Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 3045–3061
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26001.3186

Existence of solutions for Caputo sequential fractional differential equations with integral boundary conditions

Murugesan Manigandan[a], Muthaiah Subramanian[b], Thangaraj Nandha Gopal[a], B. Unyong[c,*]

[a] Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore - 641 020, Tamilnadu, India

[b] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore - 641 407, Tamilnadu, India

[c] Department of Mathematics, Phuket Rajabhat University, Phuket - 83000, Thailand

(Communicated by Ali Jabbari)

Abstract

The purpose of this paper is to investigate the existence and uniqueness of solutions to the Caputo sequential fractional differential equations and inclusions with integral boundary conditions. When it comes to proving the existence of solutions, the Krasnoselskii's fixed point theorem is employed. Further, the Banach's contraction principle and the Leray-Schauder alternative are employed to prove the uniqueness of the results. Further, for the multi-valued case, we employ the nonlinear alternative for Kakutani maps, and Convitz and Nadler's fixed point theorem. We emphasize our results with numerical examples.

Keywords: Fractional Differential Equations (FDEs), Fractional Differential Inclusions, Existence, Integral boundary conditions (IBCs), Fixed point
2020 MSC: 26A33, 34A08, 34B15

1 Introduction

Fractional differential equations (FDEs) have been used in a variety of domains, including biology, applied science, physics, and bioengineering. Fractional derivatives (FDs) of FDEs include Riemann-Liouville, Grunwald-Letnikov, Caputo, Hadamard, and others. For foundational notions in the theory of fractional calculus (FCs) and FDEs, we recommend the article and books [11, 17, 18, 23] as well as the sources given therein [2, 5, 29].

FCs have gotten a lot of attention and popularity in the last few decades. Its extensive theoretical development and applicability in a variety of technical sciences and technical fields are easy to see. Aerodynamics, operations research, biological science, and other fields are examples. FCs have been proven to be an effective modeling technique for a variety of real-world problems [6, 9, 10, 21, 28].

Many researchers have recently studied boundary value problems (BVPs), introducing a range of circumstances such as multi-point, classical, non-local, periodic/anti-periodic, fractional order, and integral boundary conditions [1, 3, 4, 8, 19, 20, 22, 25].

*Corresponding author
Email addresses: yogimani22@gmail.com (Murugesan Manigandan), subramanianmcbe@gmail.com (Muthaiah Subramanian), nandhu792002@yahoo.co.in (Thangaraj Nandha Gopal), bundit.u@pkru.ac.th (B. Unyong)

Received: January 2022 *Accepted:* June 2022

In the study of nonlinear systems and stochastic processes, differential inclusions have proven to be extremely useful. See [1, 4, 8, 19, 24, 26] for some recent BVPs results for fractional differential inclusions.

For instance, multi-point boundary value problems for fractional differential equations were studied by Mujeeb ur Rehman et al [27], is of the form

$$\begin{aligned} {}^{c}\mathcal{D}^{\alpha}y(\tau) &= g(\tau, y(\tau), \mathcal{D}^{\beta}y(\tau)), \qquad \tau \in [0,1], \\ y(1) \;=\; 0, \;\; \mathcal{D}^{\beta}y(1) &- \sum_{j=1}^{k-2}\omega_j \mathcal{D}^{\beta}y(\xi_j) = y_0, \end{aligned}$$

where ${}^{c}\mathcal{D}^{\alpha}$ is the Caputo fractional derivative (CFDs) of order $1 < \alpha \leq 2$, and g: $[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and ω_j $(j = 1, 2, ..., k-2)$ are non-negative real constants.

The boundary value problem (BVP) of the nonlinear fractional differential equation of order $q \in (1, 2]$ with three-point integral boundary conditions (IBCs) was recently solved by the authors in [7], is given by

$$\begin{aligned} &{}^{c}\mathcal{D}^{q}\mathfrak{x}(\tau) = \mathfrak{f}(\tau, \mathfrak{x}(\tau)), \quad 0 < \tau < 1, \quad 1 < q \leq 2, \\ &\mathfrak{x}(0) = 0, \mathfrak{x}(1) = \alpha\int_0^{\eta}\mathfrak{x}(\varrho)d\varrho, \quad 0 < \eta < 1, \end{aligned}$$

where ${}^{c}\mathfrak{D}^{q}$ is the Caputo fractional derivative, $\mathfrak{f} : [0,1]\times\mathbb{R}\to\mathbb{R}$, α is a real positive number .

In [9], the authors discussed a existence of solution of a fractional differential equation of the form

$$({}^{c}\mathfrak{D}^{\varpi} + \varphi{}^{c}\mathfrak{D}^{\varpi-1})\mathfrak{x}(\tau) = \mathfrak{f}(\tau, \mathfrak{x}(\tau)), \quad 2 < \tau \leq 3,$$

supplemented with IBCs

$$\mathfrak{x}(0) = 0, \quad \mathfrak{x}'(0) = 0, \quad \mathfrak{x}(\zeta) = a\int_o^{\eta}\frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\mathfrak{x}(\mathfrak{s})d\mathfrak{s}, \beta > 0,$$

where ${}^{c}\mathfrak{D}^{\varpi}$ is the Caputo fractional derivative of order $\varpi, 0 < \eta < \zeta < 1$, $\mathfrak{f} : [0,1]\times\mathbb{R}\to\mathbb{R}$,and φ, a are non-negative number. The existence and uniqueness results are established using the Banach's contraction mapping principle, Krasnoselkii's fixed point theorem, and the Leray-Schauder nonlinear alternative. The boundary condition implies that the value of the unknown function at any point $\zeta \in (\eta, 1)$ is proportional to the unknown function's the Riemann-Liouville fractional integral. To the reader, we offer a series of articles on coupled systems of fractional differential equations.

More recently, Bashir Ahmad et al. [8], studied an sequential fractional differential equations and inclusions

$$\begin{aligned} ({}^{c}\mathfrak{D}^{\varpi} + \varphi{}^{c}\mathfrak{D}^{\varpi-1})\mathfrak{x}(\tau) = \mathfrak{f}(\tau, \mathfrak{x}(\tau), {}^{c}\mathfrak{D}^{\delta}_{0+}\mathfrak{x}(\tau), \mathcal{I}^{\gamma}\mathfrak{x}(\tau)), \tau \in \mathcal{J} := [0,1], \\ ({}^{c}\mathfrak{D}^{\varpi} + \varphi{}^{c}\mathfrak{D}^{\varpi-1})\mathfrak{x}(\tau) \in \mathfrak{F}(\tau, \mathfrak{x}(\tau), {}^{c}\mathfrak{D}^{\delta}_{0+}\mathfrak{x}(\tau), \mathcal{I}^{\gamma}\mathfrak{x}(\tau)), \tau \in \mathcal{J} := [0,1], \end{aligned}$$

with multi-point boundary conditions

$$\mathfrak{x}(0) = 0, \quad \mathfrak{x}'(0) = 0, \quad \sum_{i=1}^{m} a_i\mathfrak{x}(\zeta_i) = \lambda\int_0^{\eta}\frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\mathfrak{x}(\vartheta)d\vartheta,$$

where ${}^{c}\mathfrak{D}^{\varpi}$ denotes the Caputo derivatives of fractional order, $2 < \varpi \leq 3, 0 < \delta, \gamma < 1, \varphi > 0, \beta > 0$ $\mathcal{I}^{(\cdot)}$ denotes the left Riemann -Liouville integral of fractional order (.), $\mathfrak{f} : [0,1]\times\mathbb{R}^3\to\mathbb{R}$ is given continuous function, $\mathfrak{F} : [0,1]\times\mathbb{R}^3\to\mathbb{R}$ is a multi-valued map and $\lambda, a_i, i = 1, 2, \cdot m$ are constant. Authors discussed both existence and uniqueness results via standard fixed point theorems for single-valued and multi-valued maps to obtain the desired results.

On the basis of our current understanding, we introduce and investigate the existence of solutions to the Caputo sequential fractional differential equation and inclusion

$$({}^{c}\mathfrak{D}^{\varpi} + \varphi{}^{c}\mathfrak{D}^{\varpi-1})\mathfrak{x}(\tau) = \mathfrak{f}(\tau, \mathfrak{x}(\tau)), \tau \in \mathcal{J} := [0,1], \tag{1.1}$$

$$({}^{c}\mathfrak{D}^{\varpi} + \varphi{}^{c}\mathfrak{D}^{\varpi-1})\mathfrak{x}(\tau) \in \mathfrak{F}(\tau, \mathfrak{x}(\tau)), \tau \in \mathcal{J} := [0,1], \tag{1.2}$$

with the IBCs

$$\mathfrak{x}(0) = 0, \quad \mathfrak{x}'(0) = 0, \quad \mathfrak{x}''(0) = 0, \quad \mathfrak{x}(1) + \mathfrak{x}(\nu) = \lambda\int_0^{\eta}\mathfrak{x}(\vartheta)d\vartheta, \tag{1.3}$$

where ${}^{c}\mathcal{D}^{\varpi}$ denotes the Caputo fractional derivative of order ϖ, $\varphi > 0$, $\nu \in (0,1]$, $\mathfrak{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and λ, φ are appropriate positive real constants.

We prove the existence and uniqueness of the systems (1.1) and (1.3) in this work using basic concepts from fixed point theory. The existence and uniqueness results are obtained using the Banach contraction mapping concept, whereas the existence result is obtained using the Leray-Schauder method. (1.2-1.3) while using Covitz and Nadler's fixed point theorem and the nonlinear alternative for Kakutani maps. It is essential to understand the concepts between fractional derivatives and non-sequential Riemann-Liouville derivatives. If you're interested in some recent work on sequential fractional differential equations, see [19] and [9]. The overview of this article is given as follows: Section 2, discusses the basic definitions of fractional calculus. On the other hand, an auxiliary finding about the linear version of (1.1) and (1.3) is explained. Section 3, establish the existence and uniqueness of the given problem, the Banach fixed point theorem and the Leray-Schauder alternative are utilized. Section 4 shows the existence of convex and non-convex valued maps in the system (1.2) - (1.3) by applying a nonlinear alternative of Covitz and Nadler's fixed point theorem. Section 5 gives examples of direct results.

2 Preliminaries

Before providing an auxiliary lemma, we shall go over some of the fundamental concepts of FCs in greater detail [13, 17, 18, 23]. Let $(\mathcal{S}, \|\cdot\|)$ be a normed space and that $\mathcal{U}_{cl}(\mathcal{S}) = \{\mathcal{A}_1 \in \mathcal{U}(\mathcal{S}) : \mathcal{A}_1 \text{ is closed }\}$, $\mathcal{U}_{c,cp}(\mathcal{S}) = \{\mathcal{A}_1 \in \mathcal{U}(\mathcal{S}) : \mathcal{A}_1 \text{ is convex and compact}\}$. A multi-valued map $\mathcal{W} : \mathcal{S} \to \mathcal{U}(\mathcal{S})$ is

(a) convex valued if $\mathcal{W}(s)$ is convex $\forall s \in \mathcal{S}$;
(b) upper semi-continuous (u.s.c.) on $\mathcal{S}$ if, for each $w_0 \in \mathcal{S}$; the set $\mathcal{W}(w_0)$ is a non-empty closed subset of $\mathcal{S}$ and if, for each open set $\mathcal{T}$ of $\mathcal{S}$ containing $\mathcal{W}(w_0)$, there exists an open neighborhood $\mathcal{T}_0$ of w_0 such that $\mathcal{W}(\mathcal{T}_0) \subset \mathcal{T}$;
(c) lower semi-continuous (l.s.c.) if the set $\{m \in \mathcal{S} : \mathcal{W}(m) \cap \mathcal{A} \neq \oslash\}$ is open for any open set $\mathcal{A}$ in $\mathcal{F}$;
(d) completely continuous (c.c) if $\mathcal{W}(\mathcal{A})$ is relatively compact (r.c) for every $\mathcal{A} \in \mathcal{U}_b(\mathcal{S}) = \{\mathcal{A}_1 \in \mathcal{U}(\mathcal{S}) : \mathcal{A}_1 \text{ is bounded}\}$.

A map $\mathcal{W} : [0,1] \to \mathcal{U}_{cl}(\mathbb{R})$ of multi-valued is said to be measurable if, for every $m \in \mathbb{R}$, the function $\tau \longmapsto d(m, \mathcal{W}(\tau)) = \inf\{|m-k| : k \in \mathcal{W}(\tau)\}$ is measurable. A multi-valued map $\mathcal{W} : [0,1] \times \mathbb{R} \to \mathcal{U}(\mathbb{R})$ is said to be Caratheodory if

(i) $\tau \longmapsto \mathcal{W}(\tau, s)$ is measurable for each $s \in \mathbb{R}$;
(ii) $s \longmapsto \mathcal{W}(\tau, s)$ is u.s.c for almost all $\tau \in [0,1]$.

Definition 2.1. The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^{\alpha} f(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{f(s)}{(\tau - s)^{1-\alpha}} ds, \tau > 0, \alpha > 0.$$

Provided the right-hand side is point-wise defined on $[0.\infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e^{-\tau} d\tau$.

Definition 2.2. The (R-L) fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$ is defined as

$$D_{0+}^{\alpha} f(\tau) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\tau}\right)^n \int_0^{\tau} (\tau - s)^{n-\alpha-1} f(s) ds, \tau > 0,$$

where the function k has $\mathcal{AC}$ derivative up to order $(n-1)$.

Definition 2.3. The (C-D) of order $r \in [n-1, n)$ for a function $f : [0,\infty) \to (\mathbb{R})$ can be written as

$${}^{c}D_{0+}^{r} f(\tau) = D_{0+}^{r} \left(f(\tau) - \sum_{k=0}^{n-1} \frac{\tau^k}{k!} f^{(k)}(0) \right), \tau > 0, n-1 < r < n.$$

Note that the CFDs of order $r \in [n-1, n)$ exist almost everywhere on $[0,\infty)$ if $f \in AC^n([0,\infty), (\mathbb{R}))$.

Remark 2.4. If $f \in C^n[0,\infty)$, then

$${}^{c}D_{0+}^{r} f(\tau) = \frac{1}{\Gamma(n-r)} \int_0^{\tau} \frac{f^{(n)}(s)}{(\tau - s)^{r+1-n}} ds = I^{n-r} f(n)(\tau), \tau > 0, n-1 < r < n.$$

The linear form of the problem (1.1)-(1.3) is stated by the following lemma.

Lemma 2.5. For $h \in \mathcal{C}([0,1],\mathbb{R})$, is a solution of linear sequential fractional differential equation

$$({}^{c}\mathcal{D}^{\varpi} + \varphi {}^{c}\mathcal{D}^{\varpi-1})\mathfrak{x}(\tau) = h(\tau), \tag{2.1}$$

subject to the BCs (1.3) if and only if

$$\begin{aligned}\mathfrak{x}(\tau) = \Bigg[& \frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \Bigg\{ \lambda \int_0^{\eta} \left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\tau,\mathfrak{x}(\tau)) d\varsigma \right) d\varrho \right) d\vartheta \\ & - \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\tau,\mathfrak{x}(\tau)) d\varsigma \right) d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\tau,\mathfrak{x}(\tau)) d\varsigma \right) d\varrho \Bigg\} \\ & + \int_0^{\tau} e^{-\varphi(\tau-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\tau,\mathfrak{x}(\tau)) d\varsigma \right) d\varrho \Bigg],\end{aligned} \tag{2.2}$$

Proof . Where ${}^{c}\mathcal{D}^{\varpi}$ denote the CFDs order ϖ. Rewriting as ${}^{c}\mathcal{D}^{\varpi}(\mathfrak{x}(\tau) + \varphi {}^{c}\mathcal{D}^{-1}\mathfrak{x}(\tau)) = h(\tau)$. We can write its solution as

$$\begin{aligned}\mathfrak{x}(\tau) = {} & a_0 e^{-\varphi\tau} + \frac{a_1}{\varphi}(1 - e^{-\varphi\tau}) + \frac{a_2}{\varphi^2}(\varphi\tau - 1 + e^{-\varphi\tau}) + \frac{a_3}{\varphi^3}(\varphi^2\tau^2 - 2\varphi\tau + 2 - 2e^{-\varphi\tau}) \\ & + \int_0^{\tau} e^{-\varphi(\tau-s)} \Big(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} h(\varsigma) d\varsigma \Big) d\varrho,\end{aligned} \tag{2.3}$$

where a_0, a_1, a_2, and a_3 are unknown arbitrary constants. Using the BCs (1.3) in (2.4),

$$\mathfrak{x}(0) = 0, \quad '(0) = 0, \quad \mathfrak{x}''(0) = 0, \quad \mathfrak{x}(1) + \mathfrak{x}(v) = \lambda \int_0^{\eta} \mathfrak{x}(\vartheta) d\vartheta, \tag{2.4}$$

we find that $a_0 = 0$, $a_1 = 0$, $a_2 = 0$ and

$$\begin{aligned}a_3 = \frac{\varphi^3}{\Omega} \Bigg\{ & \lambda \int_0^{\eta} \left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} h(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta - \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} h(\varsigma) d\varsigma \right) d\varrho \\ & - \int_0^{\nu} e^{-\varphi(\nu-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} h(\varsigma) d\varsigma \right) d\varrho \Bigg\},\end{aligned}$$

where,

$$\Omega = \left\{ [\varphi^2 - 2\varphi - 2e^{-\varphi} + 2 + \varphi^2\nu^2 - 2\varphi\nu^2 - 2e^{-\varphi\nu} + 2] - \left[\lambda \int_0^{\eta} (\varphi^2\vartheta^2 - 2\varphi\vartheta^2 - 2e^{-\varphi\vartheta} + 2) d\vartheta \right] \right\} \neq 0, \tag{2.5}$$

where (2.5) provides Ω. Then, (2.4) becomes (2.2) by replacing the values of a_0, a_1, a_2, and a_3. Conversely, this is a direct result of the calculation. This concludes the proof. □

3 Existence of Solutions

Let $\mathcal{X} = \mathcal{C}([0,1],\mathbb{R})$ denote the Banach Space of all continuous functions from $[0,1] \to \mathbb{R}$ with the usual norm defined by $||\mathfrak{x}|| = \sup\{|\mathfrak{x}(\tau)|, \tau \in [0,1]\} < \infty$. To make proofs easier, it is necessary to set upper and lower limits for the integrals that will be introduced in future results.

Lemma 3.1. For $h \in \mathcal{C}([0,1],\mathbb{R})$, with $||h|| = \sup_{\tau\in[0,1]}|h(\tau)|$ we have

$$(i)\left|\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta\right| = \frac{\lambda\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)||h||;$$

$$(ii)\left|\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right| = \frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})||h||;$$

$$(iii)\left|-\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right| = \frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})||h||;$$

$$(iv)\left|\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right| = \frac{\tau^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\tau})||h||.$$

Proof .

(i) Apparently

$$\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}d\varsigma = \frac{\varsigma^{(\varpi-1)}}{\Gamma(\varpi)},$$

and

$$\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\frac{\varsigma^{(\varpi-1)}}{\Gamma(\varpi)}d\varrho \le \frac{\vartheta^{(\varpi-1)}}{\Gamma(\varpi)}\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}d\varrho = \frac{\vartheta^{(\varpi-1)}}{\Gamma(\varpi)}\frac{(1-e^{-\varphi\vartheta})}{\varphi},$$

thus

$$\begin{aligned}\left|\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta\right| &\le ||h||\lambda\int_0^\eta\frac{\vartheta^{(\varpi-1)}}{\Gamma(\varpi)}\frac{(1-e^{-\varphi\vartheta})}{\varphi}d\varrho\\ &= \frac{\lambda\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)||h||.\end{aligned}$$

The proofs of (ii) and (iii) are similar. The proof is completed. □

To make things easier for us, we've established a schedule

$$\wp = \sup_{\tau\in[0,1]}\left|\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\right| = \frac{1}{|\Omega|}(\varphi^2-2\varphi-2e^{-\varphi}+2), \tag{3.1}$$

$$\Pi = \wp\left[\left\{|\lambda|\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]. \tag{3.2}$$

A fixed-point problem (1.1)-(1.3) is transformed into an equivalent problem in the context of Lemma (3.1).

$$\mathfrak{x} = \mathfrak{S}(\mathfrak{x}), \tag{3.3}$$

where $\mathfrak{S}:\mathcal{X}\to\mathcal{X}$ is defined by,

$$\begin{aligned}(\mathfrak{S}\mathfrak{x})(\tau) = &\left[\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\left\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\times\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right)d\vartheta\right.\right.\\ &-\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho-\left.\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right\}\\ &\left.+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right].\end{aligned} \tag{3.4}$$

If the operator equation (3.3) has fixed points, problem (1.1)-(1.3) has a solution.

Theorem 3.2. Assume that $\mathfrak{f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the condition

$$(\mathcal{T}_1)\ |\mathfrak{f}(\tau,\mathfrak{x})-\mathfrak{f}(\tau,\bar{\mathfrak{x}})|\leq\mathcal{L}\,|\mathfrak{x}-\bar{\mathfrak{x}}|\,. \tag{3.5}$$

for all $\tau\in[0,1]$, $\mathfrak{x},\bar{\mathfrak{x}}\in\mathbb{R}$, where $\mathcal{L}$ is the Lipschitz constant. Then the problem (1.1)-(1.3) has a unique solution if $\Pi< 1/\mathcal{L}$ where Π is given by equation (3.2).

Proof .

As the first step, we show that the operator $\mathfrak{S}$ given by (3.3) maps $\mathcal{X}$ into itself. For that, we set $\sup_{\tau\in[0,1]}|\mathfrak{f}(\tau,0)|=\mathcal{Q}<\infty$. Then, for $\mathfrak{x}\in\mathcal{X}$, we have

$$\begin{aligned}
\|\mathfrak{S}(\mathfrak{x})\| = \sup_{\tau\in[0,1]}&\Bigg|\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\times\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right)d\vartheta\\
&+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho+\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\Bigg\}\\
&+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\Bigg|\\
\leq \sup_{\tau\in[0,1]}&\left|\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\right|\\
&\times\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|f(\tau,\mathfrak{x}(\tau)-f(\tau,0)|+|f(\tau,0)|)d\varsigma\right)d\varrho\right)d\vartheta\\
&+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|f(\tau,\mathfrak{x}(\tau)-f(\tau,0)|+|f(\tau,0)|)d\varsigma\right)d\varrho\\
&+\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|f(\tau,\mathfrak{x}(\tau)-f(\tau,0)|+|f(\tau,0)|)d\varsigma\right)d\varrho\Bigg\}\\
&+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|f(\tau,\mathfrak{x}(\tau)-f(\tau,0)|+|f(\tau,0)|)d\varsigma\right)d\varrho,\\
\leq(\mathcal{L}\,\|\mathfrak{x}\|+\mathcal{Q})&\left[\wp\left\{|\lambda|\,\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}\right.\\
&\left.+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]\\
=(\mathcal{L}\,\|\mathfrak{x}\|+\mathcal{Q})&\,\Pi<\infty.
\end{aligned}$$

This shows that $\mathfrak{S}$ maps $\mathcal{X}$ into itself. Now for $\mathfrak{x}_1,\mathfrak{x}_2\in\mathcal{X}$ and for each $\tau\in[0,1]$, we obtain

$$\begin{aligned}
\|(\mathfrak{S}\mathfrak{x}_1)-(\mathfrak{S}\mathfrak{x}_2)\| &= \sup_{\tau\in[0,1]}\|(\mathfrak{S}\mathfrak{x}_1)(\tau)-(\mathfrak{S}\mathfrak{x}_2)(\tau)\|\\
&\leq \sup_{\tau\in[0,1]}\left|\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\right|\\
&\quad\times\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\,|f(\tau,\mathfrak{x}_1(\tau))-f(\tau,\mathfrak{x}_2(\tau))|\,d\varsigma\right)d\varrho\right)d\vartheta\\
&\quad+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\,|f(\tau,\mathfrak{x}_1(\tau))-f(\tau,\mathfrak{x}_2(\tau))|\,d\varsigma\right)d\varrho\\
&\quad+\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\,|f(\tau,\mathfrak{x}_1(\tau))-f(\tau,\mathfrak{x}_2(\tau))|\,d\varsigma\right)d\varrho\Bigg\}\\
&\quad+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\,|f(\tau,\mathfrak{x}_1(\tau))-f(\tau,\mathfrak{x}_2(\tau))|\,d\varsigma\right)d\varrho
\end{aligned}$$

$$\leq (\mathcal{L}\,\|\mathfrak{x}_1-\mathfrak{x}_2\|)\left[\wp\left\{|\lambda|\,\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]$$
$$\leq \mathcal{L}\Pi\,\|\mathfrak{x}_1-\mathfrak{x}_2\|\,,$$

where Π is supplied by (3.2), $\Pi < \frac{1}{\mathcal{L}}$, $\mathfrak{S}$ is a contraction. The conclusion of the theorem arises from the contraction mapping principle. The proof is complete. □

To verify that (1.1)-(1.3) has at least one solution, we require a known result from Krasnoselkii's [11].

Theorem 3.3. Assume that $\mathfrak{f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a jointly continuous function satisfying $(\mathcal{T}_1)$. In addition we suppose that the following assumption holds
$(\mathcal{T}_2)|\mathfrak{f}(\tau,\mathfrak{x})|\leq\xi(\tau)$, forever $(\tau,\mathfrak{x})\in[0,1]\times\mathbb{R}$ with $\xi\in C([0,1],\mathbb{R})$. Then the BVP has at least one solution on $[0,1]$ if,

$$\left[\wp\left\{|\lambda|\,\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}\right]<1. \tag{3.6}$$

where $\wp$ is given by equation (3.1) and Π are defined by (3.2)

Proof .

$$\sup_{\tau\in[0,1]}|\xi(\tau)|=\|\xi\|\,,$$

$$r\geq\Pi\,\|\xi\|\,,$$

where Π is given by equation (3.2) and consider $\mathcal{B}_r=\{\mathfrak{x}\in\mathcal{X};\|\mathfrak{x}\|\leq r\}$. Define the operator $\mathfrak{S}_1$ and $\mathfrak{S}_2$ on $\mathcal{B}_r$ as

$$(\mathfrak{S}_1\mathfrak{x})(\tau)=\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho,$$

$$(\mathfrak{S}_2\mathfrak{x})(\tau)=\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\left\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta\right.$$
$$\left.-\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho-\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right\}$$

It follows from that,

$$\|\mathfrak{S}_1\mathfrak{x}+\mathfrak{S}_2\mathfrak{x}\|\leq\left[p\left\{|\lambda|\,\frac{(\eta)^{\varpi-1}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]||\xi||\leq r.$$

$\mathfrak{S}_1\mathfrak{x}+\mathfrak{S}_2\mathfrak{x}\in\mathcal{B}_r$ is the result of this equation. The equation (3.6) makes it clear that the continuous function $\mathfrak{S}_2$ is a contraction of the initial value. As a result of this, it follows that the operator $\mathfrak{S}_1$ is continuous. The uniformity of the $\mathcal{B}_r$ boundary on $\mathfrak{S}_2$ is also

$$\|\mathfrak{S}_1\mathfrak{x}\|\leq\frac{(1-e^{-\varphi})\,\|\xi\|}{\varphi\Gamma(\varpi)}.$$

The compactness of the operator $\mathfrak{S}_1$ is now established. It's easy to see how this works: We define $\sup_{(\tau,\mathfrak{x}_1)\in\mathcal{B}_r}|f(\tau,\mathfrak{x})|=M_r$

$$|(\mathfrak{S}_1\mathfrak{x})(\tau_1)-(\mathfrak{S}_1\mathfrak{x})(\tau_2)|=\left|\int_0^{\tau_1}e^{-\varphi(\tau_1-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right.$$
$$\left.-\int_0^{\tau_2}e^{-\varphi(\tau_2-\varrho)}\left(\int_0^\varrho\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right|$$

$$\leq \frac{M_r}{\varphi\Gamma(\varpi)}\left(|\tau_1^{\varpi}-\tau_2^{\varpi}|+\left|\tau_1^{\varpi}e^{-\varphi\tau_1}-\tau_2^{\varpi}e^{-\varphi\tau_2}\right|\right)$$

As $\tau_2 \to \tau_1$ goes to zero, it is independent of $\mathfrak{x}$. As a result, $\mathfrak{S}_1$ is reasonably compact on $\mathcal{B}_r$. As a result, according to the Arzela-Ascoli theorem, $\mathfrak{S}_1$ is compact on $\mathcal{B}_r$. So Theorem 3.3 is satisfied in that all of its assumptions are satisfied. As a result, according to the conclusion of Theorem 3.2, BVP (1.1)-(1.3) has at least one solution on the interval $[0,1]$. This completes the demonstration. □

Lemma 3.4. Let $\mathcal{P}$ be Banach space, $\mathcal{Q}$ be a closed, convex subset of $\mathcal{P}$, $\mathcal{E}$ be an open subset of $\mathcal{Q}$ and $0 \in \mathcal{U}$. Suppose that $F : \bar{\mathcal{E}} \to C$ is a continuous, compact (that is, $F(\bar{\mathcal{E}})$ is a relatively compact subset of $\mathcal{Q}$) map. That, either
i) F has a fixed point in $\bar{\mathcal{E}}$, or
ii) There is $u \in \partial\mathcal{E}$ (the boundary of $\bar{\mathcal{E}}$ in $\mathcal{Q}$) and $\Omega \in (0,1)$ with $u = \Omega F(u)$.

Theorem 3.5. Suppose that $\mathfrak{f} : [0,1]\times\mathbb{R}\to \mathbb{R}$ is a continuous function. Further, it is assumed that the following conditions hold:

($\mathcal{T}_3$) There exist a function $\mu \in \mathcal{C}([0,1],\mathbb{R}^+)$ and a non-decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|\mathfrak{f}(\tau,\mathfrak{x})| \leq \mu(\tau)\psi(\|\mathfrak{x}\|)$ for all $(\tau,\mathfrak{x}) \in [0,1]\times\mathbb{R}$.

($\mathcal{T}_4$) there exists a constants $\mathcal{A}_1 > 0$ such that,

$$\frac{\mathcal{A}_1}{\psi(\mathcal{A}_1)\,\|\mu\|\,\Pi} > 1,$$

where Π is supplied by (3.2), A solution exists on $[0,1]$ for the boundary value problem (1.1-1.3).

Proof . consider the operator $\mathfrak{S} : \mathcal{X} \to \mathcal{X}$ with $\mathfrak{x} = \mathfrak{S}\mathfrak{x}$, where

$$\begin{aligned}(\mathfrak{S}\mathfrak{x})(\tau) =&\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\left\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta}e^{-\varphi(\vartheta-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta\right.\\&-\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\\&\left.-\int_0^{\nu}e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho\right\}+\int_0^{\tau}e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}h(\varsigma)d\varsigma\right)d\varrho.\end{aligned}$$

we show that $\mathfrak{S}$ maps bounded sets into bounded sets in $\mathcal{C}([0,1],\mathbb{R})$. For a positive number r, Let $\mathcal{B}_r = \{\mathfrak{x} \in \mathcal{C}([0,1],\mathbb{R}) : \|\mathfrak{x}\| \leq r\}$ be a bounded set in $\mathcal{C}([0,1],\mathbb{R})$. Then

$$\begin{aligned}|\mathfrak{S}(\mathfrak{x}(\tau))| \leq&\left|\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\left\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta}e^{-\varphi(\vartheta-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right)d\vartheta\right.\right.\\&+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho+\left.\int_0^{\nu}e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right\}\\&\left.+\int_0^{\tau}e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}f(\tau,\mathfrak{x}(\tau))d\varsigma\right)d\varrho\right|\\\leq&\wp\left[\left\{|\lambda|\int_0^{\eta}\left(\int_0^{\vartheta}e^{-\varphi(\vartheta-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mu(\tau)\psi(\|\mathfrak{x}\|)d\varsigma\right)d\varrho\right)d\vartheta\right.\right.\\&+\int_0^{s}e^{-\varphi(1-\varrho)}\left(\frac{(s-u)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mu(\tau)\psi(\|\mathfrak{x}\|)d\varsigma\right)d\varrho+\left.\int_0^{\nu}e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mu(\tau)\psi(\|\mathfrak{x}\|)d\varsigma\right)d\varrho\right\}\\&\left.+\int_0^{\tau}e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mu(\tau)\psi(\|\mathfrak{x}\|)d\varsigma\right)d\varrho\right]\end{aligned}$$

$$\leq\psi(\|\mathfrak{x}\|)\,(\mu)\,(\tau)\left[\wp\left\{|\lambda|\frac{(\eta)^{\varpi-1}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]$$

$$\leq \psi(\|\mathfrak{x}\|)(\mu)\Pi.$$

thus,

$$\|\mathfrak{S}\mathfrak{x}\| \leq \psi(r)\,\|\mu\|\,\Pi.$$

Next we show that $\mathfrak{S}$ that maps bounded sets into equi-continuous sets of $\mathcal{C}([0,1],\mathbb{R})$. Let $\tau_1,\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $\mathfrak{x} \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $\mathcal{C}([0,1],\mathbb{R})$. We then arrive at

$$\begin{aligned}
&|(\mathfrak{S}\mathfrak{x})(\tau_2) - (\mathfrak{S}\mathfrak{x})(\tau_1)| \\
&\leq \left| \int_0^{\tau_1} e^{-\varphi(\tau_2-\varrho)} - e^{-\varphi(\tau_1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\varsigma,\mathfrak{x}(\varsigma))d\varsigma \right) d\varrho + \int_{\tau_1}^{\tau_2} e^{-\varphi(\tau_2-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\varsigma,\mathfrak{x}(\varsigma))d\varsigma \right) d\varrho \right| \\
&+ \left| \frac{(\varphi^2(\tau_2^2-\tau_1^2) - 2\varphi(\tau_2-\tau_1) - 2(e^{-\varphi\tau_2} - e^{-\varphi\tau_1})}{\Omega} \times \left\{ \lambda \int_0^{\eta} \left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\varsigma,\mathfrak{x}(\varsigma))d\varsigma \right) d\varrho \right) d\vartheta \right.\right. \\
&\left.\left. + \int_0^{1} e^{-\varphi(1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\varsigma,\mathfrak{x}(\varsigma))d\varsigma \right) d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} f(\varsigma,\mathfrak{x}(\varsigma))d\varsigma \right) d\varrho \right\} \right| \\
&\leq \left| \int_0^{\tau_1} e^{-\varphi(\tau_2-\varrho)} - e^{-\varphi(\tau_1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} \psi(r)\mu(\varsigma)d\varsigma \right) d\varrho + \int_{\tau_1}^{\tau_2} e^{-\varphi(\tau_2-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} \psi(r)\mu(\varsigma)d\varsigma \right) d\varrho \right| \\
&+ \left| \frac{(\varphi^2(\tau_2^2-\tau_1^2) - 2\varphi(\tau_2-\tau_1) - 2(e^{-\varphi\tau_2} - e^{-\varphi\tau_1})}{\Omega} \times \left\{ \lambda \int_0^{\eta} \left(\int_0^{\vartheta} e^{-\varphi\vartheta-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} \psi(r)\mu(\varsigma)d\varsigma \right) d\varrho \right) d\vartheta \right.\right. \\
&\left.\left. + \int_0^{1} e^{-\varphi 1-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} \psi(r)\mu(\varsigma)d\varsigma \right) d\varrho + \int_0^{\nu} e^{-\varphi\nu-\varrho)} \left(\int_0^{\varrho} \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} \psi(r)\mu(\varsigma)d\varsigma \right) d\varrho \right\} \right|.
\end{aligned}$$

The right-hand side of the inequality above, regardless of $\mathfrak{x} \in \mathcal{B}_r$ as $\tau_2 - \tau_1 \to 0$, obviously tends to zero. This implies that $\mathfrak{S} : \mathcal{C}([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$ is totally continuous because $\mathfrak{S}$ satisfies the aforementioned conditions.

The Leray-Schauder nonlinear alternative will yield the desired outcome (Lemma 3.4). Once the set of all solutions to equations $\mathfrak{x} = \beta\mathfrak{S}(\mathfrak{x})$ for $\beta \in (0,1)$ has been proven to be bounded.

Let $\mathfrak{x}$ be the answer. For $\tau \in [0,1]$, and using the computations used to prove that $\mathfrak{S}$ is bounded, we obtain

$$\begin{aligned}
|\mathfrak{x}(\tau)| &= |\beta(\mathfrak{S}\mathfrak{x})(\tau)| \\
&\leq \psi(\|\mathfrak{x}\|)\,\|\mu\| \left[p \left\{ |\lambda| \frac{(\eta)^{\varpi-1}}{\varphi^2\Gamma(\varpi)} (\varphi\eta + e^{-\varphi\eta} - 1) + \frac{1}{\varphi\Gamma(\varpi)}(1 - e^{-\varphi}) + \frac{\nu^{\varpi-1}}{\varphi\Gamma(\varpi)}(1 - e^{-\varphi\nu}) \right\} + \left\{ \frac{1}{\varphi\Gamma(\varpi)}(1 - e^{-\varphi}) \right\} \right] \\
&= \psi(\|\mathfrak{x}\|)\,\|\mu\|\,\Pi.
\end{aligned}$$

as a result of this,

$$\frac{\|\mathfrak{x}\|}{\psi(\|\mathfrak{x}\|)\,\|\mu\|\,\Pi} \leq 1.$$

To put it another way, there exists a $(\mathcal{T}_4)$ in which $\mathcal{A}_1$ such that $\|\mathfrak{x}\| \neq \mathcal{A}_1$. Let's get started

$$\mathcal{V} = \mathfrak{x} \in \mathcal{C}([0,1],\mathbb{R}) : \|\mathfrak{x}\| < \mathcal{A}_1.$$

Be aware that the operator $\mathfrak{S} : \bar{\mathcal{V}} \to \mathcal{C}([0,1],\mathbb{R})$ is both continuous and completely continuous. It can be seen in the following equation: If $\mathcal{V}$ is chosen, then there is no $\mathfrak{x} \in \partial\mathcal{V}$ there is no $\mathfrak{x} \in \beta\mathfrak{S}(\mathfrak{x})$ for some $\beta \in (0,1)$. Since the nonlinear alternative of Leray-Schauder type (Lemma 3.1) leads to a solution of problem (1.1) – (1.3), we conclude that the function $\mathfrak{S}$ has the fixed point $\mathfrak{x} \in \bar{\mathcal{V}}$. Finally, we have conclusive proof. □

4 Main results (1.2) and (1.3)

4.1 The Lipschitz case

We prove the existence of solutions for the problem (1.2)-(1.3) with a non-convex valued right-hand side by applying a fixed theorem for multi-valued map due to Covitz and Nadler [[12]].

Definition 4.1. A multi-valued operator $\mathfrak{N} : \mathcal{Y} \to \mathfrak{Q}_{\mathfrak{cl}}(\mathcal{Y})$ is called

(a) $\eth$- Lipschitz if and only if there exists $\eth > 0$ such that, $\mathfrak{H}_d(\mathfrak{N}(y), \mathfrak{N}(z)) \leq \eth d(y, z)$ for each $y, z \in \mathcal{Y}$; and

(b) a contraction if and only if is $\eth$ -Lipschitz with $\eth < 1$.

Definition 4.2. Let $\mathfrak{A}$ be a subset of $[0, 1] \times \mathbb{R}$. $\mathfrak{A}$ is $\mathcal{L} \bigotimes \mathcal{B}$ measurable if $\mathfrak{A}$ belongs to the ϖ - algebra generated by all sets of the form $\mathcal{T} \times \mathcal{D}$, where $\mathcal{T}$ is Lebesgue measurable in $[0, 1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Lemma 4.3. [26] let $(\mathcal{Y}, d)$ be a complete metric space. If $\mathfrak{N} : \mathcal{Y} \to \mathfrak{Q}_{\mathfrak{cl}}(\mathcal{Y})$ is a contraction, then fix $\mathfrak{N} \neq \mu$.

Theorem 4.4. Assume that the following conditions hold:

$(\mathfrak{A}_1)$ $\mathfrak{F} : [0, 1] \times \mathbb{R} \to \mathcal{U}_{\mathfrak{cp}}(\mathbb{R})$ is such that $\mathfrak{F}(., \mathfrak{x}(\tau)) : [0, 1] \to \mathcal{U}_{\mathfrak{cp}}(\mathbb{R})$ is measurable for each $\mathfrak{x} \in \mathbb{R}$.

$(\mathfrak{A}_2)\mathfrak{H}_d(\mathfrak{F}(\tau, \mathfrak{x}), \mathfrak{F}(\tau, \mathfrak{x})) \leq \mathcal{U}(\tau)|\mathfrak{x} - \bar{\mathfrak{x}}|$ for almost all $\tau \in [0, 1]$ and $\mathfrak{x}, \bar{\mathfrak{x}} \in \mathbb{R}$ with $\mathcal{U} \in (\mathcal{C}([0, 1], \mathbb{R}^+)$ and $d(0, \mathfrak{F}(\tau, 0)) \leq \chi_1(\tau)$ for almost all $\tau \in [0, 1]$. Then the boundary value problem has at least one solution on $[0, 1]$, if

$$||\chi_1|| \left(\wp \left[\left\{ |\lambda| \frac{\eta^{(\varpi-1)}}{\varphi^2 \Gamma(\varpi)} (\varphi\eta + e^{-\varphi\eta} - 1) + \frac{1}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi}) + \frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi\nu}) \right\} + \left\{ \frac{1}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi}) \right\} \right] \right) < 1.$$

Proof . Define the operator $\Upsilon_{\mathfrak{F}} : (\mathcal{C}([0, 1], \mathbb{R}) \to \mathcal{U}(\mathcal{C}([0, 1], \mathbb{R}))$. Since set $\mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ is nonempty by Inference $\mathfrak{x} \in (\mathcal{C}([0, 1], \mathbb{R})$ for each $\mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$, $(\mathfrak{A}_1)$ has a spectrum that can be calculated. We will now see if operator $\Upsilon_{\mathfrak{F}}$ conforms to the lemma (4.3) assumptions. To show that $\Upsilon_{\mathfrak{F}}(\mathfrak{x}) \in \mathcal{U}_{\mathfrak{cl}}(((\mathcal{C}[0, 1], \mathbb{R}))$ for each $y \in (\mathcal{C}([0, 1])$, let $\mathfrak{U}_{n\geq 0} \in \Upsilon_{\mathfrak{F}}(\mathfrak{x})$,such that $\mathfrak{U}_n \to \mathfrak{U}(n \to \infty)$ in $(\mathcal{C}([0, 1], \mathbb{R})$. Then $\mathfrak{U} \in (\mathcal{C}([0, 1], \mathbb{R})$ and exists $v_n \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ such that, $\tau \in [0, 1]$,

$$\begin{aligned} \mathfrak{U}_n(\tau) = &\left[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \left\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_n(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \right. \right. \\ &- \left. \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_n(\varsigma) d\varsigma \right) d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_n(\varsigma) d\varsigma \right) d\varrho \right\} \\ &+ \left. \int_0^\tau e^{-\varphi(\tau-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_n(\varsigma) d\varsigma \right) d\varrho \right]. \end{aligned} \tag{4.1}$$

as $\mathfrak{F}$ has compact values, we pass onto a sub-sequence to obtain that v_n converges to v in $\mathfrak{L}^1([0, 1], (\mathbb{R}))$. Thus, $v \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ and for each $\tau \in [0, 1]$, we have

$$\begin{aligned} \mathfrak{U}_n(\tau) \to \mathfrak{U}(\tau) = &\left[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \left\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \right. \right. \\ &- \left. \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v(\varsigma) d\varsigma \right) d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v(\varsigma) d\varsigma \right) d\varrho \right\} \\ &+ \left. \int_0^\tau e^{-\varphi(\tau-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v(\varsigma) d\varsigma \right) d\varrho \right]. \end{aligned} \tag{4.2}$$

Hence, $\mathfrak{U} \in \Upsilon_{\mathfrak{F}}(\mathfrak{x})$. Next, we show that there exists. Such that, $\mathfrak{H}_d(\Upsilon_{\mathfrak{F}}(\mathfrak{x}), \Upsilon_{\mathfrak{F}}(\bar{\mathfrak{x}})) \leq \widehat{\theta} \|\mathfrak{x} - \bar{\mathfrak{x}}\|_{\mathfrak{x}}$ for each $\mathfrak{x}, \bar{\mathfrak{x}} \in \mathcal{AC}^4([0, 1], \mathbb{R})$. Let $\mathfrak{x}, \bar{\mathfrak{x}} \in \mathcal{AC}^4([0, 1], (\mathbb{R}))$ and $\mathfrak{h}_1 \in \Upsilon_{\mathfrak{F}}(\mathfrak{x})$. Then there exists $v_1(\tau) \in \mathfrak{F}(\tau, \mathfrak{x}(\tau))$ such that, for each $\tau \in [0, 1]$,

$$\begin{aligned} \mathfrak{h}_1(\tau) = &\left[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \left\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_1(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \right. \right. \\ &- \left. \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_1(\varsigma) d\varsigma \right) d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_1(\varsigma) d\varsigma \right) d\varrho \right\} \\ &+ \left. \int_0^\tau e^{-\varphi(\tau-\varrho)} \left(\int_0^\varrho \frac{(\varrho - \varsigma)^{(\varpi-2)}}{\Gamma(\varpi - 1)} v_1(\varsigma) d\varsigma \right) d\varrho \right]. \end{aligned} \tag{4.3}$$

$(\mathfrak{A}_2)$, We've got it $\mathfrak{H}_d(\mathfrak{F}(\tau,\mathfrak{x}),\mathfrak{F}(\tau,\bar{\mathfrak{x}})) \leq \chi_1(\tau)[|\mathfrak{x}(\tau)-\bar{\mathfrak{x}}(\tau)|]$. So, there $\mathfrak{w} \in \mathfrak{F}(\tau,\bar{\mathfrak{x}}(\tau))$ such that

$$|v_1(\tau)-\mathfrak{w}| \leq \chi_1(\tau)|\mathfrak{x}(\tau)-\bar{\mathfrak{x}}(\tau)|, \quad \tau \in [0,1].$$

Define $\mathfrak{U} : [0,1] \to \mathcal{U}(\mathbb{R}) by$

$$\mathfrak{U}(\tau) = \{\mathfrak{w} \in \mathbb{R} : |v_1(\tau)-\mathfrak{w}| \leq \chi_2(\tau)|\mathfrak{x}(\tau)-\bar{\mathfrak{x}}(\tau)|\}.$$

As the multivalued operator $\mathfrak{U}(\tau) \cap \mathfrak{F}(\tau,\bar{\mathfrak{x}}))$ is measurable, which is a measurable selection for $\mathfrak{U}(\tau) \cap \mathfrak{F}(\tau,\bar{\mathfrak{x}}))$. So $v_2(\tau) \in \mathfrak{F}(\tau,\bar{\mathfrak{x}}))$ and for each $\tau \in [0,1]$, we have $|v_1(\tau)-v_2(\tau)| \leq \chi_2(\tau)|\mathfrak{x}(\tau)-\bar{\mathfrak{x}}(\tau)|$. For each $\tau \in [0,1]$, let us define

$$\begin{aligned}\mathfrak{h}_2(\tau) = &\Bigg[\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_2(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta \\ &-\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_2(\varsigma)d\varsigma\right)d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_2(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ &+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_2(\varsigma)d\varsigma\right)d\varrho\Bigg].\end{aligned} \tag{4.4}$$

thus,

$$\begin{aligned}|\mathfrak{h}_1(\tau)-\mathfrak{h}_2(\tau)| \leq &\Bigg[\frac{(\varphi^2\tau^2-2\varphi\tau-2e^{-\varphi\tau}+2)}{\Omega}\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\right)d\vartheta \\ &+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho \\ &+\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\Bigg\} \\ &+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\Bigg] \\ \leq &\Bigg[\wp\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\right)d\vartheta \\ &+\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho \\ &+\int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\Bigg\} \\ &+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}|v_1(\varsigma)-v_2(\varsigma)|d\varsigma\right)d\varrho\Bigg] \\ \leq &||\chi_1||\left(\wp\left[\left\{|\lambda|\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}\right.\right. \\ &\left.\left.+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]\right)||\mathfrak{x}-\bar{\mathfrak{x}}||.\end{aligned} \tag{4.5}$$

Hence

$$||\mathfrak{h}_1-\mathfrak{h}_2|| \leq$$
$$||\chi_1||\left(\wp\left[\left\{|\lambda|\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]\right)||\mathfrak{x}-\bar{\mathfrak{x}}||.$$

Analogously, interchanging the role of $\mathfrak{x}$ and $\bar{\mathfrak{x}}$, we obtain

$$\mathfrak{H}_d(\Upsilon_{\mathfrak{F}}(\mathfrak{x}),\Upsilon_{\mathfrak{F}}(\bar{\mathfrak{x}}))$$
$$\leq ||\chi_1||\left(\wp\left[\left\{|\lambda|\frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta+e^{-\varphi\eta}-1)+\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})+\frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\}+\left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right]\right)||\mathfrak{x}-\bar{\mathfrak{x}}||.$$

Since $\Upsilon_{\mathfrak{F}}$ is a contraction, it follows by Lemma (2.5) that $\Upsilon_{\mathfrak{F}}$ has a fixed point $\mathfrak{x}$ by Lemma 4.3, which is a solution of (1.2 and 1.3). This completes the proof. □

4.2 The Upper Semi-Continuous case.

In this case when $\mathfrak{F}$ has convex values we prove an existence results based nonlinear alternative of Leray-Schauder type.

Lemma 4.5. [13] If $\mathcal{V} : \mathcal{Y} \to \mathfrak{Q}_{\mathfrak{cl}}(\mathcal{Z})$is u.s.c., then $Gr(\mathcal{V})$ is a closed subset of $\mathcal{Y} \times \mathcal{Z}$; i.e., for every sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$ and $\{\mathcal{Y}_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ if when $n \to \infty, y_n \to y_*, z_n \to z_*$ and $z_n \in \mathcal{V}(y_n)$, then $z_* \in \mathcal{V}(y_*)$. Conversely, if $\mathcal{V}$ completely continuous and has a closed graph, then it is upper semi-continuous.

Lemma 4.6. Let $\mathcal{Y}$ be a Banach space. Let $\mathfrak{F} : [0,1] \times \mathbb{R}^2 \to \mathfrak{Q}_{cp,c}(\mathcal{Y})$ be n $\mathfrak{L}^1$-Caratheodory multi-valued map and let Θ be a linear continuous mapping from $\mathfrak{L}^1([0,1],\mathcal{Y})$ to $\mathcal{C}([0,1],\mathcal{Y})$. Then the operator

$$\Theta \circ \mathfrak{S}_{\mathfrak{F},y} : \mathcal{C}([0,1],\mathcal{Y}) \to \mathfrak{Q}_{\mathfrak{cp},\mathfrak{c}}(\mathcal{C}([0,1],\mathcal{Y})), y \rightarrowtail (\Theta \circ \mathfrak{S}_{\mathfrak{F},y})(y) = \Theta \circ (\mathfrak{S}_{\mathfrak{F},y,z}),$$

is a closed graph operator in $\mathcal{C}([0,1],\mathcal{Y} \times \mathcal{C}([0,1],\mathcal{Y})$.

Lemma 4.7. Let $\mathfrak{E}$ be a Banach space, $\mathcal{C}$ a closed convex subset of $\mathfrak{E}, \mathfrak{U}$ an open subset of $\mathcal{C}$ and $0 \in \mathfrak{U}$. Suppose that $\mathfrak{F}$: $\bar{\mathfrak{U}} \to \mathfrak{Q}_{\mathfrak{cp},\mathfrak{c}}(\mathcal{C})$ is a upper semi-continuous compact map, then either
(i)$\mathfrak{F}$ has a fixed point in $\bar{\mathfrak{U}}$, or
(ii) there is a $\mathfrak{u} \in \partial\mathfrak{U}$ and $\lambda \in [0,1]$ with $u \in \lambda\mathfrak{F}(\mathfrak{u})$.

Definition 4.8. A subset $\mathfrak{A}$ of $\mathfrak{L}^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathfrak{A}$ and measurable $\mathcal{T} \subset [0,1] = \mathcal{T}$, the function $u\mathcal{Y}_{\mathcal{T}} + v\mathcal{Y}_{J-\mathcal{T}} \in \mathfrak{A}$, where $\mathcal{Y}_{\mathcal{T}}$ stands for the characteristic function of $\mathcal{T}$.

Theorem 4.9. Assume that;
($\mathfrak{H}_1$) $\mathfrak{F} : [0,1] \times \mathbb{R} \to \mathcal{U}(\mathbb{R})$ is $\mathfrak{L}^1$-Caratheodory and has nonempty compact and convex values;
($\mathfrak{H}_2$) There exists a function $\mu \in (\mathcal{C}([0,1],\mathbb{R}^+)$, and a non decreasing, sub-homogeneous function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathfrak{F}(\tau,\mathfrak{x})\|_{\mathcal{U}} := \sup\{|\mathfrak{w}| : \mathfrak{w} \in \mathfrak{F}(\tau,\mathfrak{x})\} \le \mu(\tau)\Upsilon(\|\mathfrak{x}\|) \quad \text{for each} \quad (\tau,\mathfrak{x}) \in [0,1] \times \mathbb{R};$$

($\mathfrak{H}_3$) There exists a contact $\mathfrak{M} > 0$ such that

$$\frac{\mathfrak{M}}{\Pi||\mu||\Upsilon(\mathfrak{M})} > 1,$$

where Π are defined by equation (3.2). Then the boundary value problem (BVP) (1.2) − (1.3) has at least one solution on $[0,1]$.

Proof . Define an operator $\Upsilon_{\mathfrak{F}} : \mathcal{C}([0,1],\mathbb{R}) \to \mathcal{U}(\mathcal{C}([0,1],\mathbb{R}))$ by $\Upsilon_{\mathfrak{F}}(\mathfrak{x}) = \{\mathfrak{h} \in \mathcal{C}([0,1],\mathbb{R})$ as $\mathfrak{h}(\tau) = \mathfrak{N}(\mathfrak{x})(\tau)\}$ where

$$\begin{aligned}\mathfrak{N}(\mathfrak{x})(\tau) = \Bigg[&\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^\eta\left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta\\ &-\int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ &+\int_0^\tau e^{-\varphi(\tau-\varrho)}\left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\Bigg], \quad v \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}.\end{aligned} \tag{4.6}$$

We will show that $\Upsilon_{\mathfrak{F}}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Upsilon_{\mathfrak{F}}$ is convex for each $\mathfrak{x} \in \mathcal{C}([0,1],\mathbb{R})$. This step is obvious since $\mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ is convex $\mathfrak{F}$ has convex values, and therefore we omit the proof.

In the second step, we show that $\Upsilon_{\mathfrak{F}}$ maps bounded sets into bounded sets in $\mathcal{C}([0,1],\mathbb{R})$. For a positive number ρ, let $\mathfrak{B}_\rho = \{\mathfrak{x} \in \mathcal{C}([0,1],\mathbb{R}) : ||\mathfrak{x}|| \leq \rho\}$ be a bounded ball in $\mathcal{C}([0,1],\mathbb{R})$. Then, for each $\mathfrak{h} \in \Upsilon_{\mathfrak{F}}(\mathfrak{x}), \mathfrak{x} \in \mathfrak{B}_\rho$, there exists $v \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ such that

$$\begin{aligned}\mathfrak{h}(\tau) = \Bigg[&\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \Bigg\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \\ &- \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \Bigg\} \\ &+ \int_0^\tau e^{-\varphi(\tau-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \Bigg]. \end{aligned} \tag{4.7}$$

Then, for $\tau \in [0,1]$ we have

$$\begin{aligned}|\mathfrak{h}(\tau)| \leq \Bigg[&\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \Bigg\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} |v(\varsigma)| d\varsigma \right) d\varrho \right) d\vartheta \\ &- \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} |v(\varsigma)| d\varsigma \right) d\varrho - \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} |v(\varsigma)| d\varsigma \right) d\varrho \Bigg\} \\ &+ \int_0^\tau e^{-\varphi(\tau-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} |v(\varsigma)| d\varsigma \right) d\varrho \Bigg] \\ \leq & ||\mu|| \Upsilon(||\mathfrak{x}||) \left(\wp \left[\left\{ |\lambda| \frac{\eta^{(\varpi-1)}}{\varphi^2 \Gamma(\varpi)} (\varphi\eta + e^{-\varphi\eta} - 1) + \frac{1}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi}) + \frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi\nu}) \right\} + \left\{ \frac{1}{\varphi\Gamma(\varpi)} (1 - e^{-\varphi}) \right\} \right] \right), \\ \leq & \Pi ||\mu|| \Upsilon(||\mathfrak{x}||_{\mathfrak{X}}) \end{aligned}$$

Consequently,

$$||\mathfrak{h}|| \leq \Pi ||\mu|| \Upsilon(||\mathfrak{x}||_{\mathfrak{X}}).$$

Now we show that $\Upsilon_{\mathfrak{F}}$ maps bounded sets into equi-continuous sets of $\mathcal{C}([0,1],\mathbb{R})$. Let $\tau_1,\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $\mathfrak{x} \in \mathfrak{B}_\rho$. For each $\mathfrak{h} \in \Upsilon_{\mathfrak{F}}(\mathfrak{x})$. We obtain

$$\begin{aligned}|\mathfrak{h}(\tau_2) - \mathfrak{h}(\tau_1)| \leq \Bigg[\Bigg| &\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \Bigg\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \\ &+ \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho + \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \Bigg\} \Bigg| \\ &+ \left| \int_0^\tau \left(e^{-\varphi(\tau_2-\varrho)} - e^{-\varphi(\tau_1-\varrho)} \right) \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right. \\ &\left. + \int_{\tau_1}^{\tau_2} e^{-\varphi(\tau_2-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right| \Bigg] \\ \leq \Bigg[\Bigg| &\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega} \Bigg\{ \lambda \int_0^\eta \left(\int_0^\vartheta e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right) d\vartheta \\ &+ \int_0^1 e^{-\varphi(1-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho + \int_0^\nu e^{-\varphi(\nu-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \Bigg\} \Bigg| \\ &+ \left| \int_0^\tau \left(e^{-\varphi(\tau_2-\varrho)} - e^{-\varphi(\tau_1-\varrho)} \right) \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right. \\ &\left. + \int_{\tau_1}^{\tau_2} e^{-\varphi(\tau_2-\varrho)} \left(\int_0^\varrho \frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)} v(\varsigma) d\varsigma \right) d\varrho \right| \Bigg]. \end{aligned}$$

Obviously, the right hand side of the above inequalities tends to zero independently of $\mathfrak{x} \in \mathfrak{B}_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\Upsilon_{\mathfrak{F}}$ satisfies the above assumption, therefore it follows by the Arzela-Ascoli Theorem [**?**], that $\Upsilon_{\mathfrak{F}} : \mathcal{U}(\mathcal{C}([0,1],\mathbb{R}) \to \mathcal{U}(\mathcal{C}([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\Upsilon_{\mathfrak{F}}$ is upper semi-continuous. To this end it is sufficient to show that $\Upsilon_{\mathfrak{F}}$ has a close graph, by Lemma 4.5. Let $\mathfrak{x}_n \to \mathfrak{x}_*$, $\mathfrak{h}_n \in \Upsilon_{\mathfrak{F}}(\mathfrak{x}_n)$ and $\mathfrak{h}_n \to \mathfrak{h}_*$. Then we need to show that $\mathfrak{h}_* \in \Upsilon_{\mathfrak{F}}(y_*)$. Associated with $\mathfrak{h}_n \in \Upsilon_{\mathfrak{F}}(\mathfrak{x}_n)$, there exists $v_n \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}_n}$ such that for each $\tau \in [0,1]$,

$$\begin{aligned}\mathfrak{h}_n(\tau) = \Bigg[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_n(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta \\ - \int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_n(\varsigma)d\varsigma\right)d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_n(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ + \int_0^{\tau} e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_n(\varsigma)d\varsigma\right)d\varrho\Bigg].\end{aligned} \tag{4.8}$$

Thus its suffices to show that there exists $v_* \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}_*}$ such that for each $\tau \in [0,1]$,

$$\begin{aligned}\mathfrak{h}_*(\tau) = \Bigg[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta \\ - \int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ + \int_0^{\tau} e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\Bigg].\end{aligned} \tag{4.9}$$

Let us consider the linear operator $\Theta : \mathfrak{L}^1([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$ given by

$$\begin{aligned}v \rightarrowtail \Theta(v)(\tau) = \Bigg[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta \\ - \int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ + \int_0^{\tau} e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v(\varsigma)d\varsigma\right)d\varrho\Bigg].\end{aligned}$$

Observe that

$$\begin{aligned}||\mathfrak{h}_n(\tau) - \mathfrak{h}_*(\tau)|| = \Bigg[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}(v_n(\varsigma) - v_*(\varsigma))d\varsigma\right)d\varrho\right)d\vartheta \\ - \int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}(v_n(\varsigma) - v_*(\varsigma))(\varsigma)d\varsigma\right)d\varrho \\ - \int_0^{\nu} e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}(v_n(\varsigma) - v_*(\varsigma))(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ + \int_0^{\tau} e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}(v_n(\varsigma) - v_*(\varsigma))(\varsigma)d\varsigma\right)d\varrho\Bigg] \to 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus, it follows by Lemma 4.6, that $\Theta \circ \mathfrak{S}_{\mathfrak{F}}$ is a closed graph operator. Further, we have $\mathfrak{h}_n(\tau) \in \Theta(\mathfrak{S}_{\mathfrak{F},\mathfrak{x}_n})$. Since $\mathfrak{x}_n \to \mathfrak{x}_*$, therefore, we have

$$\begin{aligned}\mathfrak{h}_*(\tau) = \Bigg[\frac{(\varphi^2\tau^2 - 2\varphi\tau - 2e^{-\varphi\tau} + 2)}{\Omega}\Bigg\{\lambda\int_0^{\eta}\left(\int_0^{\vartheta} e^{-\varphi(\vartheta-\varrho)} \times \left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\right)d\vartheta \\ - \int_0^1 e^{-\varphi(1-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho - \int_0^{\nu} e^{-\varphi(\nu-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\Bigg\} \\ + \int_0^{\tau} e^{-\varphi(\tau-\varrho)}\left(\int_0^{\varrho}\frac{(\varrho-\varsigma)^{(\varpi-2)}}{\Gamma(\varpi-1)}v_*(\varsigma)d\varsigma\right)d\varrho\Bigg], \quad \text{for some} \quad h_* \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}.\end{aligned}$$

Finally, we show there exists an open $\mathfrak{Z} \subseteq \mathcal{C}([0,1],\mathbb{R})$ with $\mathfrak{x} \notin \Upsilon_{\mathfrak{F}}(\mathfrak{x})$ for any $\theta \in (0,1)$ and $\mathfrak{x} \in \partial(\mathfrak{Z})$. Let $\theta \in (0,1)$ and $\mathfrak{x} \in \theta\Upsilon_{\mathfrak{F}}(\mathfrak{x})$. Then there exists $v \in \mathfrak{L}^1([0,1],\mathbb{R})$ with $v \in \mathfrak{S}_{\mathfrak{F},\mathfrak{x}}$ such that for $\tau \in [0,1]$, we can obtain

$$\begin{aligned}||\mathfrak{x}||_{\mathfrak{X}} =&\wp \left[\left\{|\lambda| \frac{\eta^{(\varpi-1)}}{\varphi^2\Gamma(\varpi)}(\varphi\eta + e^{-\varphi\eta} - 1) + \frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi}) + \frac{\nu^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\nu})\right\} + \left\{\frac{1}{\varphi\Gamma(\varpi)}(1-e^{-\varphi})\right\}\right] \\ \leq& ||\mu||\Upsilon(||\mathfrak{x}||_{\mathfrak{X}}),\end{aligned}$$

which implies that

$$\frac{||\mathfrak{x}||_{\mathfrak{X}}}{\Pi||\mu||\Upsilon(||\mathfrak{x}||_{\mathfrak{X}})} \leq 1.$$

In view of $(\mathfrak{H}_3)$, there exists $\mathfrak{M}$ such that $||\mathfrak{x}|| \neq \mathfrak{M}$. Let us put it in place

$$\mathfrak{Z} = \{\mathfrak{x} \in \mathcal{C}([0,1],\mathbb{R}) : ||\mathfrak{x}|| < \mathfrak{M}\}.$$

In the upper case, operator $\Upsilon_{\mathfrak{F}} : \bar{\mathfrak{Z}} \to \mathcal{U}(\mathcal{C}([0,1],\mathbb{R}))$ is semi-continuous, while in the lower case, it is continuous. It's nothing like $\mathfrak{x} \in \partial\mathfrak{Z}$ such that $\mathfrak{x} \in \theta\Upsilon_{\mathfrak{F}}(\mathfrak{x})$ about any of $\mathfrak{Z}$, options $\theta \in (0,1)$. As a result, we can deduce that $\Upsilon_{\mathfrak{F}}$ has a fixed point $\mathfrak{x} \in \bar{\mathfrak{Z}}$, which is a solution to the problem (1.2 amd 1.3) by the nonlinear form of the Leray-Schauder alternative (Lemma 4.7). This contributes to the proof. □

5 Example

The following is an example of a Theorem 3.2 that is illustrated.

Example 5.1. Consider the problem

$$\begin{cases} {}^cD^{3/2}(D+2)\mathfrak{x}(\tau) = \mathcal{L}\frac{1}{2}(\sqrt{\tau^2+1} + sin(\tau) + \mathfrak{x}(\tau) + tan^{-1}\mathfrak{x}(\tau)), \quad 0 \leq \tau \leq 1, \\ \mathfrak{x}(0) = 0, \mathfrak{x}'(0) = 0, \mathfrak{x}''(0) = 0, \mathfrak{x}(1) + \mathfrak{x}(\nu) = \lambda \int_0^\eta \mathfrak{x}(\vartheta)d\vartheta. \end{cases} \tag{5.1}$$

Here, $\varpi = 5/2$, $f(\tau,\mathfrak{x}(\tau)) = \mathcal{L}\frac{1}{2}(\sqrt{\tau^2+1} + sin(\tau) + \mathfrak{x}(\tau) + tan^{-1}\mathfrak{x}(\tau))$, $\nu = 1/2$, $\eta = 1/3$ Clearly

$$|f(\tau,\mathfrak{x}) - f(\tau,z)| \leq \frac{\mathcal{L}}{2}\left|\mathfrak{x} - z + tan^{-1}\mathfrak{x} - tan^{-1}z\right| \leq \mathcal{L}|\mathfrak{x} - z|.$$

Using the provided values, we can calculate that Ω is approximately 1.54196, $\wp$ is approximately 1.1121513, and Π is approximately 1.0628829. As long as the Theorem 3.2 is followed, the example problem (5.1) has a unique solution for $\mathcal{L} < 1/\Pi \approx 0.94083$.

Example 5.2. Using the nonlinear function f given by, we then demonstrate the applicability of Theorem 3.3

$$f(\tau,\mathfrak{x}(\tau)) = \frac{1}{9+\tau}(\sin(\mathfrak{x})(\tau)) + \frac{1}{10}. \tag{5.2}$$

Using this formula, the value of $\mathcal{L}= \frac{1}{9}$ and $\mathcal{L}\wp\Pi \approx 0.132435$. The conditions of Theorem 3.3 are manifestly met. That theorem's conclusion implies at least one solution in $[0,1]$ for the problem (5.1) with the given value of f.

Example 5.3. For the illustration of Theorem 4.4 , let us choose

$$\begin{cases} {}^cD^{3/2}(D+2)\mathfrak{x}(\tau) \in \mathfrak{F}(\tau,\mathfrak{x}(\tau)), \quad 0 \leq \tau \leq 1, \\ \mathfrak{x}(0) = 0, \mathfrak{x}'(0) = 0, \mathfrak{x}''(0) = 0, \mathfrak{x}(1) + \mathfrak{x}(\nu) = \lambda \int_0^\eta \mathfrak{x}(\vartheta)d\vartheta. \end{cases} \tag{5.3}$$

Here, $\varpi = 5/2$, $\nu = 1/2$, $\varphi = 1$, $\eta = 1/3$. Clearly, $\mathfrak{F}(\tau,\mathfrak{x}(\tau)) = \left[0, \frac{1}{12+\tau^2}\left(\frac{|\mathfrak{x}|}{8(4+|\mathfrak{x}|)}\right) + \frac{1}{(15+\tau)}\right]$

$$\mathfrak{H}_d(\mathfrak{F}(\tau,\mathfrak{x}),\mathfrak{F}(\tau,\mathfrak{x})) \leq \frac{1}{12+\tau^2}||\mathfrak{x} - \bar{\mathfrak{x}}||_{\mathfrak{X}}. \tag{5.4}$$

Letting $\chi_1(\tau) = \frac{1}{12+\tau^2}$. It is easy to check that $d(0,\mathfrak{F}(\tau,0)) \leq \chi_1(\tau)$ holds for all $\tau \in [0,1]$ and that $||\chi_1||\Pi \leq 0.97430 < 1$. As the hypothesis of theorem are satisfied. we conclude that the problem (5.3) with $\mathfrak{F}$ given by (5.4) has at least one solution on $[0,1]$.

6 Conclusion

We have constructed particular existence and uniqueness results for a boundary value problem of the Caputo type sequential fractional differential equations and inclusions with integral boundary conditions using Banach's contraction mapping principle, Krasnoselkii's fixed point theorem, and the Leray-Schauder alternative. In the multi-valued case we proved existence results for both convex and non-convex multi-valued map via the nonlinear alternative for Covitz and Nadler's point theorem. We realize that new outcomes follow from altering the variables involved in a given problem. The results of this work, for example, apply to a sequential fractional differential equation and inclusions with integral boundary conditions of the form $\mathfrak{x}(0) = 0, \mathfrak{x}'(0) = 0, \mathfrak{x}"(0) = 0, \mathfrak{x}(1) = \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\mathfrak{x}(s)ds.$

References

[1] R.P. Agarwal, M. Benchohra, J.J. Nieto and A. Ouahab, *Fractional differential equations and Inclusions*, Springer, Berlin, Germany, 2010.

[2] B. Ahmad, A. Alsaedi, A. Alshariff, *Existence results for fractional-order differential equations with nonlocal multi-point-strip conditions involving Caputo derivative*, Adv. Differ. Equ. **2015** (2015), no. 1, 348.

[3] B. Ahmad and J.J. Nieto. *Sequential fractional differential equations with three-point boundary conditions*, Comput. Math.Appl. **64** (2012), no. 10, 3046–3052.

[4] B. Ahmad and S.K. Ntouyas, *Boundary value problems for q-difference inclusions*, Abstr. Appl. Anal. **2011** (2011),(No. SI1), 1–15.

[5] B. Ahmad, S.K. Ntouyas, R.P. Agarwal and A. Alsaedi, *New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions*, Boundary Value Prob. **2013** (2013), no. 1, 275.

[6] B. Ahmad, S.K. Ntouyas, R.P. Agarwal and A. Alsaedi, *Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions*, Fraction Cal. Appl. Anal. **18** (2015), no. 1, 261–280.

[7] B. Ahmad, S.K. Ntouyas and A. Alsaedi, *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Differ. Equ. **2011** (2011), 1–1.

[8] B. Ahmad and L. Rodica, *Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions* , Appl. Math. Comput. **339** (2018), 516–534.

[9] A. Alsaedi, S.K. Ntouyas, R.P. Agarwal and B. Ahmad, *On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions*, Adv. Differ. Equ. **2015** (2015), no. 1, 1–12.

[10] Y. Bai and H. Kong, *Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations via the method of upper and lower solutions*, J. Nonlinear Sci. Appl. **10** (2017), no. 1, 5744–5752.

[11] T.A. Burton, *A fixed-point theorem of Krasnoselskii*, Appl. Math. Lett. **11** (1998), no. 1, 85–88.

[12] H. Covitz and S.B. Nadler, *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), no. 1, 5–11.

[13] K. Deimling, *Multivalued differential equations*, de Gruyter, Berlin, 1992.

[14] H. Dutta, O.A. Ahmet and A. Atangana, *Fractional order analysis: theory, methods and applications*, John Wiley and Sons, 2020.

[15] J.W. Green and F.A. Valentine, *On the arzela-ascoli theorem*, Math. Mag. **34** (1961), no. 4, 199–202.

[16] A. Granas and J. Dugundji, *Fixed point theory*, Vol. 14. New York, Springer, 2003.

[17] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo , *Theory and applications of fractional differential equations*, Eelsevier, 2006.

[18] J. Klafter, S.C. Lim and R. Metzler *Fractional dynamics*, Recent Advances, World Scientific, 2012.

[19] M. Manigandan, S. Muthaiah, T. Nandhagopal, R. Vadivel, B. Unyong and N. Gunasekaran, *Existence results*

for coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order, AIMS Math. **7** (2022), no. 1, 723–755.

[20] M. Manigandan, M. Subramanian, P. Duraisamy and T.N. Gopal, *On Caputo-Hadamard type fractional differential equations with nonlocal discrete boundary conditions* Discontinuity Nonlinearity Complexity **10** (2021), no. 2, 185–194.

[21] S. Muthaiah and D. Baleanu, *Existence of solutions for nonlinear fractional differential equations and inclusions depending on lower-order fractional derivatives*, Axioms **9** (2020), no. 2, 44.

[22] S. Muthaiah, D. Baleanu, M. Murugesan and D. Palanisamy, *Existence of solutions for the Caputo-Hadamard fractional differential equations and inclusions*, J. Phys.: Conf. Ser. IOP Pub. **1850** (2021), no. 1, 012107.

[23] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto, 1999.

[24] S. Sitho, S.K. Ntouyas, A. Samadi and J. Tariboon, *Boundary value problems for ψ-Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions*, Mathematics **9** (2021), no. 9, 1001.

[25] M. Subramanian, M. Manigandan and T. Nandha Gopal, *Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations,* Adv. Theory Nonlinear Anal. Appl. **3** (2019), no. 3, 162–173.

[26] J. Tariboon, S.K. Ntouyas, B. Ahmad and A. Alsaedi, *Existence Results for sequential Riemann–Liouville and Caputo fractional differential inclusions with generalized fractional integral conditions*, Mathematics **8** (2020), no. 6, 1044.

[27] M. ur Rehman and R.A. Khan, *A numerical method for solving boundary value problems for fractional differential equations*, Appl. Math. Model. 36 (2012), no. 3, 894–907.

[28] Y. Xu, *Fractional boundary value problems with integral and anti-periodic boundary conditions*, Bull. Malay. Math. Sci. Soc. **39** (2016), no. 2, 571–587.

[29] İ. Yaslan, *Multi-point boundary value problems of higher-order nonlinear fractional differential equations*, Int. J. Nonlinear Anal. Appl. **12** (2021), no. 1, 337–349.