is maximal monotone. Let $(\upsilon, g) \in G(M + B)$, *i.e.,* $g - B\upsilon \in M\upsilon$. Since $x_{n_k} \rightharpoonup x^*$ and noting Step 3, without loss of generality, we may assume that $\mathcal{V}^l x_{n_k} \rightharpoonup x^*$, in particular we have $y_{n_k} = \mathcal{V}^N x_{n_k} \rightharpoonup x^*$. From $\|y_n - \rho_n\| \to 0$, we can prove that $\rho_{n_k} \rightharpoonup x^*$. Again since $\rho_{n_k} = J_\lambda^M(I - \lambda B)\xi_{n_k}$, we have $\xi_{n_k} - \lambda B\xi_{n_k} \in (I + \lambda M)\rho_{n_k}$ *i.e.,* $\frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k} - \lambda B\xi_{n_k}) \in M\rho_{n_k}$. By virtue of the maximal monotonicity of M, we have

$$\langle \upsilon - \rho_{n_k}, g - B\upsilon - \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k} - \lambda B\xi_{n_k})\rangle \geq 0,$$

and so

$$\begin{aligned} \langle \upsilon - \rho_{n_k}, g\rangle &\geq \langle \upsilon - \rho_{n_k}, B\upsilon + \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k} - \lambda B\xi_{n_k})\rangle \\ &= \langle \upsilon - \rho_{n_k}, B\upsilon - B\rho_{n_k} + B\rho_{n_k} - B\xi_{n_k} + \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k})\rangle \\ &\geq 0 + \langle \upsilon - \rho_{n_k}, B\rho_{n_k} - B\xi_{n_k}\rangle + \langle \upsilon - \rho_{n_k}, \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k})\rangle. \end{aligned}$$

Since $\|\xi_n - \rho_n\| \to 0, \|B\xi_n - B\rho_n\| \to 0$ and $\rho_{n_k} \rightharpoonup x^*$, we have

$$\lim_{k\to\infty} \langle \upsilon - \rho_{n_k}, g\rangle = \langle \upsilon - x^*, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $M + B$ that $\theta \in (M + B)(x^*)$, that is, $x^* \in VI(H, B, M)$. Consequently, we have

$$x^* \in \Omega.$$

Finally, we prove that x^* is the unique solution of variational inequality (3.2).

We first prove that $x_{n_k} \to x^*$. Since for all $z \in \Omega$,

$$\begin{aligned}
\|x_n - z\|^2 &= \langle x_n - z, x_n - z\rangle \\
&= \langle \alpha_n \gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - z, x_n - z\rangle \\
&= \langle \alpha_n(\gamma f(u_n) - Az) + \beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(q_n - z), x_n - z\rangle \\
&\le \alpha_n \langle \gamma f(u_n) - Az, x_n - z\rangle + \beta_n \|x_n - z\|^2 \\
&\quad + (1 - \beta_n - \alpha_n \bar{\gamma})\|q_n - z\|\|x_n - z\| \\
&= (1 - \alpha_n \bar{\gamma})\|x_n - z\|^2 + \alpha_n \langle \gamma f(u_n) - Az, x_n - z\rangle, \qquad (3.19)
\end{aligned}$$

it follows that

$$\begin{aligned}
\|x_n - z\|^2 &\le \frac{1}{\bar{\gamma}}\langle \gamma f(u_n) - Az, x_n - z\rangle \\
&\le \frac{1}{\bar{\gamma}}\langle \gamma f(u_n) - \gamma f(z) + \gamma f(z) - Az, x_n - z\rangle \\
&\le \frac{1}{\bar{\gamma}}\Big\{\gamma h\|x_n - z\|^2 + \langle \gamma f(z) - Az, x_n - z\rangle\Big\}.
\end{aligned}$$

Therefore

$$\|x_n - z\|^2 \le \frac{1}{\bar{\gamma} - \gamma h}\Big\langle \gamma f(z) - Az, x_n - z\Big\rangle. \qquad (3.20)$$

Now replacing n in (3.20) with n_k and letting $k \to \infty$ and $x_{n_k} \rightharpoonup x^*$, we have $x_{n_k} \to x^*$.

On the other hand, since

$$x_n = \alpha_n \gamma f\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\Big) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds,$$

we have

$$\begin{aligned}
\alpha_n(A - \gamma f)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\Big) &= -\Big\{(1-\beta_n)(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds)\Big\} \\
&\quad + \alpha_n A \frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds \\
&= -(1-\beta_n)\Big(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_\lambda^M(I - \lambda B))^2 \mathcal{V}^N ds\Big)x_n \\
&\quad + \alpha_n A \frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds.
\end{aligned}$$

Hence for any $z \in \Omega$, we have

$$\begin{aligned}
&\alpha_n \Big\langle (A - \gamma f)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\Big), x_n - z\Big\rangle \\
&= -(1-\beta_n)\Big\langle \Big(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_\lambda^M(I - \lambda B))^2 \mathcal{V}^N ds\Big)x_n \\
&\quad - \Big(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_\lambda^M(I - \lambda B))^2 \mathcal{V}^N ds\Big)z, x_n - z\Big\rangle \\
&\quad + \alpha_n \Big\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds, x_n - z\Big\rangle.
\end{aligned}$$

Then

$$\begin{aligned}&\Big\langle(A-\gamma f)\Big(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\Big),x_n-z\Big\rangle\\&=-\frac{1-\beta_n}{\alpha_n}\Big\langle\Big(I-\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^N ds\Big)x_n\\&\quad-\Big(I-\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^N ds\Big)z,x_n-z\Big\rangle\\&\quad+\Big\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n-T(s)\rho_n)ds,x_n-z\Big\rangle.\end{aligned}\tag{3.21}$$

It is easily seen that $I-\frac{1}{t_n}\int_0^{t_n}T(s)(J_\lambda^M(I-\lambda B))^2\mathcal{V}^N ds$ is monotone. Thus from (3.21) we have that

$$\Big\langle(A-\gamma f)\Big(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\Big),x_n-z\Big\rangle\le\Big\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n-T(s)\rho_n)ds,x_n-z\Big\rangle.\tag{3.22}$$

Now in (3.22) replacing n by n_k and letting $k\to\infty$ and $x_{n_k}\to x^*$, from Step 3 and Step 5, we have

$$\|x_n-\rho_n\|\to 0.$$

Then

$$\frac{1}{t_{n_k}}\int_0^{t_{n_k}}(T(s)x_{n_k}-T(s)\rho_{n_k})ds\to 0.$$

So, we have for all $z\in\Omega$,

$$\langle(A-\gamma f)x^*,x^*-z\rangle\le 0.$$

That is, x^* is the solution of the variational inequality (3.2). It follows from [19] that x^* is a unique solution of (3.2).

Step 7. Next, we prove that

$$\limsup_{n\to\infty}\langle\gamma f(x^*)-Ax^*,x_n-x^*\rangle\le 0.\tag{3.23}$$

First, we prove that

$$\limsup_{n\to\infty}\Big\langle\frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds-x^*,\gamma f(x^*)-Ax^*\Big\rangle\le 0.\tag{3.24}$$

Indeed, there exists a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\Big\langle\frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds-x^*,\gamma f(x^*)-Ax^*\Big\rangle\\&=\lim_{i\to\infty}\Big\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)\rho_{n_i}ds-x^*,\gamma f(x^*)-Ax^*\Big\rangle.\end{aligned}$$

We may also assume that $\rho_{n_i}\rightharpoonup w$. This together with (3.9) and (3.16) show that

$$q_{n_i}=\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)\rho_{n_i}ds\to w.$$

Since $\|x_n-q_n\|\to 0$, we have $x_{n_i}\rightharpoonup w$. Again by the same way as given in Step 6, we can prove that $w\in\Omega$. Hence, we have

$$\begin{aligned}&\limsup_{n\to\infty}\Big\langle\frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds-x^*,\gamma f(x^*)-Ax^*\Big\rangle\\&=\lim_{i\to\infty}\Big\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)\rho_{n_i}ds-x^*,\gamma f(x^*)-Ax^*\Big\rangle\\&=\lim_{i\to\infty}\langle q_{n_i}-x^*,\gamma f(x^*)-Ax^*\rangle\\&=\langle w-x^*,\gamma f(x^*)-Ax^*\rangle\\&\le 0.\end{aligned}$$

On the other hand, from $\|x_n - q_n\| \to 0$ and (3.24), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle &= \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - q_n + q_n - x^*\rangle \\
&\le \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - q_n\rangle \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, q_n - x^*\rangle \\
&\le 0.
\end{aligned}$$

Step 8. Finally, we prove that $x_n \to x^*$. Indeed from (3.1),(3.5) and (3.7), we have

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|\alpha_n(\gamma f(u_n) - Ax^*) - \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(q_n - x^*)\|^2 \\
&\le \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(q_n - x^*)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(u_n) - Ax^*, x_n - x^*\rangle \\
&\le [\beta_n\|x_n - x^*\| + \|((1-\beta_n)I - \alpha_n A)(q_n - x^*)\|]^2 \\
&\quad + 2\alpha_n\gamma\langle f(u_n) - f(x^*), x_n - x^*\rangle + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \\
&\le [\beta_n\|x_n - x^*\| + \|(1-\beta_n - \alpha_n\bar{\gamma})\|\rho_n - x^*\|]^2 \\
&\quad + 2\alpha_n\gamma h\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \\
&\le ((1-\alpha_n\bar{\gamma})^2 + 2\alpha_n\gamma h)\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle.
\end{aligned}$$

This implies that

$$\|x_n - x^*\|^2 \le \frac{2}{2(\bar{\gamma} - \gamma h) - \bar{\gamma}^2}\Big\langle \gamma f(x^*) - Ax^*, x_n - x^*\Big\rangle. \tag{3.25}$$

Combining (3.23) and (3.25), we obtain that $x_n \to x^*$. This completes the proof. □

Acknowledgements

The authors wish to thank the anonymous referees for the careful reports and valuable comments that helped significantly improve the presentation of the paper.

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