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Some $\psi-$fixed point theorems of Wardowski kind in $\mathcal{G}$-metric spaces with application to integral equations

Sahil Arora

Department of Mathematics, K.R.M.D.A.V. College, Nakodar-144040, Punjab, India

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Abstract

In this manuscript, we introduce new notions of generalized $(\mathfrak{f}^*, \psi)$-contraction and utilize this concept to prove some fixed point results for lower semi-continuous ψ-mapping satisfying certain conditions in the frame of G-metric spaces. Our results improve the results of [6] and [8] by omitting the continuity condition of $F \in \Im$ with the aid of the ψ-fixed point. We give an illustrative example to help accessibility of the got results and to show the genuineness of our results. Also, many existing results in the frame of metric spaces are established. Moreover, as an application, we employ the achieved result to earn the existence and uniqueness criteria of the solution of a type of non-linear integral equation.

Keywords: Generalized $(\mathfrak{f}^*, \psi)$-contraction, $\mathcal{G}$-metric space, ψ-fixed point, lower semi-continuous function, integral equation
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1 Introduction

The Banach contraction rule is one of the predominant outcomes in analysis and has continuously been at the front line of making and providing remarkable speculations for its researchers. Numerous authors have summed up and used the Banach principle in their relevant research, see [11, 12]. Along these lines, we can without much of a stretch presume that the biggest part of the fixed point theory has been involved by different speculations of the Banach contraction rule.In 2012, Wardowski [12] gave a new contraction known as F-contraction and proved fixed point theorem concerning F-contractions. In this manner, Wardowski conclude the Banach contraction principle in a different way from the eminent results from the literature.Piri and Kumam [9] also established Wardowski type fixed point theorems in complete metric spaces.Motivated by the perception of Dung and Hang [3], in 2016, Piri and Kumam [9] generalized the concept of generalized F-contraction and established some fixed point theorems for such kind of functions in complete metric spaces, by addition of four terms $d(f^2x, x), d(f^2x, fx), d(f^2x, y), d(f^2x, fy)$. Subsequently, Kumam and Piri [10] replace the condition $(F3)$ with $(F3^{'})$ in the definition of F-contraction given by Wardowski [12]. $(F3^{'})$: F is continuous on $(0, \infty)$. They gave the notation $\mathfrak{F}$ to denote the class of all maps $F : \mathbb{R}_+ \to \mathbb{R}$ which fulfil the conditions $(F1), (F2)$ and $(F3^{'})$. Piri and Kumam also proved some useful fixed point results for metric spaces. Now, the conditions $(F3)$ and $(F3^{'})$ are not associated with each other. For example $F(\beta) = \frac{-1}{\beta^q}$, where $q \geq 1$, then F meet the conditions (F1) and (F2) but it does not fulfil (F3), while it fulfils the condition $(F3^{'})$. In view of

Email address: drprofsahilarora@gmail.com (Sahil Arora)

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this, it is significant to observe the sequel of Wardowski [12] with the functions $F \in \Im$ rather than F $\in \mathfrak{F}$. In 2017, Gornicki [4] established some results for F-expansion mapping in the context of metric and G-metric spaces. On the other hand one of the significant and imperative ideas of the fixed point result was presented by Jleli et al. [5] and concluded some results for partial metric spaces. They also established various $\varphi-$fixed point results for graphic and weak $(F, \varphi)-$contraction mappings in the edge of metric spaces. In 2019, Saleh et al. [8] introduced a new contraction $(\mathfrak{f}^*, \psi)$-contraction and investigated some results for ψ-fixed point in $(\mathcal{H}, d)$. Recently, Kumar and Arora[6] introduced new notions of generalized F-contractions of type (S) and type(M) and established some fixed point theorems using these new notions in the frame of in G-metric spaces.

2 Preliminaries

Mustafa and Sims introduced the perception of G-metric space and gave an important generalization of a metric space as follows:

Definition 2.1. [7] Let $\mathcal{H}$ be a non empty set and $\mathcal{G} : \mathcal{H}^3 \to [0, \infty)$ be a map such that for all $x, y, z, a \in \mathcal{H}$, satisfies the following properties:

(i) $\mathcal{G}(x, y, z) = 0$ if $x = y = z$;

(ii) $0 < \mathcal{G}(x, x, y)$ whenever $x \neq y$, for all $x, y \in \mathcal{H}$;

(iii) $\mathcal{G}(x, x, y) \leq \mathcal{G}(x, y, z)$, $y \neq z$;

(iv) $\mathcal{G}(x, y, z) = \mathcal{G}(x, z, y) = \mathcal{G}(y, x, z) = \mathcal{G}(z, x, y) = \mathcal{G}(y, z, x) = \mathcal{G}(z, y, x)$;

(v) $\mathcal{G}(x, y, z) \leq \mathcal{G}(x, a, a) + \mathcal{G}(a, y, z)$.

Then the function $\mathcal{G}$ is said to be a $\mathcal{G}$-metric on $\mathcal{H}$ and the pair $(\mathcal{H}, \mathcal{G})$ is known as a $\mathcal{G}-$metric space.

In 1922, Banach established a useful result in fixed point theory regarding a contraction mapping, known as the Banach contraction principle.

Definition 2.2. [1] Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping. Let $d(fx, fy) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. Then f is called a contraction known as Banach contraction.

Wardowski defined the F-contraction as follows.

Definition 2.3. [12] Let (X, d) be a metric space and let $f : X \to X$ be a self-mapping. Then, f is called an F-contraction on (X, d), if there exist $F \in \Im$ and $\gamma > 0$ such that for all $x, y \in X$ for which $d(fx, fy) > 0$, then $\gamma + F(d(fx, fy)) \leq F(d(x, y))$, where $\Im$ is class of all mappings $F : (0, \infty) \to R$ such that

(F1) F is strictly increasing function, that is, for all $a, b \in (0, \infty)$, if $a < b$,then $F(a) < F(b)$.

(F2) For every sequence a_n of natural numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists $q \in (0, 1)$ such that $lim_{a\to 0^+}(a^q F(a)) = 0$.

Wardowski [12] gave some examples of $\Im$ as follow:

1. $F(\zeta) = \ln \zeta$.
2. $F(\zeta) = -\frac{1}{\zeta^{\frac{1}{2}}}$.
3. $F(\zeta) = \ln(\zeta) + \zeta$.
4. $F(\zeta) = \ln(\zeta^2 + \zeta)$.

It is obvious that F satisfies (F1)-(F3) ((F3) for any $q \in (0, 1)$).

Remark 2.4. Let $F : R_+ \to R$ be defined as $F = \ln(\beta)$, then $F \in \Im$. Now, F-contraction changes to a Banach contraction. Consequently, the Banach contractions are special case of F-contractions. There are F-contractions which are not Banach contractions(see[12]).

F-weak contraction was established by Wardowski and Dung in 2014 which is defined as follows:

Definition 2.5. [11]Let (X,d) be a metric space and $\mathcal{T}:X\to X$ be a function. $\mathcal{T}$ is known as F-weak contraction on (X,d) if there exists $F\in\Im$ and $\gamma>0$ such that for all $x,y\in X$, $d(\mathcal{T}x,\mathcal{T}y)>0$, then

$$\gamma+F(d(\mathcal{T}x,\mathcal{T}y))\leq F\left(\max\left\{d(x,y),d(\mathcal{T}x,x),d(y,\mathcal{T}y),\frac{d(x,\mathcal{T}y)+d(y,\mathcal{T}x)}{2}\right\}\right).$$

Theorem 2.6. [11] Let (X,d) be a complete metric space and let $\mathcal{T}:X\to X$ be an F-weak contraction. If F or $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a unique fixed point $x^*\in X$ and the sequence $\{\mathcal{T}^nx\}$ converges to x^*, for every $x\in X$, where n varies from 1 to ∞.

Hang and Dung [3] investigated the concept of generalized F-contraction and proved useful fixed point results for such kind of functions.

Definition 2.7. [3] Let (X,d) be a metric space and $f:X\to X$ be a self-mapping. f is called a generalized F-contraction on (X,d) if there exist $F\in\Im$ and $\delta>0$ such that for all
$x,y\in X$, $d(fx,fy)>0$, which implies that

$$\delta+F(d(fx,fy))\leq F\Big(\max\Big\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2},\frac{d(f^2x,x)+d(f^2x,fy)}{2},$$
$$d(f^2x,fx),d(f^2x,y),d(f^2x,fy)\Big\}\Big).$$

In 2017, Gornicki introduced the notion of F-expansion as follows:

Definition 2.8. [4] Let (X,d) be a metric space and let $f:X\to X$ be a self-mapping. Then, f is called an F-expansion on (X,d), if there exist $F\in\Im$ and $\gamma>0$ such that for all $x,y\in X$ for which $d(fx,fy)>0$, then $F(d(fx,fy))\geq F(d(x,y))+\gamma$, where $\Im$ is class of all mappings $F:(0,\infty)\to R$ such that

(F1) F is strictly increasing function, that is, for all $a,b\in(0,\infty)$, if $a<b$, then $F(a)<F(b)$.

(F2) For every sequence a_n of natural numbers, $\lim_{n\to\infty}a_n=0$ if and only if $\lim_{n\to\infty}F(a_n)=-\infty$.

(F3) There exists $q\in(0,1)$ such that $\lim_{a\to0^+}(a^qF(a))=0$.

3 Main Results

Motivated by Kumar and Arora [6], we introduce generalized $(\mathfrak{f}^*,\psi)$-contraction with the aid of the ψ-fixed point in the framework of $\mathcal{G}$-metric space. Also some ψ-fixed point results have been established by replacing the conditions $(F1)$, $(F3)$ and $(F3^{'})$ of [9] by a single condition (E).

Throughout this paper, we denote by $\mathcal{H}$, $\mathcal{R}$, $\mathcal{R}+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ the nonempty set, the set of real numbers, the set of positive real numbers, the set of integers and the set of positive integers, respectively. Also,
$F_h=\{\Omega\in\mathcal{H}:h\Omega=\Omega\}\,and\,K_\psi=\{\Omega\in\mathcal{H}:\psi(\Omega)=0\}$.

Let $\mathcal{F}_\mathcal{E}$ be the class of all continuous functions $\mathfrak{f}^*:(0,\infty)\times[0,\infty)^2\to\mathcal{R}$, which gratify the accompanying condition:
(E) For all sequences $\{e_n\},\{f_n\}$ and $\{g_n\}\in\mathcal{R}_+$,

$$\lim_{n\to\infty}\mathfrak{f}^*(e_n,f_n,g_n)=-\infty\ \text{ if and only if }\lim_{n\to\infty}e_n^2+e_n+f_n+g_n=0.$$

Proposition 3.1. If $\{e_n\},\{f_n\}$ and $\{g_n\}\in\mathcal{R}_+$, then

$$\lim_{n\to\infty}e_n^2+e_n+f_n+g_n=0\quad\text{if and only if}\quad\lim_{n\to\infty}e_n=0,\lim_{n\to\infty}f_n=0\quad\text{and}\quad\lim_{n\to\infty}g_n=0.$$

Now, we introduce new notion of generalized $(\mathfrak{f}^*,\psi)$-contraction function in $(\mathcal{H},\mathcal{G})$.

Definition 3.2. Let $\psi:\mathcal{H}\to[0,\infty)$ be a map in $(\mathcal{H},\mathcal{G})$. A function $h:\mathcal{H}\to\mathcal{H}$ is called generalized$(\mathfrak{f}^*,\psi)$-contraction if there exist $\mathfrak{f}^*\in\mathcal{F}_\mathcal{E}$ and $\chi>0$ such that $h\Omega\neq h\mho$ implies

$$\chi+\mathfrak{f}^*(\mathcal{G}(h\Omega,h\mho,h\wp),\psi(h\Omega),\psi(h\mho))\leq\mathfrak{f}^*(S_h(\Omega,\mho,\wp),\psi(\Omega),\psi(\mho)),\tag{3.1}$$

where

$$S_h(\Omega,\mho,\wp)=\max\{\mathcal{G}(\Omega,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\mho,h\wp,h\wp),\mathcal{G}(\wp,h\mho,h\mho),\mathcal{G}(\wp,h\Omega,h\Omega),\mathcal{G}(\Omega,h\wp,h\wp)\}.$$

Lemma 3.3. Let $\psi : \mathcal{H} \to [0, \infty)$ and $h : \mathcal{H} \to \mathcal{H}$ be $(\mathfrak{f}^*, \psi)$-contraction in $(\mathcal{H}, \mathcal{G})$, where $\mathfrak{f}^* \in \mathcal{F}_{\mathcal{E}}$, then
(i) Ω_n is Cauchy;
(ii) $\lim_{n\to\infty} \mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}) = \lim_{n\to\infty} \psi(\Omega_n)$.

Proof . Choose $\Omega_0 \in \mathcal{H}$ and define a sequence $\{\Omega_n\}$ by $\Omega_1 = h\Omega_0$, $\Omega_2 = h\Omega_1$, $\dots, h\Omega_{n+1} = \Omega_{n+2}$, for every $n \in \mathbb{Z}_+$. Inserting $\Omega = \Omega_{n-1}$, $\mho = \Omega_n$ and $\wp = \Omega_n$ in (3.1), we acquire

$$\chi + \mathfrak{f}^*(\mathcal{G}(h\Omega_{n-1}, h\Omega_n, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) \leq \mathfrak{f}^*(S_h(\Omega_{n-1}, \Omega_n, \Omega_n), \psi(\Omega_{n-1}), \psi(\Omega_n)), \tag{3.2}$$

for all $\Omega_{n-1}, \Omega_n \in \mathcal{H}$, where

$$\begin{aligned}
S_h(\Omega_{n-1}, \Omega_n, \Omega_n) =& \max\{\mathcal{G}(\Omega_{n-1}, h\Omega_n, h\Omega_n), \mathcal{G}(\Omega_n, h\Omega_{n-1}, h\Omega_{n-1}), \mathcal{G}(\Omega_n, h\Omega_n, h\Omega_n), \mathcal{G}(\Omega_n, h\Omega_n, h\Omega_n),\\
& \mathcal{G}(\Omega_n, h\Omega_{n-1}, h\Omega_{n-1}), \mathcal{G}(\Omega_{n-1}, h\Omega_n, h\Omega_n)\}\\
=& F(\max\{\mathcal{G}(\Omega_{n-1}, \Omega_{n+1}, \Omega_{n+1}), \mathcal{G}(\Omega_n, \Omega_n, \Omega_n), \mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}), \mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}), \mathcal{G}(\Omega_n, \Omega_n, \Omega_n),\\
& \mathcal{G}(\Omega_{n-1}, \Omega_{n+1}, \Omega_{n+1})\})\\
=& F(\max\{\mathcal{G}(\Omega_{n-1}, \Omega_{n+1}, \Omega_{n+1}), \mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1})\}).
\end{aligned}$$

If there exists $n \in \mathbb{Z}_+$ such that

$$\max\{G(\Omega_{n-1}, \Omega_{n+1}, \Omega_{n+1}), G(\Omega_n, \Omega_{n+1}, \Omega_{n+1})\} = G(\Omega_n, \Omega_{n+1}, \Omega_{n+1})$$

then (3.2) becomes

$$\chi + \mathfrak{f}^*(\mathcal{G}(h\Omega_{n-1}, h\Omega_n, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) \leq \mathfrak{f}^*(\mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}), \psi(\Omega_{n-1}), \psi(\Omega_n)).$$

Thus,

$$\chi + \mathfrak{f}^*(\mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}), \psi(\Omega_n), \psi(\Omega_{n+1})) \leq \mathfrak{f}^*(\mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1}), \psi(\Omega_{n-1}), \psi(\Omega_n)). \tag{3.3}$$

Since $\chi > 0$, we get a contradiction. Thus,

$$\max\{\mathcal{G}(\Omega_{n-1}, \Omega_{n+1}, \Omega_{n+1}), \mathcal{G}(\Omega_n, \Omega_{n+1}, \Omega_{n+1})\} = \mathcal{G}(\Omega_{n-1}, \Omega_n, \Omega_n).$$

From (3.2), we get

$$\begin{aligned}
\mathfrak{f}^*(\mathcal{G}(h\Omega_{n-1}, h\Omega_n, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) &\leq \mathfrak{f}^*(\mathcal{G}(\Omega_{n-1}, \Omega_n, \Omega_n)), \psi(\Omega_{n-1}), \psi(\Omega_n)) - \chi\\
&\leq \mathfrak{f}^*(\mathcal{G}(\Omega_{n-2}, \Omega_{n-1}, \Omega_{n-1}), \psi(\Omega_{n-2}), \psi(\Omega_{n-1})) - 2\chi\\
&\leq \mathfrak{f}^*(\mathcal{G}(\Omega_{n-3}, \Omega_{n-2}, \Omega_{n-2}), \psi(\Omega_{n-3}), \psi(\Omega_{n-2})) - 3\chi\\
&\ \vdots\\
&\leq \mathfrak{f}^*(\mathcal{G}(\Omega_0, \Omega_1, \Omega_1), \psi(\Omega_0), \psi(\Omega_1)) - n\chi.
\end{aligned} \tag{3.4}$$

Let $n \to \infty$ in (3.4), we acquire

$$\lim_{n\to\infty} \mathfrak{f}^*(\mathcal{G}(h\Omega_{n-1}, h\Omega_n, h\Omega_n), \psi(h\Omega_{n-1}), \psi(h\Omega_n)) = -\infty.$$

With the assistance of (E), we have

$$\lim_{n\to\infty} \mathcal{G}(\Omega_{n-1}, \Omega_n, \Omega_n) = \lim_{n\to\infty} \psi(\Omega_n) = 0. \tag{3.5}$$

Assume that $\{\Omega_n\}$ is not Cauchy in $\mathcal{H}$.Then, there exist $\delta > 0$ and subsequences $\{\Omega_{n_e}\}$ and $\{\Omega_{r_e}\}$ of $\{\Omega_n\}$ such that $\mathcal{G}(\Omega_{n_e}, \Omega_{r_e}, \Omega_{r_e}) \geq \delta$ and $\mathcal{G}(\Omega_{n_e}, \Omega_{r_e-1}, \Omega_{r_e-1}) < \delta$, for each $r_e > n_e > e$, where $e \in \mathbb{Z}_+$. Now,

$$\begin{aligned}
\delta &\leq \mathcal{G}(\Omega_{n_e}, \Omega_{r_e}, \Omega_{r_e})\\
&\leq \mathcal{G}(\Omega_{r_e}, \Omega_{r_e-1}, \Omega_{r_e-1}) + \mathcal{G}(\Omega_{r_e-1}, \Omega_{n_e}, \Omega_{n_e})\\
&\leq \mathcal{G}(\Omega_{r_e}, \Omega_{r_e-1}, \Omega_{r_e-1}) + \delta.
\end{aligned}$$

With the assistance of (3.5) and making $e \to \infty$, we acquire

$$\lim_{e\to\infty} \mathcal{G}(\Omega_{n_e}, \Omega_{r_e}, \Omega_{r_e}) = \delta. \tag{3.6}$$

Further for each $e \geq n_1$, there exists $n_1 \in \mathbb{Z}_+$ ensuring that

$$\mathcal{G}(\Omega_{r_e}, \Omega_{r_e+1}, \Omega_{r_e+1}) < \frac{\delta}{8} \text{ and } \mathcal{G}(\Omega_{n_e}, \Omega_{n_e+1}, \Omega_{n_e+1}) < \frac{\delta}{8}. \tag{3.7}$$

Now, we show that $\mathcal{G}(\Omega_{r_e+1}, \Omega_{n_e+1}, \Omega_{n_e+1}) > 0$, for every $e \geq n_1$. Assume that there exists $g \geq n_1$ such that

$$\mathcal{G}(\Omega_{r_g+1}, \Omega_{n_g+1}, \Omega_{n_g+1}) = 0. \tag{3.8}$$

With the aid of (3.6), (3.7) and (3.8), we acquire

$$\begin{aligned}
\delta &\leq \mathcal{G}(\Omega_{r_g}, \Omega_{n_g}, \Omega_{n_g}) \\
&\leq \mathcal{G}(\Omega_{r_g}, \Omega_{r_g+1}, \Omega_{r_g+1}) + \mathcal{G}(\Omega_{r_g+1}, \Omega_{n_g}, \Omega_{n_g}) \\
&\leq \mathcal{G}(\Omega_{r_g}, \Omega_{r_g+1}, \Omega_{r_g+1}) + \mathcal{G}(\Omega_{r_g+1}, \Omega_{n_g+1}, \Omega_{n_g+1}) + \mathcal{G}(\Omega_{n_g+1}, \Omega_{n_g}, \Omega_{n_g}) \\
&< \frac{\delta}{8} + 0 + \frac{\delta}{8} = \frac{\delta}{4},
\end{aligned}$$

which is a contradiction. Consequently,

$$\mathcal{G}(\Omega_{r_e+1}, \Omega_{n_e+1}, \Omega_{n_e+1}) > 0. \tag{3.9}$$

for every $e \geq n_1$. Further inserting $\Omega = \Omega_{r_e}$ and $\mho = \Omega_{n_e}$ in (3.1), we acquire

$$\chi + \mathfrak{f}^*(\mathcal{G}(h\Omega_{r_e}, h\Omega_{n_e}, h\Omega_{n_e}), \psi(h\Omega_{r_e}), \psi(h\Omega_{n_e})) \leq \mathfrak{f}^*(S_h(\Omega_{r_e}, \Omega_{n_e}, \Omega_{n_e}), \psi(\Omega_{r_e}), \psi(\Omega_{n_e})),$$

where

$$\begin{aligned}
S_h(\Omega_{r_e}, \Omega_{n_e}, \Omega_{n_e}) =& \max\{\mathcal{G}(\Omega_{r_e}, h\Omega_{n_e}, h\Omega_{n_e}), \mathcal{G}(\Omega_{n_e}, h\Omega_{r_e}, h\Omega_{r_e}), \mathcal{G}(\Omega_{n_e}, h\Omega_{n_e}, h\Omega_{n_e}), \mathcal{G}(\Omega_{n_e}, h\Omega_{n_e}, h\Omega_{n_e}), \\
& \mathcal{G}(\Omega_{n_e}, h\Omega_{r_e}, h\Omega_{r_e}), \mathcal{G}(\Omega_{r_e}, h\Omega_{n_e}, h\Omega_{n_e})\} \\
=& \max\{\mathcal{G}(\Omega_{r_e}, \Omega_{n_{e+1}}, \Omega_{n_{e+1}}), \mathcal{G}(\Omega_{n_e}, \Omega_{r_{e+1}}, \Omega_{r_{e+1}}), \mathcal{G}(\Omega_{n_e}, \Omega_{n_{e+1}}, \Omega_{n_{e+1}}), \mathcal{G}(\Omega_{n_e}, \Omega_{n_{e+1}}, \Omega_{n_{e+1}}), \\
& \mathcal{G}(\Omega_{n_e}, \Omega_{r_{e+1}}, \Omega_{r_{e+1}}), \mathcal{G}(\Omega_{r_e}, \Omega_{n_{e+1}}, \Omega_{n_{e+1}})\}.
\end{aligned}$$

By (3.5), (3.6) and continuity property of $\mathfrak{f}^*$, we acquire

$$\chi + \mathfrak{f}^*(\delta, 0, 0) \leq \mathfrak{f}^*(\delta, 0, 0),$$

a contradiction, which shows that $\{\Omega_n\}$ is a Cauchy sequence in $\mathcal{H}$. □

Theorem 3.4. Let $h : \mathcal{H} \to \mathcal{H}$ be generalized $(\mathfrak{f}^*, \psi)$-contraction and $\psi : \mathcal{H} \to [0, \infty)$ be lower semi-continuous mapping having $F_h \subseteq K_\psi$ in complete $(\mathcal{H}, \mathcal{G})$. Then, h has a unique ψ-fixed point.

Proof . Let $\Omega_0 \in \mathcal{H}$ be any point. Inserting $h^n\Omega_0 = \Omega_{n+1}$, for every $n \in \mathbb{Z}_+$. If there exists $n \in \mathbb{Z}_+$ such that $h\Omega_n = \Omega_n$. Using given assumption of Theorem 3.4, we get that Ω_n is the ψ-fixed point of h. Assume that $\mathcal{G}(h\Omega_n, \Omega_n, \Omega_n) > 0$, for every $n \in \mathbb{Z}_+$. From Lemma 3.3, we get that $\{\Omega_n\}$ is a Cauchy sequence. But $(\mathcal{H}, \mathcal{G})$ is complete, which indicates that there exixts $\Omega \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \Omega_n = \Omega. \tag{3.10}$$

By Lemma 3.3 and lower semi-continuity property of ψ, we have

$$0 \leq \psi(\Omega) \leq \lim_{n\to\infty} \inf \psi(\Omega_n) = 0,$$

which yields that

$$\psi(\Omega) = 0. \tag{3.11}$$

Let $\mathcal{D} = \{n \in \mathbb{Z}_+ : h\Omega = \Omega_n\}$. When $\mathcal{D}$ is infinite set, then there exists a subsequence $\{\Omega_{n_e}\}$ of $\{\Omega_n\}$ having $h\Omega = \lim_{e\to\infty} \Omega_{n_e}$, which indicates that $h\Omega = \Omega$. Further, when $\mathcal{D}$ is finite set, then $\mathcal{G}(h\Omega, \Omega_n, \Omega_n) > 0$, for infinte $n \in \mathbb{Z}_+$. Consequently, there exists a subsequence $\{\Omega_{n_e}\}$ of $\{\Omega_n\}$ such that $\mathcal{G}(h\Omega, \Omega_{n_e}, \Omega_{n_e}) > 0$, for every $e \in \mathbb{Z}_+$. Since h is a generalized $(\mathfrak{f}^*, \psi)$-contraction, we have

$$\chi + \mathfrak{f}^*(\mathcal{G}(h\Omega_{n_e}, h\Omega, h\Omega), \psi(h\Omega_{n_e}), \psi(h\Omega)) \leq \mathfrak{f}^*(S_h(\Omega_{n_e}, \Omega, \Omega), \psi(\Omega_{n_e}), \psi(\Omega)),$$

where

$$S_h(\Omega_{n_e}, \Omega, \Omega) = \max\{\mathcal{G}(\Omega_{n_e}, h\Omega, h\Omega), \mathcal{G}(\Omega, h\Omega_{n_e}, h\Omega_{n_e}), \mathcal{G}(\Omega, h\Omega, h\Omega), \mathcal{G}(\Omega, h\Omega, h\Omega), \mathcal{G}(\Omega, h\Omega_{n_e}, h\Omega_{n_e}), \mathcal{G}(\Omega_{n_e}, h\Omega, h\Omega)\}.$$

From (3.10), (3.11), condition(E), continuity of $\mathfrak{f}^*$ and Lemma 3.3 as $e \to \infty$, we get $\chi + \mathfrak{f}^*(\mathcal{G}(\Omega, h\Omega, h\Omega), \psi(\Omega), \psi(h\Omega)) \leq \mathfrak{f}^*(\mathcal{G}(\Omega, h\Omega, h\Omega), \psi(\Omega), \psi(\Omega))$, which implies that $\chi + \mathfrak{f}^*(\mathcal{G}(\Omega, h\Omega, h\Omega), 0, 0) \leq \mathfrak{f}^*(\mathcal{G}(\Omega, h\Omega, h\Omega), 0, 0)$, which is a contradiction because $\chi > 0$. Thus,

$$\mathcal{G}(\Omega, h\Omega, h\Omega) = 0. \tag{3.12}$$

Equations (3.11) and (3.12) yields that Ω is the ψ-fixed point of h. Now, we show that the ψ-fixed point of h is unique. Let Ω_1, Ω_2 be the distinct ψ-fixed points of h. Thus, $\mathcal{G}(h\Omega_1, h\Omega_2, h\Omega_2) = \mathcal{G}(\Omega_1, \Omega_2, \Omega_2) > 0$. Now, by setting $\Omega = \Omega_1$ and $\mho = \Omega_2$ in (3.1), we obtain

$$\chi + \mathfrak{f}^*(\mathcal{G}(h\Omega_1, h\Omega_2, h\Omega_2), \psi(h\Omega_1), \psi(h\Omega_2)) \leq \mathfrak{f}^*(S_h(\Omega_1, \Omega_2, \Omega_2), \psi(\Omega_1), \psi(\Omega_2)), \tag{3.13}$$

where

$$\begin{aligned} S_h(\Omega_1, \Omega_2, \Omega_2) =& \max\{\mathcal{G}(\Omega_1, h\Omega_2, h\Omega_2), \mathcal{G}(\Omega_2, h\Omega_1, h\Omega_1), \mathcal{G}(\Omega_2, h\Omega_2, h\Omega_2), \mathcal{G}(\Omega_2, h\Omega_2, h\Omega_2), \mathcal{G}(\Omega_2, h\Omega_1, h\Omega_1), \\ & \mathcal{G}(\Omega_1, h\Omega_2, h\Omega_2)\} \\ =& \max\{\mathcal{G}(\Omega_1, \Omega_2, \Omega_2), \mathcal{G}(\Omega_2, \Omega_1, \Omega_1), \mathcal{G}(\Omega_2, \Omega_2, \Omega_2), \mathcal{G}(\Omega_2, \Omega_2, \Omega_2), \mathcal{G}(\Omega_2, \Omega_1, \Omega_1), \mathcal{G}(\Omega_1, \Omega_2, \Omega_2)\} \\ =& \mathcal{G}(\Omega_1, \Omega_2, \Omega_2). \end{aligned}$$

By (3.13), we have

$$\chi + \mathfrak{f}^*(\mathcal{G}(\Omega_1, \Omega_2, \Omega_2), 0, 0) \leq \mathfrak{f}^*(\mathcal{G}(\Omega_1, \Omega_2, \Omega_2), 0, 0),$$

which is a contradiction. Thus, $\Omega_1 = \Omega_2$, which indicates that h has a unique ψ-fixed point in $\mathcal{H}$. □

Example 3.5. Consider $\mathcal{H} = [0, 6]$ associated with the usual metric $\mathcal{G}$. We define $h : \mathcal{H} \to \mathcal{H}$ by

$$h(\Omega) = \begin{cases} 0, & \text{if } 0 \leq \Omega < 5.5, \\ s\ln(\frac{\Omega}{9}), & 5.5 \leq \Omega < 6, \end{cases}$$

for all $\Omega \in \mathcal{H}$ and $s < 1$. Now, it is clear that h is not continuous at $\Omega = 5.5$. Let $\mathfrak{f}^* : (0, \infty) \times [0, \infty)^2 \to \mathbb{R}$ be defined as $\mathfrak{f}^*(e, f, g) = \ln(e + e^2 + f + g)$, for all $e, f, g \in [0, \infty)$ and $g \neq 0$. Let $\psi : \mathcal{H} \to [0, \infty)$ be defined as $\psi(\Omega) = \Omega$, for every $\Omega \in \mathcal{H}$. It is clear that ψ is lower semi-continuous and $\mathfrak{f}^* \in \mathcal{F}_{\mathcal{E}}$. Now, we assert that

$$\mathcal{G}(h\Omega, h\mho, h\mho) + \psi(h\Omega) + \psi(h\mho) \leq e^{-\chi}(S_h(\Omega, \mho, \mho) + \psi(\Omega) + \psi(\mho)), \tag{3.14}$$

where

$$S_h(\Omega, \mho, \mho) = \max\{\mathcal{G}(\Omega, h\mho, h\mho), \mathcal{G}(\mho, h\Omega, h\Omega), \mathcal{G}(\mho, h\mho, h\mho), \mathcal{G}(\mho, h\mho, h\mho), \mathcal{G}(\mho, h\Omega, h\Omega), \mathcal{G}(\Omega, h\mho, h\mho)\},$$

for all $\Omega, \mho \in \mathcal{H}$ and $h\mho \neq h\Omega$. Three cases arise:

Case 1: If $\Omega, \mho \in [0, 5.5)$, then (3.14) holds trivially.

Case 2: If $\Omega, \mho \in [5.5, 6]$, then

$$\begin{aligned} \mathcal{G}(h\Omega, h\mho, h\mho) + \psi(h\Omega) + \psi(h\mho) &= 2\max\{h\Omega, h\mho, h\mho\} \\ &\leq 2\max\{s\Omega, s\mho, s\mho\} \\ &= s(2\max\{\Omega, \mho, \mho\}). \end{aligned} \tag{3.15}$$

Now,

$$S_h(\Omega,\mho,\mho)=\max\{\mathcal{G}(\Omega,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\mho,h\mho,h\mho),\mathcal{G}(\mho,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\Omega,h\mho,h\mho)\}. \tag{3.16}$$

With the aid of (3.15) and (3.16), we acquire

$$\mathcal{G}(h\Omega,h\mho,h\mho)+\psi(h\Omega)+\psi(h\mho)\le s(\mathcal{G}(\Omega,\mho,\mho)+\psi(\Omega)+\psi(\mho)).$$

Case 3: If $\Omega\in[5.5,6]$ and $\mho\in[0,5.5)$, then

$$\begin{aligned}\mathcal{G}(h\Omega,h\mho,h\mho)+\psi(h\Omega)+\psi(h\mho)&=2\max\{h\Omega,h\mho,h\mho\}\\&=2\max\{h\Omega,0,0\}\\&\le 2\max\{s\Omega,0.0\}\\&=s(2\max\{\Omega,\mho,\mho\}).\end{aligned} \tag{3.17}$$

Now,

$$S_h(\Omega,\mho,\mho)=\max\{\mathcal{G}(\Omega,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\mho,h\mho,h\mho),\mathcal{G}(\mho,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\Omega,h\mho,h\mho)\}. \tag{3.18}$$

From (3.17) and (3.18), we get

$$\mathcal{G}(h\Omega,h\mho,h\mho)+\psi(h\Omega)+\psi(h\mho)\le s(\mathcal{G}(\Omega,\mho,\mho)+\psi(\Omega)+\psi(\mho)).$$

Thus, h and ψ gratify all the conditions of Theorem 3.4 with $\chi=-\ln s>0$. Consequently, h has a unique ψ-fixed point, which is zero.

Corollary 3.6. Let $\psi:\mathcal{H}\to[0,\infty)$ be lower semi-continuous mapping having $F_h\subseteq K_\psi$ and $h:\mathcal{H}\to\mathcal{H}$ be a map gratifying the subsequent condition

$$h\Omega\ne h\mho\quad\text{then}\quad \chi+\mathfrak{f}^*(\mathcal{G}(h\Omega,h\mho,h\wp),\psi(h\Omega),\psi(h\mho))\le\mathfrak{f}^*(\mathcal{G}(\Omega,\mho,\wp),\psi(\Omega),\psi(\mho)), \tag{3.19}$$

where $\mathfrak{f}^*\in\mathcal{F}_{\mathcal{E}}$ and $\chi>0$ in complete $(\mathcal{H},\mathcal{G})$. Then, h has a unique ψ-fixed point.

Corollary 3.7. [6] Let $(\mathcal{H},\mathcal{G})$ be a complete $\mathcal{G}-$metric space and $h:\mathcal{H}\to\mathcal{H}$ satisfies the following condition

$$h\Omega\ne h\mho\quad\text{then}\quad \chi+\mathfrak{f}^*(\mathcal{G}(h\Omega,h\mho,h\wp))\le\mathfrak{f}^*(S_h(\Omega,\mho,\wp)),$$

where

$$S_h(\Omega,\mho,\wp)\;=\;\max\{\mathcal{G}(\Omega,h\mho,h\mho),\mathcal{G}(\mho,h\Omega,h\Omega),\mathcal{G}(\mho,h\wp,h\wp),\mathcal{G}(\wp,h\mho,h\mho),\mathcal{G}(\wp,h\Omega,h\Omega),\mathcal{G}(\Omega,h\wp,h\wp)\},$$

$\mathfrak{f}^*\in\Im$ and $\chi>0$. Then, h has a unique fixed point in $\mathcal{H}$.

Proof . By letting ψ as a zero constant function on $\mathcal{H}$ in Theorem 3.4, we get the result. □

We can deduce the results for F-weak contraction of Wardowski type with the aid of Theorem 3.4 as follows.

Corollary 3.8. [11] Let (X,d) be a complete metric space and let $\mathcal{T}:X\to X$ be an F-weak contraction. If F or $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a unique fixed point $x^*\in X$ and the sequence $\{\mathcal{T}^n x\}$ converges to x^*, for every $x\in X$, where n varies from 1 to ∞.

Corollary 3.9. [8] Let $(\mathcal{H},d)$ be a complete metric space and $\psi:\mathcal{H}\to[0,\infty)$ be lower semi-continuous mapping having $F_h\subseteq K_\psi$. Assume that $h:\mathcal{H}\to\mathcal{H}$ satisfies the following condition

$$h\Omega\ne h\mho\quad\text{then}\quad \chi+\mathfrak{f}^*(d(h\Omega,h\mho),\psi(h\Omega),\psi(h\mho))\le\mathfrak{f}^*(d(\Omega,\mho),\psi(\Omega),\psi(\mho)), \tag{3.20}$$

where $\mathfrak{f}^*\in\mathcal{F}_{\mathcal{E}}$ and $\chi>0$. Then, h has a unique ψ-fixed point.

Proof . With the aid of Theorem 3.4, result can easily be deduced. □

Corollary 3.10. (Banach Contraction Principle in $(\mathcal{H},\mathcal{G})$) Let $(\mathcal{H},\mathcal{G})$ be a complete metric space and $h:\mathcal{H}\to\mathcal{H}$ be a map gratifying

$$\mathcal{G}(h\Omega,h\mho,h\mho)\leq\lambda\mathcal{G}(\Omega,\mho,\mho),$$

for some $\lambda\in(0,1)$ and $\Omega,\mho\in\mathcal{H}$. Then, h has a unique fixed point.

Proof . If $\mathfrak{f}^*(e,f,g)=\ln(e+f+g)$ and $\psi(\Omega)=0\,\forall\,\Omega\in\mathcal{H}$ in Theorem 3.4, we can deduce the result. □

Corollary 3.11. [2] Let $(\mathcal{H},d)$ be a complete metric space and $h:\mathcal{H}\to\mathcal{H}$ be a map gratifying

$$d(h\Omega,h\mho)\leq\lambda max\{d(\Omega,\mho),\frac{d(\Omega,h\mho)+d(\mho,h\Omega)}{2},\frac{d(\Omega,h\Omega)+d(\mho,h\mho)}{2}\},$$

for some $\lambda\in(0,1)$ and $\Omega,\mho\in\mathcal{H}$. Then, h has a unique fixed point.

Proof . If $\mathfrak{f}^*(e,f,g)=\ln(e+f+g)$ and $\psi(\Omega)=0$, for all $\Omega\in\mathcal{H}$ in Theorem 3.4, we get the result for $(\mathcal{H},d)$. □

4 Application to integral equation

In this section, we apply our results to prove that solution of the following non-linear integral equation exists and unique.

$$x(t)=\eta(t)+\int_0^t F(t,s,x(s))ds, \tag{4.1}$$

for some $t\in[0,M]$, where $M>0, F:[0,M]\times[0,M]\times\mathcal{R}\to\mathcal{R}$ and $\eta:[0,M]\to\mathcal{R}$. Let $X=C([0,M],\mathcal{R})$ be set of all continuous functions, $x:[0,M]\to\mathcal{R}$ with norm

$$\|x\|=\sup_{t\in[0,M]}e^{-t}|x(t)|.$$

Theorem 4.1. Let F be a continuous function such that

$$|\ F(t,s,x(s))-F(t,s,y(s))\ |\leq\frac{|\ x(s)-y(s)\ |}{\chi\|x(s)-y(s)\|+1},$$

for all $x,y\in C([0,M],\mathcal{R}),x,y\in[0,M]$ and for some $\chi>0$. Then, (4.1) has a unique solution.

Proof . Let $\mathcal{G}:X\times X\times X\to\mathcal{R}^+$ be defined as

$$\mathcal{G}(x,y,z)=\sup_{t\in[0,M]}|x(t)-y(t)|+\sup_{t\in[0,M]}|y(t)-z(t)|+\sup_{t\in[0,M]}|x(t)-z(t).|$$

Clearly, $(X,\mathcal{G})$ is a G-complete metric space. Let $h:X\to X$ be defined as

$$h(x(t))=\eta(t)+\int_0^t F(t,s,x(s))ds,$$

for all $x\in X$. Now, we define two functions $\mathfrak{f}^*:(0,\infty)\times[0,\infty)^2\to\mathcal{R}$ and $\psi:X\to[0,\infty)$ such that $\mathfrak{f}^*(e,f,g)=\frac{-1}{(e+f+g)}$, for all $e,f,g\in[0,\infty),e\neq0$ and $\psi(x)=0,$for all $x\in X$. Now, for $x,y\in X$ with $hx\neq hy$, we have

$$\begin{aligned}|\ h(x(t))-h(y(t))\ |&=|\int_0^t F(t,s,x(s))ds-\int_0^t F(t,s,y(s))ds\ |\\&\leq\int_0^t|\ F(t,s,x(s))-F(t,s,y(s))\ |\ ds\\&\leq\int_0^t\frac{1}{\chi\|x(s)-y(s)\|+1}(|\ x(s)-y(s)\ |\ e^{-s})e^s ds\\&\leq\frac{\mathcal{G}(x,y,z)}{\chi\mathcal{G}(x,y,z)+1},\end{aligned}$$

which indicates that

$$sup \mid h(x(t)) - h(y(t)) \mid e^{-t} \leq \frac{\mathcal{G}(x,y,z)}{\chi\mathcal{G}(x,y,z)+1},$$

or

$$\chi + \frac{1}{\mathcal{G}(x,y,z)} \leq \frac{1}{\mathcal{G}(hx,hy,hz)},$$

or

$$\chi + \frac{1}{\mathcal{G}(x,y,z)+\psi(x)+\psi(y)} \leq \frac{1}{\mathcal{G}(hx,hy,hz)+\psi(hx)+\psi(hy)},$$

or

$$\chi + \frac{-1}{\mathcal{G}(hx,hy,hz)+\psi(hx)+\psi(hy)} \leq \frac{-1}{\mathcal{G}(x,y,z)+\psi(x)+\psi(y)},$$

or

$$\chi + \mathfrak{f}^*(\mathcal{G}(hx,hy,hz),\psi(hx),\psi(hy)) \leq \mathfrak{f}^*(\mathcal{G}(x,y,z),\psi(x),\psi(y)),$$

which shows that h is generalized $(\mathfrak{f}^*, \psi)$-contraction. Therefore, all the requirements of Corollary 3.6 are fulfilled. Hence, integral equation (4.1) has a unique solution. □

5 Conclusion

In this paper, the ψ-fixed point results are investigated with the aid of generalized $(\mathfrak{f}^*, \psi)$-contractive and functions of in the context of complete G-metric space. In this way, the relationship of the contractive functions of Wardowski kind with previous concept the ψ-fixed point is investigated through indispensable theorems. Moreover, we established the results by substituting the continuity condition of $\mathfrak{f}^*$ by lower semi-continuity of ψ, for detail please see [6, 9]. Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Our results can be utilized to find solution of fractional non-linear differential and integral equations.

References

[1] B. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundam. Math. **3** (1922), 133–145.

[2] L. Ciric, *Fixed points for generalized multi-valued mappings*, Mat. Vesnik **24** (1972), 265–272.

[3] N.V. Dung and V.L. Hang, *A fixed point theorem for generalized F-contractions on complete metric spaces*, Vietnam J. Math. **43** (2015), 743–753.

[4] J. Gornicki, *Fixed point theorems for F-expanding mappings*, Fixed Point Theory Appl. **9** (2017), 1–10.

[5] M. Jleli, B. Samet, and C. Vetro, *Fixed point theory in partial metric spaces via φ-fixed point concept in metric spaces*, J. Inequal. Appl. **426** (2014), no. 1, 1–9.

[6] M. Kumar and S. Arora, *Fixed point theorems for modified generalized F-contraction in G-metric spaces*, Bol. Soc. Paran. Mat. **40** (2022), 1–8.

[7] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 289–297.

[8] H.N. Saleh, M. Imdad and W.M. Alfaqih, *Some metrical φ-fixed point results of Wardowski type with applications to integral equations*, Bol. Soc. Paran. Mat. **40** (2022), 1–11.

[9] H. Piri and P. Kumam, *Wardowski type fixed point theorems in complete metric spaces*, Fixed Point Theory Appl. **45** (2016), 1–12.

[10] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl. **210** (2014), 1–13.

[11] D. Wardowski and N.V. Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonst. Math. **47** (2014), 146–155.

[12] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **94** (2012), 1–6.