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Existence result for double phase problem involving the $(p(x), q(x))$-Laplacian-like operators

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Abstract

The paper study the existence of at least one weak solutions for Dirichlet boundary value problem involving the $(p(x), q(x))$-Laplacian-like operators of the following form:

$$\begin{cases} -\Delta^l_{p(x)} - \Delta^l_{q(x)} = \lambda g(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta^l_{r(x)}$ is the $r(x)$-Laplacian-like operators, Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a real parameter and g is Carathéodory function satisfies the assumption of growth. The existence is proved by using Berkovits' topological degree.

Keywords: Dirichlet problems, double phase problems, $(p(x), q(x))$- Laplacian-like operators, topological degree methods
2020 MSC: 35J60, 35D30, 46E30, 47H11, 47H30

1 Introduction

In this work, we consider the following nonlinear problems for the $(p(x), q(x))$-Laplacian-like operators:

$$\begin{cases} -\Delta^l_{p(x)} - \Delta^l_{q(x)} = \lambda g(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta^l_{p(x)} := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big),$$

and

$$\Delta^l_{q(x)} := \operatorname{div}\Big(|\nabla u|^{q(x)-2}\nabla u + \frac{|\nabla u|^{2q(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2q(x)}}}\Big),$$

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are the $p(x)$-Laplacian-like operators and $q(x)$-Laplacian-like operators respectively, Ω is smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$, $p(\cdot), q(\cdot) \in C_+(\overline{\Omega})$, λ is a real parameter and $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfy the assumption of growth.

The motivation for this study came from the application of similar problems in physics to model the behavior of electrorheological fluids (see [19, 23]), specifically capillarity phenomena, which is dependent on solid-liquid interfacial characteristics such as surface tension, contact angle, and solid surface geometry. In the context of the study of capillarity phenomena, recently, problem like (1.1) has begun to receive more and more attention, for instance [10, 25, 26, 29, 9, 20, 6, 21, 2, 13].

Many researchers have investigated problems relating to (1.1), for example, W. Ni et al. [14, 15] study the following equation:

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N.$$

The operator $-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ is most often denoted by the specified mean curvature operator.

Not that, if $\lambda > 0$, g independent of ∇u and without the term $\Delta^l_{q(x)}$, then we obtain the following problem:

$$\begin{cases} -\Delta^l_{p(x)} = \lambda g(x,u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this case, M.Rodrigues [22], by using Mountain Pass lemma and Fountain theorem, established the existence of non-trivial solutions of (1.2).

In the present paper, we study the existence of weak solution to the problem (1.1), by using another approach based on the topological degree for a class of demicontinuous operators of generalized (S_+) type [5] and the theory of the variable-exponent Sobolev spaces.

This article is organized as follows. In section 2 we present some necessary preliminary about Sobolev spaces with variable exponent and an outline of Berkovits' topological degree theory. In section 3 we give our basic assumptions, some technical lemmas and finally, we prove the existence of weak solutions of (1.1).

2 Preliminaries

2.1 Lebesgue-Sobolev spaces with variable exponent

In this subsection we give some definitions and results about Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [7, 12, 16, 17, 18] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \quad \text{and} \quad p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\{\lambda > 0 : \rho_{p(x)}\left(\frac{u}{\lambda}\right) \leq 1\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx, \quad \text{for all} \quad u \in L^{p(x)}(\Omega).$$

Proposition 2.1. [7] Let (u_n) and $u \in L^{p(\cdot)}(\Omega)$, then

$$|u|_{p(x)} < 1 (resp. = 1; > 1) \;\Leftrightarrow\; \rho_{p(x)}(u) < 1 (resp. = 1; > 1) \tag{2.1}$$

$$|u|_{p(x)} > 1 \;\Rightarrow\; |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{2.4}$$

Remark 2.2. According to (2.2) and (2.3), we have

$$|u|_{p(x)} \le \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{2.6}$$

Proposition 2.3. [12] The spaces $L^{p(x)}(\Omega)$ is a separable and reflexive Banach spaces.

Proposition 2.4. [12] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality

$$|\int_\Omega uv \; dx| \le \left(\frac{1}{p-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. If $r_1, r_2 \in C_+(\overline{\Omega})$ with $r_1(x) \le r_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \Big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.6. [8, 24] If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x - y| \le \dfrac{1}{2}$ one has

$$|p(x) - p(y)| \le \frac{\alpha}{-\log|x-y|}, \tag{2.8}$$

then we have the poincaré inequality, i.e. the exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \quad \text{for all} \quad u \in W_0^{1,p(\cdot)}(\Omega). \tag{2.9}$$

In this paper we will use the following equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$. Furthermore, we have the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$(see [12]).

Proposition 2.7. [7, 12] The spaces $\left(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ and $\left(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ are separable and reflexive Banach spaces.

Remark 2.8. The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|u|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$.

2.2 Review on some classes of mappings and topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [1, 5, 27, 4, 11].

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle\cdot,\cdot\rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 2.9. Let Y be real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be

1. bounded, if it takes any bounded set into a bounded set.
2. demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
3. compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.10. A mapping $F : \Omega \subset X \to X^*$ is said to be

1. of class (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
2. quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 2.11. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

1. F of class $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.
2. F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

In the following, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \left\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\right\},$$
$$\mathcal{F}_{T,B}(\Omega) := \left\{F : \Omega \to X : F \text{ is bounded, demicontinuous and of class } (S_+)_T\right\},$$
$$\mathcal{F}_T(\Omega) := \left\{F : \Omega \to X :\ F \text{ is demicontinuous and of class } (S_+)_T\right\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$. Now, let $\mathcal{O}$ be the collection of all bounded open set in X and we define

$$\mathcal{F}(X) := \left\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\right\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 2.12. [11, Lemma 2.3] Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true:

1. If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.
2. If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.

Definition 2.13. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy
$\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \text{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.14. [11, Lemma 2.5] *The above affine homotopy is of class* $(S_+)_T$.

Next, as in [11] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 2.15. Let

$$M := \big\{(F,E,h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. (*Normalization*) For any $h \in E$, we have
$$d(I,E,h) = 1.$$
2. (*Additivity*) Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E}\backslash(E_1 \cup E_2))$, then we have
$$d(F,E,h) = d(F,E_1,h) + d(F,E_2,h).$$
3. (*Homotopy invariance*) If $\mathcal{H} :\ [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \mathcal{H}(t, \partial E)$ for all $t \in [0,1]$, then
$$d(\mathcal{H}(t,\cdot), E, h(t)) = const \quad \text{for all} \quad t \in [0,1].$$
4. (*Existence*) If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.

Definition 2.16. [11, Definition 3.3] The above degree is defined as follows:

$$d(F,E,h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [5] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$.

3 Assumptions and main results

In this section, we will discuss the existence of weak solutions of the problem (1.1). We assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p, q \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.8), with $2 \leq q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ \leq \infty$, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(M_1) g is a Carathódory function.

(M_2) There exists $\varrho > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that $|g(x,\zeta,\xi)| \leq \varrho(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1})$, for a.e. $x \in \Omega$ and all $(\zeta,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Definition 3.1. $u \in W_0^{1,p(x)}(\Omega)$ is a weak solutions of (1.1) if

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v dx + \int_\Omega \Big(|\nabla u|^{q(x)-2}\nabla u + \frac{|\nabla u|^{2q(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2q(x)}}}\Big)\nabla v dx$$
$$= \int_\Omega \lambda g(x,u,\nabla u) v dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

First, let us consider the following functional:

$$\mathcal{I}(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\Big)dx + \int_\Omega \frac{1}{q(x)}\Big(|\nabla u|^{q(x)} + \sqrt{1+|\nabla u|^{2q(x)}}\Big)dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. From [22], $\mathcal{I}$ is a continuously Gâteaux differentiable and let $\mathcal{F} := \mathcal{I}'(u) \in W^{-1,p'(x)}(\Omega)$ such that

$$\langle \mathcal{F}u, v\rangle = \langle F_1 u, v\rangle + \langle F_2 u, v\rangle,$$

where

$$\langle F_1 u, v\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v dx, \quad \text{and} \quad \langle F_2 u, v\rangle = \int_\Omega \Big(|\nabla u|^{q(x)-2}\nabla u + \frac{|\nabla u|^{2q(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2q(x)}}}\Big)\nabla v dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$, where $\langle \cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. It follows from [22, Proposition 3.1.] that F_1 and F_2 are continuous bounded, strictly monotone operators, and are of class (S_+). We have also the following result:

Lemma 3.2. The mapping

$$\begin{aligned} &\mathcal{F} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{F}u, v\rangle = \langle F_1 u, v\rangle + \langle F_2 u, v\rangle, \end{aligned}$$

is continuous, bounded, strictly monotone operator, and *is of class* (S_+).

Lemma 3.3. If $(M_1) - (M_2)$ hold, then the operator

$$\begin{aligned} &\mathcal{S} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{S}u, v\rangle = -\lambda \int_\Omega g(x,u,\nabla u) v dx, \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega) \end{aligned}$$

is compact.

Proof . We'll take two steps to prove this lemma.

Step 1 : Let us define the operator $\Phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda g(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous. Let $u \in W_0^{1,p(x)}(\Omega)$. According to (M_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned} |\Phi u|_{p'(x)} &\le \rho_{p'(x)}(\Phi u) + 1 \\ &= \int_\Omega |\lambda g(x,u(x),\nabla u(x))|^{p'(x)} dx + 1 \\ &= \int_\Omega |\lambda|^{p'(x)} |g(x,u(x),\nabla u(x))|^{p'(x)} dx + 1 \\ &\le \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega |\varrho\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)} dx + 1 \\ &\le C\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{(q(x)-1)p'(x)} + |\nabla u|^{(q(x)-1)p'(x)}\Big) dx + 1 \\ &\le const \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{p(x)} + |\nabla u|^{p(x)}\Big) dx + 1 \\ &\le const\Big(\rho_{p'(x)}(\gamma) + \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u)\Big) + 1 \\ &\le const\Big(|\gamma|_{p'(x)}^{p'^+} + |\gamma|_{p'(x)}^{p'^-} + |u|_{p(x)}^{p^+} + |u|_{p(x)}^{p^-} + |\nabla u|_{p(x)}^{p^+} + |\nabla u|_{p(x)}^{p^-}\Big) + 1 \\ &\le const\Big(|\gamma|_{p'(x)}^{p'^+} + |u|_{p(x)}^{p^+} + |\nabla u|_{p(x)}^{p^+}\Big) + 1 \\ &\le const\Big(|\gamma|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{p^+} + |u|_{1,p(x)}^{p^-}\Big) + 1. \end{aligned}$$

and as a result Φ is bounded on $W_0^{1,p(x)}(\Omega)$. It remains to show that Φ is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \ \text{ and } \ \nabla u_k(x) \to \nabla u(x), \tag{3.1}$$

$$|u_k(x)| \le \phi(x) \ \text{ and } \ |\nabla u_k(x)| \le |\psi(x)|, \tag{3.2}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Thanks to (M_1) and (3.1), we obtain,

$$g(x, u_k(x), \nabla u_k(x)) \to g(x, u(x), \nabla u(x)) \ \text{ as } k \longrightarrow \infty \ \text{ and a.e. } \ x \in \Omega.$$

From (M_2) and (3.2), we have

$$|g(x, u_k(x), \nabla u_k(x))| \le \varrho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. Seeing that

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |g(x, u_k(x), \nabla u_k(x)) - g(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, by Lebesgue's theorem and (2.4), we conclude that

$$\Phi u_k \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega),$$

that means

$$\Phi u_n \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega).$$

Then Φ is continuous.

Step 2: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. We then define

$$I^* o \Phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

Taking into account that I is compact, then I^* is compact. Hence $\mathcal{S} = I^* \circ \Phi$ is compact. The proof is completed. □

Theorem 3.4. If (M_1) and (M_2) hold, then the problem (1.1), has at least one weak solution in the spaces $W_0^{1,p(x)}(\Omega)$.

Proof . Note that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1) if and only if

$$\mathcal{F}u = -\mathcal{S}u, \quad \text{for all } \ u \in W_0^{1,p(x)}(\Omega), \tag{3.3}$$

where the operators $\mathcal{F}$ and $\mathcal{S}$, are defined as in Lemmas 3.2 and 3.3 respectively. From Lemma 3.2 and [28, Theorem 26 A], the inverse operator

$$\mathcal{G} := \mathcal{F}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is bounded, continuous, strictly monotone and of class (S_+). Consequently, following Zeidler's terminology [28] and Lemma 3.3, the equation (3.3) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{G}v \ \text{ and } \ v + \mathcal{S} \circ \mathcal{G}v = 0, \ \ u \in W_0^{1,p(x)}(\Omega) \text{ and } v \in W^{-1,p'(x)}(\Omega). \tag{3.4}$$

To solve (3.3) it is thus enough to solve (3.4). We will apply the Berkovits topological degree introducing in Section 2.2. Let us set

$$\mathcal{B} := \Big\{ v \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ v + t\mathcal{S} \circ \mathcal{G}v = 0 \Big\}.$$

Now, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$. Let us put $u := \mathcal{G}v$ for all $v \in \mathcal{B}$. Taking into account that $|\mathcal{G}v|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:

First case : If $|\nabla u|_{p(x)} \le 1$. Then $|\mathcal{G}v|_{1,p(x)} \le 1$, that means $\Big\{ \mathcal{G}v : v \in \mathcal{B} \Big\}$ is bounded.

Second case : If $|\nabla u|_{p(x)} > 1$. Then, we deduce from (2.2) and (M_2) that

$$\begin{aligned}
|\mathcal{G}v|^{p^-}_{1,p(x)} &= |\nabla u|^{p-}_{p(x)} \\
&\le \rho_{p(x)}(\nabla u) \\
&\le \langle \mathcal{F}u,\ u\rangle \\
&= \langle v,\ \mathcal{G}v\rangle \\
&= -t\langle \mathcal{S}o\mathcal{G}v,\ \mathcal{G}v\rangle \\
&= t\int_\Omega \lambda g(x,u,\nabla u)u dx \\
&\le t\varrho|\lambda|\Big(\int_\Omega |\gamma(x)u(x)|dx + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&= t\varrho|\lambda|\Big(\int_\Omega |\gamma(x)u(x)|dx + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)
\end{aligned}$$

This, (2.7) and (2.6) yield

$$|\mathcal{G}v|^{p^-}_{1,p(x)} \le const\Big(|\gamma|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big).$$

Using Young's inequality, we see that

$$\begin{aligned}
&const\Big(|\gamma|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\qquad \le const\Big(|\gamma|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big).
\end{aligned}$$

Therefore, $|\mathcal{G}v|^{p^-}_{1,p(x)} \le const\Big(|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{p(x)}\Big)$, according to $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{G}v|^{p^-}_{1,p(x)} \le const\Big(|\mathcal{G}v|_{1,p(x)} + |\mathcal{G}v|^{q^+}_{1,p(x)} + |\mathcal{G}v|^{q^-}_{1,p(x)} + |\mathcal{G}v|^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{G}v : v \in \mathcal{B}\Big\}$ is bounded. Since $\mathcal{S}$ is bounded, then $\mathcal{S}\circ\mathcal{G}v$ is bounded, and thanks to (3.4), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$. However, $\exists\ b > 0$ such that

$$|v|_{-1,p'(x)} < b \ \text{ for all } \ v \in \mathcal{B},$$

which leads to

$$v + t\mathcal{S}\circ\mathcal{G}v \neq 0, \ \ v \in \partial B_b(0) \ \text{ and } \ t \in [0,1].$$

On another side $I + \mathcal{S}\circ\mathcal{G}$ is bounded, then by Lemma 2.12, we conclude that

$$I + \mathcal{S}\circ\mathcal{G} \in \mathcal{F}_{\mathcal{G},B}(\overline{\mathcal{B}_b(0)}) \ \text{ and } \ I = \mathcal{F}\circ\mathcal{G} \in \mathcal{F}_{\mathcal{G},B}(\overline{\mathcal{B}_b(0)}).$$

Now, we define the affine homotopy

$$\begin{aligned}
\mathcal{H} : [0,1]\times\overline{B_b(0)} &\to W^{-1,p'(x)}(\Omega) \\
(t,v) &\mapsto \mathcal{H}(t,v) := v + t\mathcal{S}\circ\mathcal{G}v.
\end{aligned}$$

Hence, by the properties of the degree d seen in Theorem 2.15, we get

$$d(I + \mathcal{S}\circ\mathcal{G}, B_b(0), 0) = d(I, B_b(0),\ 0) = 1 \neq 0,$$

what implies that there exists $v \in B_b(0)$ which verifies

$$v + \mathcal{S}\circ\mathcal{G}v = 0.$$

Finally, $u = \mathcal{G}v$ is a weak solutions of (1.1). The proof is completed. □

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