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# On continuity and categorical property of interval-valued topological spaces

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**Abstract**

An interval set (or an interval-valued set), is a special set, which is an effective tool for illustrating and describing obscure information systems and partially known problems. Recently, Kim et al.[5] defined the topological structure for interval-value sets and studied many properties of them. In this work, we discuss some characteristics and relations of continuity in interval-valued topological spaces with some necessary illustrative examples. Then we provide a categorical framework for interval-valued topological spaces $\mathcal{IV}$-$\mathcal{TOP}$. Many functors and subcategories of $\mathcal{IV}$-$\mathcal{TOP}$ are defined and studied. Furthermore, the relationships between the $\mathcal{IV}$-$\mathcal{TOP}$ and its subcategories are investigated. We show that the category $\mathcal{TOP}$ is isomorphic to the category $\mathcal{IV}$-$\mathcal{TOP}_1$. Moreover, we demonstrate that $\mathcal{TOP}$ and $\mathcal{IV}$-$\mathcal{TOP}_1$ are bireflective full subcategories of $\mathcal{IV}$-$\mathcal{TOP}$.

Keywords: $IV$-sets, $IV$-topology, $IV$-product, $IV$-continuous maps, Category theory
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## 1 Introduction

In 1993, Yao [12] defined and studied the concept of interval sets as a new kind of sets, represented by a pair of sets namely, its lower and upper bounds. The interval-valued sets(briefly, $IVS$) attracts more authors and gradually develops into a complete theory system and it has been applied to many real-life problems, such as decision making, machine learning, and expert system as in ([10, 13, 14, 15, 18]).

In the past recent decades, authors have done more work on uncertain concepts at the same time, many methods for get uncertain knowledge from uncertain information systems appeared. Zadeh introduced fuzzy set and interval-valued fuzzy set theories as in ([16, 17]). Palwak [6] defined the concept of rough sets. Then intuitionistic (fuzzy) sets were proposed by Coker and Atanassov as in ([1, 2]) respectively, also, Kandil et al. [3] defined and studied the double (flou) sets. An $IVS$ is not only considered the appropriate technique to describe the partially known notion, but also can be used to investigate the approximation set of the uncertain problems. Therefore, the $IVS$ is a more

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general mathematical tool to deal with the uncertain information. Recently, Kim et al. [5] defined the notion of an interval-valued topological spaces(briefly, $IVTS$) and studied many properties and concepts of $IVSs$ and topological structures for them.

Category theory is a powerful tool for understanding and organizing mathematical concepts, and it has applications in many areas of topology as in ([7, 8, 11]). This paper is concerned with the theoretical study to investigate some properties of the continuity and categorical property of interval-valued topological spaces $\mathcal{IV}$-$\mathcal{TOP}$, we define many functors, subcategories of $\mathcal{IV}$-$\mathcal{TOP}$ and investigate some of their properties. The relationships between the category $\mathcal{IV}$-$\mathcal{TOP}$ and its subcategories are studied, We show that the categories $\mathcal{TOP}$ and $\mathcal{IV}$-$\mathcal{TOP}_1$ are isomorphic. We also, show that the category of topological spaces $\mathcal{TOP}$ and the category $\mathcal{IV}$-$\mathcal{TOP}_1$ are bireflective full subcategories of $\mathcal{IV}$-$\mathcal{TOP}$.

In the next, we give some results and definitions that may be used in the sequel. The notation $IV$ refers to interval-valued. For categorical terminologies used in section. 3, we refer to [9].

A category is an object $\Im = (\psi, hom, I, \circ)$ defined by the next conditions:
(1) A family $\psi$, whose elements are called $\Im$-objects,
(2) For every pair $(A, B)$ of $\Im$-objects, a set $hom(A, B)$, whose elements are called $\Im$-morphisms from $A$ to $B$,
(3) For every $\Im$-object $A$, a morphism $I_A : A \to A$, called the $\Im$-identity on $A$,
(4) A composition law associating with every $\Im$-morphism $f : A \to B$ and every $\Im$-morphism $g : B \to C$ an $\Im$-morphism $g \circ f : A \to C$, called the composite of $f$ and $g$, subject to the next conditions:
(i) composition is associative,
(ii) $\Im$-identities act as identities with respect to composition,
(iii) the sets $hom(A, B)$ are pairwise disjoint.

**Definition 1.1.** [12] Let $U$ be an universal set and $2^U$ be its power set. An $IVS$ is an object having the form $\mathcal{A} = [A_1, A_2] = \{A \in 2^U : A_1 \subseteq A \subseteq A_2\}$, where it is assumed $A_1, A_2 \in 2^U$ and $A_1 \subseteq A_2$. The family of all $IVS$ on $U$ is denoted by $IVS(U)$. The $IVS$ $\tilde{U} = (U, U)$ (resp. $\widetilde{\emptyset} = [\emptyset, \emptyset]$)is called the universal $IVS$ (resp. the empty $IVS$ ). Any $A \in 2^U$ is obviously an $IVS$ in the form $\widetilde{\mathcal{A}} = [A, A]$. Thus an $IVS$ may be considered as a generalization of ordinary set.

**Definition 1.2.** [12] For two $IVSs$ $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$ on $U$, we have:

(i) $\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$.

(ii) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$, *i.e.* $\mathcal{A} = \mathcal{B} \Longleftrightarrow A_1 = B_1$ and $A_2 = B_2$,

(iii) $\mathcal{A} \sqcup \mathcal{B} = [\, A_1 \cup B_1,\ A_2 \cup B_2]\,$,

(iii) $\mathcal{A} \sqcap \mathcal{B} = [\, A_1 \cap B_1,\ A_2 \cap B_2]\,$,

(iv) $\mathcal{A}^c = [A_2^c, A_1^c]$ , where $A_i^c = U - A_i,\ i = 1, 2$

(v) $\mathcal{A} - \mathcal{B} = [A_1 - B_2, A_2 - B_1]$.

**Definition 1.3.** [5] Let $IVS\,(U)$ and $IVS(V)$ be two classes of $IVSs$ on $U$ and $V$ respectively. The map $f : IVS(U) \longrightarrow IVS(V)$ is called an $IV$-map. If $\mathcal{A} \in IVS(U)$, then $f\,(\mathcal{A})$ is an $IVS$ in $V$, given by $f\,(\mathcal{A}) = [f\,(A_1)\,, f\,(A_2)]\,$. And if $\mathcal{B} \in IVS(V)$, then the preimage $f^{-1}\,(\mathcal{B})$ of $\mathcal{B}$ is an $IVS$ in $U$, given by $f^{-1}\,(\mathcal{B}) = \left[f^{-1}\,(B_1)\,, f^{-1}\,(B_2)\right]$.

For more details about the properties of the $IV$-maps see [**?** ].

**Definition 1.4.** [5] The family $\eta \subset IVS(U)$ is called an interval valued topology (briefly, $IVT$) on $U$ if it is satisfies the next conditions:
(i) $\widetilde{\phi}, \widetilde{U} \in \eta$,
(ii) The union of any number of $IVSs$ in $\eta$ is in $\eta$,
(iii) The intersection of any two $IVSs$ in $\eta$ is in $\eta$.

In this case $(U,\eta)$ is called an interval valued topological space (briefly, $IVTS$). The class of all $IVTSs$ on $U$, denoted by $IVTS(U)$. Any $\mathcal{A}\in\eta$ is called an $IV$-open set (briefly, $IVOS$) and the complement of $\mathcal{A}$, denoted by $\mathcal{A}^c$, is called $IV$-closed set (briefly, $IVCS$). $\eta_c$ refers to the set of all $IVCSs$.

**Definition 1.5.** (i) For an $IVTS$ $(U,\eta)$ and any $\mathcal{A}\in IVS(U)$. The $IV$-closure (resp. $IV$-interior) of $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$ (resp. $\mathcal{A}^{\circ}$) and defined by $\overline{\mathcal{A}}=\sqcap\{\mathcal{B}\in\eta_c{:}\mathcal{A}\sqsubseteq\mathcal{B}\}$ ( *resp.* $\mathcal{A}^{\circ}=\sqcup\{\mathcal{B}\in\eta:\mathcal{B}\sqsubseteq\mathcal{A}\}$) [5].
(ii) Let $\eta_1,\eta_2\subseteq 2^U$. The $IV$-product of $\eta_1$ and $\eta_1$ is denoted by $\eta_1\widehat{\times}\eta_2$ and defined by $\eta_1\widehat{\times}\eta_2=\{[A_1,A_2]\in\eta_1\times\eta_2: A_1\subseteq A_2\}$ [7].

**Theorem 1.6.** [3] For any $IVTS$ $(U,\,\eta)$. The next collections are topologies on $U$ generated by $\eta$:

(1) $\tau_1=\{A_1:\mathcal{A}=[A_1,A_2]\in\eta\}$,

(2) $\tau_2=\{A_2:\mathcal{A}=[A_1,A_2]\in\eta\}$,

(3) $\tau_3=\{A:\widetilde{\mathcal{A}}=[A,A]\in\eta\}$. Moreover, $\tau_3\subseteq\tau_1\cap\tau_2$.

**Definition 1.7.** An $IVTS$ $(U,\eta)$ is called $G$-$IVTS$ iff $\eta=\tau_1\widehat{\times}\tau_2$ which is the greatest $IVT$ constructed by $IV$-product of $\tau_1,\tau_2$. Moreover in general $\eta\subseteq\tau_1\widehat{\times}\tau_2$.

**Theorem 1.8.** [3] Let $(U,\tau)$ be a topological space. The next collections are $G$-$IVTSs$ on $U$ induced by $\tau$:

(1) $\eta_1=\{\mathcal{A}=[A_1,A_2]:A_1\in\tau,\ A_2\in 2^U\}$,

(2) $\eta_2=\{\mathcal{A}=[A_1,A_2]:A_1\in 2^U,\ A_2\in\tau\}$,

(3) $\eta_3=\{\mathcal{A}=[A,U]:A\in\tau\}\cup\{\widetilde{\emptyset}\}$,

(4) $\eta_4=\{\mathcal{A}=[\emptyset,A]:A\in\tau\}\cup\{\tilde{U}\}$,

(5) $\widetilde{\eta}=\{\widetilde{\mathcal{A}}=[A,A]:A\in\tau\}$.

**Remark 1.9.** For $\eta_i$ , $i=1,2,3,4$, we have $\eta_3\subseteq\eta_1,\eta_4\subseteq\eta_2$, and $\widetilde{\eta}\subseteq\eta_1\cap\eta_2$.

## 2 Some properties of continuity in $IVTSs$

**Definition 2.1.** [5] Let $(U,\eta)$ and $(V,\delta)$ be two $IVTSs$. An $IV$-map $f:(U,\eta)\longrightarrow(V,\delta)$ is said to be:
(i) $IV$-continuous iff $f^{-1}(\mathcal{B})\in\eta$ whenever $\mathcal{B}\in\delta$ [or equivalently, $f^{-1}(\mathcal{B})\in\eta_c$ for all $IVCS$ $\mathcal{B}$ in $V$],
(ii) $IV$-open(resp. $IV$-closed) iff $f(\mathcal{A})\in\delta$ whenever $\mathcal{A}\in\eta$ (resp. $f(\mathcal{A})\in\delta_c$ whenever $\mathcal{A}\in\eta_c$),
(iii) $IV$-homeomorphism iff $f$ is bijective, $f$ and $f^{-1}$ are $IV$-continuous.

**Theorem 2.2.** Let $f:(U,\eta)\longrightarrow(V,\delta)$ be an $IV$-map. The next statements are equivalent:
(1) $f$ is an $IV$-continuous map.
(2) $\overline{f^{-1}(\mathcal{B})}\sqsubseteq f^{-1}(\overline{\mathcal{B}})$ for any $\mathcal{B}$ in $(V,\delta)$,
(3) $f^{-1}(\mathcal{B}^{\circ})\sqsubseteq(f^{-1}(\mathcal{B}))^{\circ}$ for any $\mathcal{B}$ in $(V,\delta)$,
(4) $f\left(\overline{\mathcal{A}}\right)\sqsubseteq\overline{f(\mathcal{A})}$ for any $\mathcal{A}$ in $(U,\eta)$.

**Proof .**The proofs for (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) are similar to that in an ordinary setting.
For the case (1) $\Longleftrightarrow$ (4). Let $f$ be an $IV$-continuous and $\mathcal{A}\in IVS(U)$. Since $f\left(\overline{\mathcal{A}}\right)\in\delta_c$ and $f^{-1}(\overline{f(\mathcal{A})})\in\eta_c$. Thus $\overline{\mathcal{A}}\sqsubseteq\overline{(f^{-1}f(\mathcal{A}))}\sqsubseteq\overline{(f^{-1}\overline{f(\mathcal{A})})}=f^{-1}\overline{f(\mathcal{A})})$. Hence $f\left(\overline{\mathcal{A}}\right)\sqsubseteq ff^{-1}\overline{f(\mathcal{A})}\sqsubseteq\overline{f(\mathcal{A})}$.
Conversely, Let $\mathcal{B}\in IVS(V)$. From (2), we have $f\left(\overline{f^{-1}(\mathcal{B})}\right)\sqsubseteq\left(\overline{ff^{-1}(\mathcal{B})}\right)\sqsubseteq\mathcal{B}$. So that $\left(\overline{f^{-1}(\mathcal{B})}\right)\sqsubseteq f^{-1}f(\overline{f^{-1}(\mathcal{B})}\sqsubseteq f^{-1}(\overline{\mathcal{B}})$, and the result follows from (3). $\square$

**Remark 2.3.** The conditions in the above theorem are not equivalent to the condition $(f(\mathcal{A}))^{\circ}\sqsubseteq f(\mathcal{A}^{\circ})$ for any $\mathcal{A}$ in $(U,\eta)$. This can be shown by the next example.

**Example 2.4.** Let $X = Y = \{a,b,c\}$, $\eta = \{\tilde{X}, \widetilde{\phi}, \mathcal{A}\}$ and $\delta = \{\tilde{X}, \widetilde{\phi}, \mathcal{B}_1, \mathcal{B}_2\}$ where, $\mathcal{A} = [\{a,c\}, \{a,c\}]$, $\mathcal{B}_1 = [\{a\}, \{a\}]$, and $\mathcal{B}_2 = [\{a,b\}, \{a,b\}]$. Then $\eta$ and $\delta$ are $IVTs$ on $X$. Now define $f : (X,\eta) \longrightarrow (Y,\delta)$ by $f(a) = f(c) = a$ and $f(b) = b$, then $f^{-1}(\mathcal{B}_1) = \mathcal{A} \in \eta$, $f^{-1}(\mathcal{B}_2) = \tilde{X} \in \eta$. Thus $f$ is $IV$-continuous but not one-one. One the other hand, if $\mathcal{D} = [\{a,b\}, \{a,b\}]$ is an $IVS$ in $(X,\eta)$, then $f(\mathcal{D}^\circ) = \widetilde{\emptyset}$ and $(f(\mathcal{D}))^\circ = \mathcal{B}_2$. Since $\mathcal{B}_2 \not\sqsubseteq \widetilde{\emptyset}$, then $(f(\mathcal{A}))^\circ \not\sqsubseteq f(\mathcal{A}^\circ)$.

However, we obtain the following properties for a bijection map $f$.

**Theorem 2.5.** Let $f : (U,\eta) \longrightarrow (V,\delta)$ be bijection, then $f$ is $IV$-continuous if and only if $(f(\mathcal{A}))^\circ \sqsubseteq f(\mathcal{A}^\circ)$.

**Proof .** From Theorem (2.2), it suffices to prove that the condition (3) equivalent to the condition $(f(\mathcal{A}))^\circ \sqsubseteq f(\mathcal{A}^\circ)$. Suppose $\mathcal{A} \in IVS(U)$, then $f(\mathcal{A}) \in IVS(V)$. So $f^{-1}(\ f(\mathcal{A}^\circ)) \sqsubseteq (f^{-1} f(\mathcal{A}))^\circ$. Since $f$ is one-one, we have $f^{-1}\left((f(\mathcal{A}))^\circ\right) \sqsubseteq (f^{-1} f(\mathcal{A}))^\circ = \mathcal{A}^\circ$. Hence $ff^{-1}((f(\mathcal{A}))^\circ) \sqsubseteq f(\mathcal{A}^\circ)$. Also, $f$ is onto, then $(f(\mathcal{A}))^\circ = ff^{-1}((f(\mathcal{A}))^\circ) \sqsubseteq f(\mathcal{A}^\circ)$.
Conversely, Let $\mathcal{B} \in IVS(V)$, then $f^{-1}(\mathcal{B}) \in IVS(U)$. Since $f$ is onto, we have $\mathcal{B}^\circ = (ff^{-1}(\mathcal{B}))^\circ \sqsubseteq f\left((f(\mathcal{B}))^\circ\right)$. Again $f$ is one-one, then $f^{-1}(\mathcal{B}^\circ) \sqsubseteq f^{-1}f((f^{-1}(\mathcal{B}))^\circ) = f^{-1}(\mathcal{B}))^\circ$. □

Now from Theorem (2.2) and the above theorem we obtain the next corollary.

**Corollary 2.6.** Let $f : (U,\eta) \longrightarrow (V,\delta)$ be a bijection map. Then the conditions in Theorem (2.2) are equivalent to the condition $(f(\mathcal{A}))^\circ \sqsubseteq f(\mathcal{A}^\circ)$.

**Theorem 2.7.** Let $f : (U,\eta) \longrightarrow (V,\delta)$ be an $IV$-map, then the next items are equivalent:
(1) $f$ is an $IV$-open map,
(2) $f(\mathcal{A}^\circ) \sqsubseteq (f(\mathcal{A}))^\circ$ for all $\mathcal{A} \in IVS(U)$,
(3) $\left(f^{-1}(\mathcal{B})\right)^\circ \sqsubseteq f^{-1}(\mathcal{B}^\circ)$ for all $\mathcal{B} \in IVS(V)$.

**Proof .** (1) $\Longrightarrow$ (2) Let $\mathcal{A} \in IVS(U)$, then $\mathcal{A}^\circ \in \eta$. Since $f$ is an $IV$-open map, then $f(\mathcal{A}^\circ) \in \delta$. Therefore $f(\mathcal{A}^\circ) = (f(\mathcal{A}^\circ))^\circ \sqsubseteq (f(\mathcal{A}))^\circ$.
(2) $\Longrightarrow$ (3) Let $\mathcal{B} \in IVS(V)$, then $f^{-1}(\mathcal{B}) \in IVS(X)$. From (2), we have $f((f^{-1}(\mathcal{B}))^\circ) \sqsubseteq f^{-1}f((f^{-1}(\mathcal{B}))^\circ) \sqsubseteq f^{-1}(\mathcal{B}^\circ)$.
(3) $\Longrightarrow$ (1) Let $\mathcal{A} \in \eta$. Then $\mathcal{A}^\circ = \mathcal{A}$ and $f(\mathcal{A}) \in IVS(V)$. From (3), we have $\mathcal{A} = \mathcal{A}^\circ \subseteq \left(f^{-1}f(\mathcal{A})\right)^\circ \sqsubseteq f^{-1}(f(\mathcal{A}))^\circ$. Thus $f(\mathcal{A}) \subseteq ff^{-1}((f(\mathcal{A}))^\circ) \subseteq (f(\mathcal{A}))^\circ \subseteq f(\mathcal{A})$. Hence $f(\mathcal{A}) = (f(\mathcal{A}))^\circ$ and so, $f(\mathcal{A}) \in \delta$. Therefore $f$ is an $IV$-open map. □

**Theorem 2.8.** Let $f : (U,\eta) \longrightarrow (V,\delta)$ be an $IV$-map, then $f$ is $IV$-closed if and only if $\overline{f(\mathcal{A})} \sqsubseteq f(\mathcal{A})$ for all $\mathcal{A} \in IVS(U)$.

**Proof .** The proof is similar to that in the above theorem. □

**Example 2.9.** Let $U = \{a,b,c\}$. Consider the $IVT$ $\eta = \{\tilde{U}, \widetilde{\phi}, \mathcal{A}_1, \mathcal{A}_2\}$ on $U$ where, $\mathcal{A}_1 = [\{a\}, \{a\}], \mathcal{A}_2 = [\{a,b\}, \{a,b\}]$. Define $f : (U,\eta) \longrightarrow (U,\eta)$ by $f(a) = f(b) = a, f(c) = b$, then $f(\mathcal{A}_1) = [\{a\}, \{a\}] = \mathcal{A}_1 \in \eta$, $f(\mathcal{A}_2) = [\{a\}, \{a\}] = \mathcal{A}_1 \in \eta$, $f(\tilde{U}) = [\{a,b\}, \{a,b\}] = \mathcal{A}_2 \in \eta$, and $f(\widetilde{\phi}) = \widetilde{\phi}$. Thus $f$ is an $IV$-open map. But $f(\mathcal{A}_2^c) = [\{b\}, \{b\}] \notin \eta_c$. Here $f$ is not an $IV$-closed map. On the other hand, let us define $h : (U,\eta) \longrightarrow (U,\eta)$ by $h(a) = h(b) = b, h(c) = c$, then we have, $h(\mathcal{A}_1^c) = [\{b,c\}, \{b,c\}] = \mathcal{A}_1^c \in \eta_c, h(\mathcal{A}_2^c) = [\{c\}, \{c\}] = \mathcal{A}_2^c \in \eta_c$, $h(\tilde{U}) = [\{b,c\}, \{b,c\}] = \mathcal{A}_1^c \in \eta_c$, and $h(\widetilde{\phi}) = \widetilde{\phi}$. Thus $h$ is $IV$-closed. However, $h(\mathcal{A}_1) = [\{b\}, \{b\}] \notin \eta$. Hence $h$ is not an $IV$-open map.

However, we obtain the next theorem for a bijection map $f$.

**Theorem 2.10.** Let $f : (U,\eta) \longrightarrow (V,\delta)$ be a bijection map, then $f$ is $IV$-open if and only if $f$ is $IV$-closed.

**Proof .** Let $f(\mathcal{F}) \in \eta_c$, then $\mathcal{F} = \mathcal{D}^c$ for some $IV$-open set $\mathcal{D} = [D_1, D_2]$ in $U$. Since $f$ is onto, we have $f(\mathcal{F}) = f(\mathcal{D}^c) = f([D_2^c, D_1^c]) = [f(D_2^c), f(D_1^c)] = [(f(D_2))^c,\ (f(D_1))^c]$. Also, $f$ is $IV$-open, then $f(\mathcal{F}) = [f(D_1), f(D_2)]^c$ which is $IV$-closed in $V$. Therefore $f$ is an $IV$-closed map.
The proof of the converse can be done in a similar way. □

**Theorem 2.11.** Let $(U,\eta)$ and $(V,\delta)$ be any two $IVTSs$. If the map $f:(U,\eta)\longrightarrow(V,\delta)$ is an $IV$-continuous map, then the $IV$-maps $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$ are continuous, where $(\tau_i,\tau_i^*\ ,i=1,2)$ are defined as in Theorem (1.6).

**Proof .** Let $f:(U,\eta)\longrightarrow(V,\delta)$ be an $IV$-continuous map and let $B_1\in\tau_1^*$, then there is $B_2\in\tau_2^*$ such that $\mathcal{B}=[B_1,B_2]\in\delta$ and so, $f^{-1}(\mathcal{B})=[f^{-1}(B_1),f^{-1}(B_2)]\in\eta$, consequently $f^{-1}(B_1)\in\tau_1$. This means that $f:(U,\tau_1)\longrightarrow(V,\tau_1^*)$ is continuous. The proof of the case $i=2$ is similar. □

The next example shows that the converse of the above theorem may not be true in general.

**Example 2.12.** Let $(U,\tau)$ be any topological space, then the identity map $I_U:(U,\tau)\longrightarrow(U,\tau)$ is continuous, but clearly, $I_U:(U,\widetilde{\eta})\longrightarrow(U,\tau\widehat{\times}\tau)$ is not $IV$-continuous, where $\widetilde{\eta}=\{\widetilde{\mathcal{A}}=[A,A]:A\in\tau\}$.

**Theorem 2.13.** Let $(U,\eta)$ be an $G$-$IVTS$ and $(V,\delta)$ be any $IVTS$, then $f:(U,\eta)\longrightarrow(V,\delta)$ is an $IV$-continuous map if and only if $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$ are continuous,

**Proof .** "⇒" It follows from that of theorem (2.11).
Conversely, let $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,\ i=1,2$ be continuous maps and $\mathcal{A}=[A_1,A_2]\in\delta$, then $A_1\in\tau_1^*$ and $A_2\in\tau_2^*$ this implies $f^{-1}(A_1)\in\tau_1$and $f^{-1}(A_2)\in\tau_2$ and so, $f^{-1}(\mathcal{A})=[f^{-1}(A_1),f^{-1}(A_2)]\in\eta$. Consequently, $f:(U,\eta)\longrightarrow(V,\delta)$ is an $IV$-continuous map. □

**Theorem 2.14.** Let $f:(U,\eta)\longrightarrow(V,\delta)$ be an $IV$-open (resp. $IV$-closed) map, then the maps $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$ are open (resp. closed) maps.

**Proof .** Let $f:(U,\eta)\longrightarrow(V,\delta)$ be an $IV$-open map and $A_1\in\tau_1$, then there is $A_2\in\tau_2$ such that, $\mathcal{A}=[A_1,A]\in\eta$ and so, $f(\mathcal{A})=[f(A_1),f(A_2)]\in\delta$, consequently $f(A_1)\in\tau_1^*$ and $f(A_2)\in\tau_2^*$ this means that $f:(U,\tau_1)\longrightarrow(V,\tau_1^*)$ is open map. The proof of the other cases is similar. □

The following example shows that the converse of the above theorem may be not true in general.

**Example 2.15.** Let $X=\{a,b,c\}$ and $Y=\{x,y,z\}$. Consider the $IVT$ $\eta=\{\tilde{X},\widetilde{\phi},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6\}$ on $X$, where $\mathcal{A}_1=[\emptyset,X]\,,\mathcal{A}_2=[\emptyset,\{b\}]$, $\mathcal{A}_3=[\emptyset,\{a,c\}]$, $\mathcal{A}_4=[\{a\},\{a,c\}]$, $\mathcal{A}_5=[\{b,c\},X]$, and $\mathcal{A}_6=[\{a\},X]$ and the $IVT$ $\delta=\{\tilde{Y},\widetilde{\phi},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5\}$ on $Y$, where $\mathcal{B}_1=[\emptyset,Y]\,,\mathcal{B}_2=[\{x\},Y]\,,\mathcal{B}_3=[\{y,z\},Y]\,,\mathcal{B}_4=[\emptyset,Y]$, and $\mathcal{B}_5=[\emptyset,\{x,z\}]$. Define the map $f:X\longrightarrow Y$ by $f(a)=x$, $f(b)=y$ ,and $f(c)=z$. One can check that $f:(X,\tau_i)\longrightarrow(Y,\tau_i^*)\,,i=1,2$ are open maps. But $f:(X,\eta)\longrightarrow(Y,\delta)$ in not $IV$-open, because $f(\mathcal{A}_4)=[f(\{a\}),f(\{a,c\})]=[\{x\},\{x,z\}]\notin\delta$.

**Theorem 2.16.** Let $(V,\delta)$ be an $G$-$IVTS$ and $(U,\eta)$ be any $IVTS$, then $f:(U,\eta)\longrightarrow(V,\delta)$ is an $IV$-open (resp. $IV$-closed) map if and only if $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$, are open ( resp. closed) maps, where $(\tau_i,\tau_i^*,i=1,2)$ are defined as in Theorem (1.6).

**Proof .** "⇒" It follows from that of Theorem (2.14).
Conversely, let $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$ be open (closed) maps, and let $\mathcal{A}=[A_1,A_2]\in\eta$. Then $A_1\in\tau_1$and $A_2\in\tau_2$. Thus $f(A_1)\in\tau_1^*$ and $f(A_2)\in\tau_2^*$ this implies $f(\mathcal{A})=[f(A_1),f(A_2)]\in\delta$. Consequently, $f$ is an $IV$-open map. □

**Theorem 2.17.** For two $G$-$IVTSs$ $(U,\eta)$ and $(V,\delta)$, we have $f:(U,\eta)\longrightarrow(V,\delta)$ is an $IV$-homeomorphism if and only if $f:(U,\tau_i)\longrightarrow(V,\tau_i^*)\,,i=1,2$ are homeomorphism.

**Proof .** It follows of that in Theorem (2.11) and the above theorem. □

## 3 On Categorical property of *IVTSs*

In this section, we study the relationship between the category of Interval-valued topological spaces and that of crisp topological spaces. Let $\mathcal{IV}$-$\mathcal{TOP}$ be the category of $IVTSs$ with $IV$-continuous maps and $\mathcal{TOP}$ be the category of topological spaces ($TSs$) with continuous maps.

First, let us define some functors between $\mathcal{IV}$-$\mathcal{TOP}$ and $\mathcal{TOP}$.

**Theorem 3.1.** For the categories $\mathcal{IV}$-$\mathcal{TOP}$ and $\mathcal{TOP}$, we define:
(i) $G_1 : \mathcal{IV}\text{-}\mathcal{TOP} \longrightarrow \mathcal{TOP}$ by $G_1(U,\eta) = (U, G_1(\eta))$ , $G_1(f) = f$ where,

$$G_1(\eta) = \{A_1 : \mathcal{A} = [A_1, A_2] \in \eta\}.$$

(ii) $G_2 : \mathcal{IV}\text{-}\mathcal{TOP} \longrightarrow \mathcal{TOP}$ by $G_2(U,\eta) = (U, G_2(\eta)), G_2(f) = f$ where,

$$G_2(\eta) = \{A_2 : \mathcal{A} = [A_1, A_2] \in \eta\}.$$

Then $G_1$ and $G_2$ are functors.

**Proof .** By Theorem (1.6), we have $G_i(\eta), i = 1, 2$ are $IVTSs$ on $U$. Furthermore, it follows from Theorem (2.11), if the map $f : (U, \eta) \longrightarrow (V, \delta)$ is $IV$-continuous, then the map $f : (U, G_i(\eta)) \longrightarrow (V,\ G_i(\eta^*))$ , $i = 1, 2$ are continuous. Therefore $G_1, G_2$ are functors. □

**Theorem 3.2.** For the categories $\mathcal{IV}$-$\mathcal{TOP}$ and $\mathcal{TOP}$, we define:
(i) $\mathcal{F}_1 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ by $\mathcal{F}_1(U,\tau) = (U, \mathcal{F}_1(\tau)), \mathcal{F}_1(f) = f$ where,

$$\mathcal{F}_1(\tau) = \{\mathcal{A} = [A_1, A_2] : A_1 \in \tau, A_1 \subseteq A_2\}.$$

(ii) $\mathcal{F}_2 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ by $\mathcal{F}_2(U,\tau) = (U, \mathcal{F}_2(\tau)), \mathcal{F}_2(f) = f$ where,

$$\mathcal{F}_2(\tau) = \{\mathcal{A} = [A_1, A_2] : A_2 \in \tau,\ A_1 \subseteq A_2\}.$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are functors.

**Proof .** (i) Clearly, $\mathcal{F}_1(\tau)$ is an $IVTS$ on $U$ this follows directly from Theorem (1.7). It remains to prove that if $f : (U, \tau) \longrightarrow (V, \tau^*)$ is continuous, then $f : (U, \mathcal{F}_1(\tau)) \longrightarrow (V, \mathcal{F}_1(\tau^*))$ is $IV$-continuous. Assume $\mathcal{B} = [B_1, B_2] \in \mathcal{F}_1(\tau^*)$, we have $B_1 \in \tau^*$ and so, $f^{-1}(B_1) \in \tau$. Since $B_1 \subseteq B_2$, we get $f^{-1}(B_1) \subseteq f^{-1}(B_2)$, so that $f^{-1}(\mathcal{B}) = [f^{-1}(B_1), f^{-1}(B_2)] \in \mathcal{F}_1(\tau)$ . Therefore $f : (U, \mathcal{F}_1(\tau)) \longrightarrow (V, \mathcal{F}_1(\tau^*))$ is $IV$-continuous. Hence $\mathcal{F}_1$ is a functor. The proof for the case (ii) can be proved in a similar way. □

**Theorem 3.3.** The functor $\mathcal{F}_1 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ is a left adjoint of the functor $G_1 : \mathcal{IV}\text{-}\mathcal{TOP} \longrightarrow \mathcal{TOP}$.

**Proof .** Let $(U, \tau) \in \mathcal{TOP}$ and $I_U : (U, \tau) \longrightarrow G_1(\mathcal{F}_1(U, \tau)) = (U, \tau)$ be a continuous map. To prove that $I_U$ is an universal map. Let $(V, \theta) \in \mathcal{IV}\text{-}\mathcal{TOP}$ with a continuous map $f : (U, \tau) \longrightarrow G_1(V, \theta)$. We only need to prove that the map $f^* : \mathcal{F}_1(U, \tau) = (U, \mathcal{F}_1(\tau)) \longrightarrow (V, \theta)$ is $IV$-continuous. Let $\mathcal{B} = [B_1, B_2] \in \theta$, then $B_1 \in G_1(\theta)$. Since $f : (U, \tau) \longrightarrow G_1(V, \theta)$ is continuous, we have $f^{-1}(B_1) \in \tau$ and so, $f^{-1}(\mathcal{B}) = [f^{-1}(B_1), f^{-1}(B_2)] \in \mathcal{F}_1(\tau)$. Hence $f^* : \mathcal{F}_1(U, \tau) \longrightarrow (V, \theta)$ is an $IV$-continuous map. Therefore $I_U$ is a $G_1$-universal map for $(U, \tau)$ in $\mathcal{TOP}$. The result holds. □

**Theorem 3.4.** The functor $\mathcal{F}_2 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ is a left adjoint of the functor $G_2 : \mathcal{IV}\text{-}\mathcal{TOP} \longrightarrow \mathcal{TOP}$.

**Proof .** The proof is analogous of that in the above theorem. □

**Theorem 3.5.** For the categories $\mathcal{IV}$-$\mathcal{TOP}$ and $\mathcal{TOP}$, we define:
$\mathcal{F}_0 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ by $\mathcal{F}_0(U, \tau) = (U, \mathcal{F}_0(\tau))$ and $\mathcal{F}_0(f) = f$ where,

$$\mathcal{F}_0(\tau) = \{\widetilde{\mathcal{A}} = [A, A] : A \in \tau\}.$$

Then $\mathcal{F}_0$ is a functor.

**Proof .** From Theorem (1.7), we get $\mathcal{F}_0(\tau)$ is an $IVT$ on $U$. It remains to prove that $f : (U, \mathcal{F}_0(\tau)) \longrightarrow (V,\ \mathcal{F}_0(\tau^*))$ is an $IV$-continuous map. Let $\widetilde{\mathcal{B}} = [B, B] \in \mathcal{F}_0(\tau^*)$, then $B \in \tau^*$. Since the map $f : (U, \tau) \longrightarrow (V, \tau^*)$ is continuous, then $f^{-1}(B) \in \tau$ and so, $f^{-1}(\widetilde{\mathcal{B}}) = [f^{-1}(B), f^{-1}(B)] \in \mathcal{F}_0(\tau)$ . Hence $f : (U, \mathcal{F}_0(\tau)) \longrightarrow (V,\ \mathcal{F}_0(\tau^*))$ is an $IV$-continuous map. Therefore $\mathcal{F}_0$ is a functor. □

**Theorem 3.6.** For the categories $\mathcal{IV}$-$\mathcal{TOP}$ and $\mathcal{TOP}$, we define:
(1) $\mathcal{P}_1 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ by $\mathcal{P}_1(U,\tau) = (U, G_1(\tau)), \mathcal{P}_1(f) = f$ where,

$$\mathcal{P}_1(\tau) = \{[A,U] : A \in \tau\} \cup \{\widetilde{\emptyset}\}.$$

(2) $\mathcal{P}_2 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}$ by $\mathcal{P}_2(U,\tau) = (U, \mathcal{P}_2(\tau)), \mathcal{P}_2(f) = f$ where,

$$\mathcal{P}_2(\tau) = \{[\emptyset, A] : A \in \tau\} \cup \{\tilde{U}\}.$$

Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are functors.

**Proof .** (1) Clearly, $\mathcal{P}_1(\tau)$ is $IVT$ on $U$ this follows from Theorem (1.7). It remains to prove that if $f : (U,\tau) \longrightarrow (V,\tau^*)$ is a continuous map, then $f : (U, \mathcal{P}_1(\tau)) \longrightarrow (V, \mathcal{P}_1(\tau^*))$ is an $IV$-continuous map. Let $\mathcal{B} = [B,Y] \in \mathcal{P}_1(\tau^*)$, then $B \in \tau^*$ and so, $f^{-1}(B) \in \tau$. Thus $f^{-1}(\mathcal{B}) = \left[f^{-1}(B), U\right] \in \mathcal{P}_1(\tau)$ and so, $f : (U, \mathcal{P}_1(\tau)) \longrightarrow (V, \mathcal{P}_1(\tau^*))$ is $IV$-continuous. Hence $\mathcal{P}_1$ is a functor. The proof of (2) can be done by a similar way. □

In the next, we define a subcategory of the category $\mathcal{IV}$-$\mathcal{TOP}$ as follows.

**Definition 3.7.** The class of all $IVTSs$ whose members are of the form $[A,U] \cup \widetilde{\phi}$ with $IV$-continuous maps forms a full subcategory of the category $\mathcal{IV}$-$\mathcal{TOP}$ and is denoted as $\mathcal{IV}\text{-}\mathcal{TOP}_1$.

**Theorem 3.8.** The category $\mathcal{TOP}$ is isomorphic to the category $\mathcal{IV}\text{-}\mathcal{TOP}_1$.

**Proof .** Let $\mathcal{P}_1 : \mathcal{TOP} \longrightarrow \mathcal{IV}\text{-}\mathcal{TOP}_1$ be the functor defined in Theorem (3.6). Consider the restriction $G_0 : \mathcal{IV}\text{-}\mathcal{TOP}_1 \longrightarrow \mathcal{TOP}$ of the functor $G_1$ defined in Theorem (3.1). Then $G_0$ is a functor. Clearly, $G_0\mathcal{P}_1((U,\tau) = G_0(U, \mathcal{P}_1(\tau)) = (U, G_0(\mathcal{P}_1(\tau))) = (U,\tau)\ \forall\ (U,\tau) \in \mathcal{TOP}$. Similarly, we have $\mathcal{P}_1 G_0((U,\eta) = (U,\eta)$ for all $(U,\eta) \in \mathcal{IV}\text{-}\mathcal{TOP}_1$. Hence the result holds. □

**Theorem 3.9.** The category $\mathcal{IV}\text{-}\mathcal{TOP}_1$ is a bireflective full subcategory of the category $\mathcal{IV}$-$\mathcal{TOP}$.

**Proof .** By Definition (3.7), the category $\mathcal{IV}\text{-}\mathcal{TOP}_1$ is a full subcategory of $\mathcal{IV}$-$\mathcal{TOP}$. Now let $(U,\eta) \in \mathcal{IV}\text{-}\mathcal{TOP}$ and $\eta^* = \{\mathcal{A} \in \eta : \mathcal{A} = [A,U]\} \cup \{\widetilde{\emptyset}\}$, then $(U,\eta^*) \in \mathcal{IV}\text{-}\mathcal{TOP}_1$ and $I_U : (U,\eta) \longrightarrow (U,\eta^*)$ be an $IV$-continuous map. Consider the $IVTS$ $(Y,\theta) \in \mathcal{IV}\text{-}\mathcal{TOP}_1$ with an $IV$-continuous map $f : (X,\eta) \longrightarrow (Y,\theta)$. We need only to prove that $f : (U,\eta^*) \longrightarrow (V,\theta))$ is $IV$-continuous. Indeed, let $\mathcal{B} \in \theta$. Since $(V,\theta) \in \mathcal{IV}\text{-}\mathcal{TOP}_1$, then $\mathcal{B} = [B,V]$ and $f : (U,\eta) \longrightarrow (V,\theta)$ is $IV$-continuous, so that $f^{-1}(\mathcal{B}) \in \eta$ and $f^{-1}(\mathcal{B}) = [f^{-1}(B), U] \in \eta^*$. Therefore $f : (U,\eta^*) \longrightarrow (V,\theta)$ is $IV$-continuous. The result follows. □

**Corollary 3.10.** The category $\mathcal{TOP}$ is a bireflective full subcategory of the category $\mathcal{IV}$-$\mathcal{TOP}$.

**Proof .** It follows from that of the above theorem and Theorem (3.8). □

## 4 Conclusion

This paper focuses on the theoretical study of investigation some characterizations of continuous, open and closed maps in topological structure based on $IVSs$ and for study the category of interval-valued topological spaces $\mathcal{IV}$-$\mathcal{TOP}$, we defined many functors, subcategories of $\mathcal{IV}$-$\mathcal{TOP}$ and investigated some of their properties. The relations between the category $\mathcal{IV}$-$\mathcal{TOP}$ and its subcategories are studied, we prove that the category $\mathcal{TOP}$ of topological spaces is a bireflective full subcategory of the category $\mathcal{IV}$-$\mathcal{TOP}$. In the future work, we will study a new set of separation properties in $IVTSs$ with some applications.

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