Int. J. Nonlinear Anal. Appl. 14 (2023) 5, 325–336
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.28596.3938

Solutions of fractional Lotka-Volterra and Lorenz-Stenflo equations by Sumudu decomposition method

Oluwaseun Akinlo Mogbojuri

Department of Mathematical Sciences, Adekunle Ajasin University, P.M.B 001, Akungba Akoko, Ondo State, Nigeria

(Communicated by Mugur Alexandru Acu)

Abstract

In this paper, we obtain approximate analytical solutions for the fractional predator-prey and the generalized fractional Lorenz-Stenflo systems using the Sumudu decomposition method. The fractional derivative is described in the Caputo sense. The results show that the method gives an easy-to-implement procedure and accurate approximate solutions.

Keywords: Fractional differential equation, Lotka-Volterra equation, Lorenz equation, Lorenz-Stenflo equation, Sumudu Decomposition method
2020 MSC: 34A08, 34A34, 44A05, 65L05

1 Introduction

Fractional calculus generalizes the concept of integer order derivatives and integrals to non-integer orders. Studies have shown that fractional calculus models physical phenomena more accurately than classical calculus. The fractional forms of various classical differential equations have been studied by various researchers in recent times (see, e.g., [2], [3], [4], [8], [12], [15], [19], [23], [25]).

The Lotka-Volterra equation describes two interacting populations that are in a predator-prey relationship. The predators (e.g., sharks, foxes) feed on the preys (e.g., fish, rabbits), which in turn feed on some food items. The classical Lotka-Volterra equation is given by the system of differential equations [6].

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -dx + cxy \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the populations of the preys and predators respectively; a, b, c and d are positive constants. Bildik and Deniz in [7] used the Sumudu decomposition method to obtain approximate solution of (1.1). The fractional form of (1.1) has also been studied by researchers. The homotopy perturbation method was employed to solve the fractional predator-prey system in [10] while Ghanbari used numerical techniques to obtain the solution of the fractional predator-prey model in [11].

The Lorenz equation, a three-dimensional nonlinear system was proposed by the famous meteorologist Edward Lorenz in 1963. The Lorenz system has two nonlinear terms and is given by ([13], [27]).

Email address: oluwaseun.mogbojuri@aaua.edu.ng (Oluwaseun Akinlo Mogbojuri)

Received: October 2022 *Accepted:* February 2023

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz \end{aligned} \tag{1.2}$$

where $x(t)$, $y(t)$ and $z(t)$ are proportional to the convective velocity, the temperature difference between descending and ascending flows and the mean convective flow respectively; σ, r and b are real parameters. Several methods have been employed in solving both the classical and fractional Lorenz system. The Adomian decomposition method and the differential transformation method were respectively used in [13] and [24] to obtain solutions of (1.2). Alomari et al. in [1] presented the homotopy analysis method to solve the fractional Lorenz system while Milici et al. obtained the solution of the fractional Lorenz system using the power series method in [22]. Lennart Stenflo improved on the Lorenz equation by deriving a four-dimensional system with four parameters known as the Lorenz-Stenflo system to model atmospheric dynamics [26]. To model the dynamics of the atmosphere more accurately, Chen and Liang [9] proposed a generalization of the Lorenz-Stenflo system with six parameters.

The Sumudu transform is an integral transform introduced by Watugala [28] to solve differential equations and control engineering problems. In recent times, researchers have used the Sumudu transform to solve problems involving ordinary, partial and fractional differential equations (see, e.g., [7], [16], [17], [18], [20], [21]).

In this paper, the Sumudu decomposition method (SDM), which is a combination of the Sumudu transform and the Adomian decomposition method, is used to solve the fractional-order predator-prey, Lorenz, and Lorenz-Stenflo systems. To the best of our knowledge, the Sumudu decomposition method has not been used to solve the fractional predator-prey system. Also, the fractional-order version of the Lorenz-Stenflo system has not appeared in the literature. The method used in this paper involves two steps: the Sumudu transform is first used to simplify the given systems of equations, and then the Adomian decomposition method is applied to the nonlinear terms. Comparison of the Sumudu decomposition method with other methods shows that it is easy to implement and gives a reliable approximate solution.

The paper is organized as follows: In section 2, we recall some important definitions and theorems. In section 3, we first construct the Sumudu decomposition method and then demonstrate how the method is used to solve the systems of equations under consideration. Section 4 gives a concluding remark on the results obtained.

2 Preliminaries

In this section, we give a brief overview of the basic concepts of fractional calculus and the Sumudu transform. These concepts will be needed subsequently in this paper.

Definition 2.1. [25] For $\alpha > 0$, the Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined as

$$\mathcal{J}^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi, & 0 < \xi < x, \\ f(x) & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. [25] For $\alpha > 0$, the Riemann-Liouville fractional derivative of order α is defined as follows ($m \geq 1$):

$$\begin{aligned} D^\alpha f(x) =& D^m \mathcal{J}^{m-\alpha} f(x) \\ =& \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\xi)^{m-\alpha-1} f(\xi)\, d\xi, & m-1 < \alpha < m, \\ \frac{d^m}{dx^m} f(x) & \alpha = m. \end{cases} \end{aligned} \tag{2.2}$$

Definition 2.3. [8] For $\alpha > 0$, the Caputo fractional derivative of order α is defined as follows ($m \geq 1$):

$$\begin{aligned} {}^c D^\alpha f(x) =& \mathcal{J}^{m-\alpha} D^m f(x) \\ =& \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\xi)^{m-\alpha-1} f^{(m)}(\xi)\, d\xi, & m-1 < \alpha < m, \\ \frac{d^m}{dx^m} f(x) & \alpha = m. \end{cases} \end{aligned} \tag{2.3}$$

Definition 2.4. [28] The Sumudu transform is defined over the set of functions as

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_1}}, \text{if} \quad t \in (-1)^j \times [0, \infty)\}. \tag{2.4}$$

by the formula

$$F(u) = \mathcal{S}[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2). \tag{2.5}$$

The Sumudu transform of some functions are given by ([5], [28]):

1. $\mathcal{S}[1] = 1$
2. $\mathcal{S}[t] = u$
3. $\mathcal{S}[t^{n-1}] = u^{n-1}\Gamma(n), n \in \mathbb{R}^+$.

Theorem 2.5. [7]: If $F(u)$ is the Sumudu transform of the function $f(t)$, then the Sumudu transform of the nth derivative of $f(t)$ is given by

$$\mathcal{S}[f^{(n)}(t)] = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \tag{2.6}$$

Theorem 2.6. [14]: If $F(u)$ is the Sumudu transform of the function $f(t)$, then the Sumudu transform of the Caputo derivative of $f(t)$ of order α is given by

$$\mathcal{S}[{}^cD^\alpha f(t)] = u^{-\alpha}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right], \quad n - 1 < \alpha < n. \tag{2.7}$$

3 Sumudu Decomposition Method

In this section, we construct the Sumudu decomposition method for solving systems of nonlinear fractional differential equations. The method is then employed to solve the fractional predator-prey, Lorenz and Lorenz-Stenflo systems.

We consider the general system of nonlinear fractional differential equations of the form:

$${}^cD^{\alpha_i} x_i(t) = g_i(t, x_1(t), x_2(t), ..., x_m(t)), \quad x_i^{(k)}(0) = \xi_{ik}. \tag{3.1}$$

$1 \leq i \leq m$, $\alpha_i > 0$, ξ_{ik} are real constants, where ${}^cD^{\alpha_i} x_i(t)$ is the Caputo derivative of $x_i(t)$ of order α_i.

Taking Sumudu transform of both sides of (3.1) yields

$$\mathcal{S}\left[{}^cD_i^\alpha x_i(t)\right] = \mathcal{S}\left[g_i(t, x_1(t), x_2(t), ..., x_m(t))\right]. \tag{3.2}$$

Applying Theorem 2.6, equation (3.2) becomes

$$\mathcal{S}\left[x_i(t)\right] = \sum_{k=0}^{m-1} u^{-\alpha_i + k} x_i^{(k)}(0) + u^{\alpha_i}\mathcal{S}\left[G_i\right], \tag{3.3}$$

where $G_i = g_i(t, x_1(t), x_2(t), ..., x_m(t))$. Taking Sumudu inverse transform of both sides of (3.3) results in:

$$x_i(t) = \sum_{k=0}^{m-1} \xi_{ik} + \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[G_i\right]\right] = \eta_{ik} + \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[L_i + N_i\right]\right]. \tag{3.4}$$

where $\sum_{k=0}^{m-1} \xi_{ik} = \eta_{ik}$ and G_i has been decomposed into two parts L_i and N_i $(1 \leq i \leq m)$ being the linear and nonlinear terms respectively.

We assume that the unknown linear function $x_i(t)$ can be represented by the infinite series [29]

$$x_i(t) = \sum_{n=0}^{\infty} x_{in}(t), \quad 1 \leq i \leq m. \tag{3.5}$$

and the nonlinear term can be decomposed as infinite series of Adomian polynomials A_n.

$$\text{i.e,} \qquad N_i = \sum_{n=0}^{\infty} A_{in}, \quad 1 \leq i \leq m. \tag{3.6}$$

$$\text{where} \qquad A_{in} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_i\left(\sum_{n=0}^{\infty}\lambda^n x_{1n}, ..., \sum_{n=0}^{\infty}\lambda^n x_{mn}\right)\right]_{\lambda=0}. \tag{3.7}$$

Substituting (3.5) and (3.6) into (3.4) with $L_i(x_{1n}, x_{2n}, ...x_{mn}) = L_i(\bar{x}_n)$, we have

$$\sum_{n=0}^{\infty} x_{in}(t) = \eta_{ik} + \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[L_i(\bar{x}_n) + \sum_{n=0}^{\infty} A_{in}\right]\right] \tag{3.8}$$

On comparing both sides of (3.8), we have

$$\begin{aligned} x_{i0}(t) &= \eta_{ik} \\ x_{i1}(t) &= \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[L_i(\bar{x}_0) + A_{i0}\right]\right] \\ x_{i2}(t) &= \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[L_i(\bar{x}_1) + A_{i1}\right]\right] \\ &\vdots \\ x_{in}(t) &= \mathcal{S}^{-1}\left[u^{\alpha_i}\mathcal{S}\left[L_i(\bar{x}_{n-1}) + A_{i(n-1)}\right]\right] \end{aligned} \tag{3.9}$$

Finally, we can approximate the analytical solution $x_i(t)$ by the truncated series as

$$x_i(t) = \lim_{N\to\infty}\sum_{n=0}^{N} x_{in}(t). \tag{3.10}$$

In what follows, we now apply the Sumudu decomposition method in solving systems of nonlinear fractional differential equations.

3.1 Fractional Lotka-Volterra System

Example 3.1. We consider the fractional version of the classical Lotka-Volterra system treated in [7] given by:

$$\begin{aligned} {}^cD^{\alpha}x(t) &= x(t)(2 - y(t)) \\ {}^cD^{\beta}y(t) &= y(t)(-3 + x(t)) \end{aligned} \tag{3.11}$$

with initial conditions

$$x(0) = 1, y(0) = 2. \tag{3.12}$$

Taking Sumudu transform of the system (3.11) and by (3.4), we obtain:

$$\begin{aligned} x(t) &= 1 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[2x(t) - x(t)y(t)\right]\right] \\ y(t) &= 2 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[-3y(t) + x(t)y(t)\right]\right] \end{aligned} \tag{3.13}$$

By (3.8), we have:

$$\begin{aligned} \sum_{n=0}^{\infty} x_n(t) &= 1 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[2\sum_{n=0}^{\infty} x_n(t) - \sum_{n=0}^{\infty} A_n\right]\right] \\ \sum_{n=0}^{\infty} y_n(t) &= 2 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[-3\sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n\right]\right] \end{aligned} \tag{3.14}$$

where A_n are the Adomian polynomials given by the formula (3.7). Thus, the first four values of A_n are as follows: $A_0 = x_0y_0$, $A_1 = x_1y_0 + x_0y_1$, $A_2 = x_0y_2 + x_1y_1 + x_2y_0$, $A_3 = x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1$.

On comparing both sides of (3.14), we obtain

$$
\begin{aligned}
x_0 &= 1, y_0 = 2\\
x_1 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[x_0 - A_0\right]\right] = 0\\
y_1 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[-3x_0 + A_0\right]\right] = -\frac{4t^{\beta}}{\Gamma(\beta+1)}\\
\\
x_2 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[2x_1 - A_1\right]\right] = \frac{4t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
y_2 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[-3y_1 + A_1\right]\right] = \frac{8t^{2\beta}}{\Gamma(2\beta+1)}\\
\\
x_3 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[2x_2 + A_2\right]\right] = -\frac{8t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\\
y_3 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[-3y_2 + A_2\right]\right] = -\frac{16t^{3\beta}}{\Gamma(3\beta+1)} + \frac{8t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\\
&\vdots
\end{aligned}
$$

By (3.10), it follows that:

$$
\begin{aligned}
x(t) &= x_0 + x_1 + x_2 + x_3 + \cdots\\
y(t) &= y_0 + y_1 + y_2 + y_3 + \cdots
\end{aligned}
$$

Hence, the solution of the system (3.11)-(3.12) is given by:

$$
\begin{aligned}
x(t) &= 1 + \frac{4t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{8t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \cdots\\
y(t) &= 2 - \frac{4t^{\beta}}{\Gamma(\beta+1)} + \frac{8t^{2\beta}}{\Gamma(2\beta+1)} - \frac{16t^{3\beta}}{\Gamma(3\beta+1)} + \frac{8t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \cdots
\end{aligned}
\tag{3.15}
$$

For $\alpha = \beta = 1$, we obtain the same solution in [7] given as:

$$
\begin{aligned}
x(t) &= 1 + 2t^2 - \frac{4}{3}t^3 + \cdots\\
y(t) &= 2 - 4t + 4t^2 - \frac{4}{3}t^3 + \cdots
\end{aligned}
\tag{3.16}
$$

For $\alpha = 0.9$, $\beta = 0.8$, we obtain:

$$
\begin{aligned}
x(t) &= 1 + 2.5895t^{1.7} - 2.4072t^{2.5} + \cdots\\
y(t) &= 2 - 4.4024t^{0.8} + 5.5960t^{1.6} - 5.3670t^{2.4} + 2.4072t^{2.5} + \cdots
\end{aligned}
\tag{3.17}
$$

The comparison between the classical and fractional solutions of the system (3.11)-(3.12) is presented in Table 1.

3.2 Fractional Lorenz-Stenflo System

We first examine and solve the fractional form of the Lorenz equation introduced earlier in Section 1. Lorenz set the parameters $\sigma = 10, r = 28, b = \frac{8}{3}$, where the system exhibits chaotic behavior [13].

Table 1: Comparison of the approximate solution of the system (3.11)-(3.12) for $\alpha = \beta = 1$ and $\alpha = 0.9, \beta = 0.8$.

t	$\alpha = \beta = 1$	$\alpha = 0.9, \beta = 0.8$
0.1	(1.018667, 1.638667)	(1.044055, 1.429078)
0.2	(1.069333, 1.349333)	(1.124806, 1.141578)
0.5	(1.333333, 0.833333)	(1.371475, 0.726157)
1.0	(1.666667, 0.666667)	(1.182300, 0.233800)

Example 3.2. We consider the fractional Lorenz system [22] given by

$$\begin{aligned} {}^cD^{\alpha}x(t) &= 10(y(t) - x(t)) \\ {}^cD^{\beta}y(t) &= x(t)(28 - z(t)) - y(t) \\ {}^cD^{\gamma}z(t) &= x(t)y(t) - \frac{8}{3}z(t)) \end{aligned} \tag{3.18}$$

with initial conditions

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \tag{3.19}$$

Taking Sumudu transform of the system (3.18) and by (3.4), we obtain:

$$\begin{aligned} x(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[10(y(t) - x(t))\right]\right] \\ y(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[28x(t) - y(t) - x(t)z(t)\right]\right] \\ z(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-\frac{8}{3}z(t) + x(t)y(t)\right]\right] \end{aligned} \tag{3.20}$$

By (3.8), it follows that:

$$\begin{aligned} \sum_{n=0}^{\infty} x_n(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[10\left(\sum_{n=0}^{\infty} y_n(t) - \sum_{n=0}^{\infty} x_n(t)\right)\right]\right] \\ \sum_{n=0}^{\infty} y_n(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[28\sum_{n=0}^{\infty} x_n(t) - \sum_{n=0}^{\infty} y_n(t) - \sum_{n=0}^{\infty} A_n\right]\right] \\ \sum_{n=0}^{\infty} z_n(t) &= 0.1 + \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-\frac{8}{3}\sum_{n=0}^{\infty} z_n(t) + \sum_{n=0}^{\infty} B_n\right]\right] \end{aligned} \tag{3.21}$$

where A_n and B_n are the Adomian polynomials given by the formula (3.7). The first four values of A_n are given as follows: $A_0 = x_0z_0$, $A_1 = x_1z_0 + x_0z_1$, $A_2 = x_0z_2 + x_1z_1 + x_2z_0$, $A_3 = x_0z_3 + x_3z_0 + x_1z_2 + x_2z_1$. The first four values of B_n are given by: $B_0 = x_0y_0$, $B_1 = x_1y_0 + x_0y_1$, $B_2 = x_0y_2 + x_1y_1 + x_2y_0$, $B_3 = x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1$. Now, on comparing both sides of (3.21), it follows that:

$$x_0 = 0.1, \qquad y_0 = 0.1, \qquad z_0 = 0.1$$

$$\begin{aligned} x_1 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[10\left(y_0 - x_0\right)\right]\right] = 0 \\ y_1 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[28x_0 - y_0 - A_0\right]\right] = \frac{2.69t^{\beta}}{\Gamma(\beta+1)} \\ z_1 &= \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-\frac{8}{3}z_0 + B_0\right]\right] = -\frac{0.2567t^{\gamma}}{\Gamma(\gamma+1)} \end{aligned}$$

$$
\begin{aligned}
x_2 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[10\left(y_1 - x_1\right)\right]\right] = \frac{26.9t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
y_2 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[28x_1 - y_1 - A_1\right]\right] = -\frac{2.69t^{2\beta}}{\Gamma(2\beta+1)} + \frac{0.02567t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\\
z_2 &= \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-\frac{8}{3}z_1 + B_1\right]\right] = \frac{0.6845t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{0.269t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[10\left(y_2 - x_2\right)\right]\right] = -\frac{26.9t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{0.2567t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} - \frac{269t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\\
y_3 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[28x_2 - y_2 - A_2\right]\right] = \frac{755.89t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{2.69t^{3\beta}}{\Gamma(3\beta+1)} - \frac{0.05257t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{0.06845t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)}\\
z_3 &= \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-\frac{8}{3}z_2 + B_2\right]\right] = -\frac{1.8253t^{3\gamma}}{\Gamma(3\gamma+1)} - \frac{0.7147t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} - \frac{0.269t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} + \frac{2.69t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}\\
&\vdots
\end{aligned}
$$

By (3.10), it follows that:

$$
\begin{aligned}
x(t) &= x_0 + x_1 + x_2 + x_3 + \cdots\\
y(t) &= y_0 + y_1 + y_2 + y_3 + \cdots\\
z(t) &= z_0 + z_1 + z_2 + z_3 + \cdots
\end{aligned}
$$

Hence, the solution of the system (3.18)-(3.19) is given by:

$$
\begin{aligned}
x(t) &= 0.1 + \frac{26.9t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{26.9t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{0.2567t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} - \frac{269t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \cdots\\
y(t) &= 0.1 + \frac{2.69t^{\beta}}{\Gamma(\beta+1)} - \frac{2.69t^{2\beta}}{\Gamma(2\beta+1)} + \frac{0.02567t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\\
&\quad + \frac{755.89t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{2.69t^{3\beta}}{\Gamma(3\beta+1)} - \frac{0.05257t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{0.06845t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} + \cdots\\
z(t) &= 0.1 + -\frac{0.2567t^{\gamma}}{\Gamma(\gamma+1)} + \frac{0.6845t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{0.269t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\\
&\quad - \frac{1.8253t^{3\gamma}}{\Gamma(3\gamma+1)} - \frac{0.7147t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} - \frac{0.269t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} + \frac{2.69t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \cdots
\end{aligned}
\tag{3.22}
$$

For $\alpha = \beta = \gamma = 1$, we obtain the solution:

$$
\begin{aligned}
x(t) &= 0.1 + 0.00t + 13.45t^2 - 49.27t^3 + \cdots\\
y(t) &= 0.1 + 2.69t - 1.3322t^2 + 126.41t^3 + \cdots\\
z(t) &= 0.1 - 0.2567t + 0.4768t^2 - 0.01983t^3 + \cdots
\end{aligned}
\tag{3.23}
$$

For $\alpha = \beta = \gamma = 0.95$, we obtain the result [22]:

$$
\begin{aligned}
x(t) &= 0.1 + 0.0000t^{0.95} + 14.7202t^{1.9} - 59.2978t^{2.85} + \cdots\\
y(t) &= 0.1 + 2.7452t^{0.95} - 1.4580t^{1.9} + 152.13t^{2.85} + \cdots\\
z(t) &= 0.1 - 0.2620t^{0.95} + 0.5218t^{1.9} - 0.0239t^{2.85} + \cdots
\end{aligned}
\tag{3.24}
$$

The generalized Lorenz-Stenflo system proposed by Chen and Liang ([9], [30]) is given by

Table 2: Comparison of the approximate solution of the system (3.18)-(3.19) by SDM and the method presented in [22] for $\alpha = \beta = \gamma = 0.95$.

t	SDM	Milici et al [22]
0.1	(0.201556, 0.604551, 0.077138)	(0.200580, 0.601168, 0.077137)
0.2	(0.187713, 2.175898, 0.067482)	(0.180693, 2.158627, 0.067435)
0.5	(-4.180199, 22.230193, 0.100878)	(-4.275788, 22.055188, 0.100084)
1.0	(-44.477600, 153.517200, 0.335900)	(-45.166800, 152.405900, 0.330040)

$$\begin{aligned}
\frac{dx}{dt} &= a(y(t) - x(t)) + sv(t) \\
\frac{dy}{dt} &= x(t)(c - z(t)) + dy(t) \\
\frac{dz}{dt} &= x(t)y(t) - bz(t)) \\
\frac{dv}{dt} &= -x(t) - rv(t)
\end{aligned} \tag{3.25}$$

where $x(t), y(t), z(t), v(t)$ are state variables; a, b, c, d, r, s are real parameters.

We now propose and solve the fractional generalized Lorenz-Stenflo system. We set the parameter values ([9], [30]): $a = 19.42, b = 1.91, c = 29.45, d = -2.86, r = 0.23, s = 9.64$ and the initial conditions $x_0 = 2.2, y_0 = 2.0, z_0 = 10.5, v_0 = 20.0$.

Example 3.3. Consider the system:

$$\begin{aligned}
{}^cD^{\alpha}x(t) &= 19.42(y(t) - x(t)) + 9.64v(t) \\
{}^cD^{\beta}y(t) &= x(t)(29.45 - z(t)) - 2.86y(t) \\
{}^cD^{\gamma}z(t) &= x(t)y(t) - 2.86z(t)) \\
{}^cD^{\mu}v(t) &= -x(t) - 0.23v(t)
\end{aligned} \tag{3.26}$$

with initial conditions

$$x(0) = 2.2, y(0) = 2.0, z(0) = 10.5, v(0) = 20.0 \tag{3.27}$$

Taking Sumudu transform of the system (3.26) and by (3.4), we obtain:

$$\begin{aligned}
x(t) &= 2.2 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[19.42(y(t) - x(t)) + 9.64v(t)\right]\right] \\
y(t) &= 2.0 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[29.45x(t) - 2.86y(t) - x(t)z(t)\right]\right] \\
z(t) &= 10.5 + \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-1.91z(t) + x(t)y(t)\right]\right] \\
v(t) &= 20.0 + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[-(x(t) + 0.23v(t))\right]\right].
\end{aligned} \tag{3.28}$$

Clearly, by (3.8), it follows that:

$$\begin{aligned}
\sum_{n=0}^{\infty} x_n(t) &= 2.2 + \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[19.42\left(\sum_{n=0}^{\infty} y_n(t) - \sum_{n=0}^{\infty} x_n(t)\right) + 9.64\sum_{n=0}^{\infty} v_n(t)\right]\right] \\
\sum_{n=0}^{\infty} y_n(t) &= 2.0 + \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[29.45\sum_{n=0}^{\infty} x_n(t) - 2.86\sum_{n=0}^{\infty} y_n(t) - \sum_{n=0}^{\infty} A_n\right]\right] \\
\sum_{n=0}^{\infty} z_n(t) &= 10.5 + \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-1.91\sum_{n=0}^{\infty} z_n(t) + \sum_{n=0}^{\infty} B_n\right]\right] \\
\sum_{n=0}^{\infty} v_n(t) &= 20.0 + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[-\left(\sum_{n=0}^{\infty} x_n(t) + \sum_{n=0}^{\infty} v_n\right)\right]\right]
\end{aligned} \tag{3.29}$$

where A_n and B_n are as given in Example 3.2. On comparing both sides of (3.29), it is easy to see that:

$$x_0 = 2.2, \qquad y_0 = 2.0, \qquad z_0 = 10.5, \qquad v_0 = 20.0$$

$$x_1 = \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[19.42\left(y_0 - x_0\right) + 9.64v_0\right]\right] = \frac{188.92t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$y_1 = \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[29.45x_0 - 2.86y_0 - A_0\right]\right] = \frac{35.97t^{\beta}}{\Gamma(\beta+1)}.$$

$$z_1 = \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-1.91z_0 + B_0\right]\right] = -\frac{15.66t^{\gamma}}{\Gamma(\gamma+1)}.$$

$$v_1 = \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[-(x_0 + 0.23v_0)\right]\right] = -\frac{6.8t^{\mu}}{\Gamma(\mu+1)}.$$

$$x_2 = \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[19.42\left(y_1 - x_1\right) + 9.64v_1\right]\right] = \frac{698.54t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{3668.83t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{65.55t^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}.$$

$$y_2 = \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[29.45x_1 - 2.86y_1 - A_1\right]\right] = \frac{3580.03t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{102.87t^{2\beta}}{\Gamma(2\beta+1)} - \frac{34.45t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}.$$

$$z_2 = \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-1.91z_1 + B_1\right]\right] = \frac{29.91t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{377.84t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{79.13t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}.$$

$$v_2 = \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[-(x_1 + 0.23v_1)\right]\right] = -\frac{188.92t^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)} + \frac{1.56t^{2\mu}}{\Gamma(2\mu+1)}.$$

$$\begin{aligned}
x_3 &= \mathcal{S}^{-1}\left[u^{\alpha}\mathcal{S}\left[19.42\left(y_2 - x_2\right) + 9.64v_2\right]\right] = \frac{55958.54t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{1997.74t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{669.02t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \\
&+ \frac{71248.68t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1272.98t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu+1)} - \frac{1821.19t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu)} + \frac{15.04t^{\alpha+2\mu}}{\Gamma(\alpha+2\mu+1)}. \\
y_3 &= \mathcal{S}^{-1}\left[u^{\beta}\mathcal{S}\left[29.45x_2 - 2.86y_2 - A_2\right]\right] = \frac{2998.44t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \frac{69524.33t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{1242.17t^{\alpha+\beta+\mu}}{\Gamma(\alpha+\beta+\mu+1)} \\
&+ \frac{294.21t^{3\beta}}{\Gamma(3\beta+1)} - \frac{272.62t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{65.80t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma)} - \frac{831.25t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \\
&+ \frac{2958.49\Gamma(\alpha+\gamma+1)t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}. \\
z_3 &= \mathcal{S}^{-1}\left[u^{\gamma}\mathcal{S}\left[-1.91z_2 + B_2\right]\right] = -\frac{57.13t^{3\gamma}}{\Gamma(3\gamma+1)} - \frac{721.67t^{\alpha+2\gamma}}{\Gamma(\alpha+2\gamma+1)} - \frac{75.35t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} + \frac{9273.15t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \\
&- \frac{131.1t^{\alpha+\gamma+\mu}}{\Gamma(\alpha+\gamma+\mu+1)} - \frac{226.31t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{7337.66t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma)} \\
&+ \frac{6795.45\Gamma(\alpha+\beta+1)t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1)}. \\
v_3 &= \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[-(x_2 + 0.23v_2)\right]\right] = -\frac{698.54t^{\alpha+\beta+\mu}}{\Gamma(\alpha+\beta+\mu+1)} + \frac{3668.83t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu+1)} + \frac{109t^{\alpha+2\mu}}{\Gamma(\alpha+2\mu+1)} - \frac{0.3588t^{3\mu}}{\Gamma(3\mu+1)}.
\end{aligned}$$

Hence, the solution of the system (3.26)-(3.27) is given by:

$$\begin{aligned}
x(t) = 2.2 &+ \frac{188.92t^{\alpha}}{\Gamma(\alpha+1)} + \frac{698.54t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{3668.83t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{65.55t^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)} \\
&+ \frac{55958.54t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{1997.74t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{669.02t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{71248.68t^{3\alpha}}{\Gamma(3\alpha+1)} \\
&+ \frac{1272.98t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu+1)} - \frac{1821.19t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu)} + \frac{15.04t^{\alpha+2\mu}}{\Gamma(\alpha+2\mu+1)} + \cdots
\end{aligned}$$

$$
\begin{aligned}
y(t) &= 2.0 + \frac{35.97t^{\beta}}{\Gamma(\beta+1)} + \frac{3580.03t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{102.87t^{2\beta}}{\Gamma(2\beta+1)} - \frac{34.45t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} \\
&+ \frac{2998.44t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \frac{69524.33t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{1242.17t^{\alpha+\beta+\mu}}{\Gamma(\alpha+\beta+\mu+1)} + \frac{294.21t^{3\beta}}{\Gamma(3\beta+1)} \\
&- \frac{272.62t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{65.80t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma)} - \frac{831.25t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \\
&+ \frac{2958.49\Gamma(\alpha+\gamma+1)t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} + \cdots \\
z(t) &= 10.5 - \frac{15.66t^{\gamma}}{\Gamma(\gamma+1)} + \frac{29.91t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{377.84t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{79.13t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - \frac{57.13t^{3\gamma}}{\Gamma(3\gamma+1)} \\
&- \frac{721.67t^{\alpha+2\gamma}}{\Gamma(\alpha+2\gamma+1)} - \frac{75.35t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} + \frac{9273.15t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} - \frac{131.1t^{\alpha+\gamma+\mu}}{\Gamma(\alpha+\gamma+\mu+1)} \\
&- \frac{226.31t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} - \frac{7337.66t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma)} + \frac{6795.45\Gamma(\alpha+\beta+1)t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1)} + \cdots \\
v(t) &= 20.0 - \frac{6.8t^{\mu}}{\Gamma(\mu+1)} - \frac{188.92t^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)} + \frac{1.56t^{2\mu}}{\Gamma(2\mu+1)} - \frac{698.54t^{\alpha+\beta+\mu}}{\Gamma(\alpha+\beta+\mu+1)} \\
&+ \frac{3668.83t^{2\alpha+\mu}}{\Gamma(2\alpha+\mu+1)} + \frac{109t^{\alpha+2\mu}}{\Gamma(\alpha+2\mu+1)} - \frac{0.3588t^{3\mu}}{\Gamma(3\mu+1)} + \cdots
\end{aligned}
\tag{3.30}
$$

For $\alpha = \beta = \gamma = \mu = 1$, we obtain the solution

$$
\begin{aligned}
x(t) &= 2.2 + 188.92t - 1517.92t^2 + 20890.89t^3 + \cdots \\
y(t) &= 2.0 + 35.97t + 1721.36t^2 - 10454.42t^3 + \cdots \\
z(t) &= 10.5 - 15.66t + 243.44t^2 + 2385.81t^3 + \cdots \\
v(t) &= 20.0 - 6.8t - 93.68t^2 + 513.16t^3 + \cdots
\end{aligned}
\tag{3.31}
$$

4 Conclusions

In this work, the Sumudu transform method combined with the Adomian decomposition method was used to solve some systems of nonlinear fractional differential equations. The approximate solutions of the two-dimensional fractional Lotka-Volterra system, the three-dimensional fractional Lorenz chaotic system and the four-dimensional fractional Lorenz-Stenflo system were obtained.

The solution obtained for the fractional Lotka-Volterra system (3.11) coincides with the solution of the classical Lotka-Volterra when $\alpha = \beta = 1$ (see [7], equations (21)-(22)). Table 1. shows a consistent relationship between the classical and fractional solutions of (3.11). In Table 2, we compare the solutions obtained in the case of the fractional Lorenz system using the SDM with the method used in ([22], page 119).

The results show that the Sumudu decomposition method is easy to implement and gives an accurate solution to systems of nonlinear fractional differential equations. Hence, the method can be extended to solve more systems of strongly nonlinear fractional differential equations.

Acknowledgements

The author thanks the anonymous reviewers for their careful reading of the manuscript and valuable comments for the improvement of the quality of the paper.

References

[1] A.K. Alomari, M.S.M. Noorani, R. Nazar and C.P. Li, *Homotopy analysis method for solving fractional Lorenz system*, Commun. Nonlinear Sci. Numer. Simul. **15** (2010), 1864–1872.

[2] A. Atangana and A. Secer, *A note on fractional order derivatives and table of fractional derivatives of some special functions*, Abstr. Appl. Anal.ysis **2013** (2013), 1–8.

[3] R.O. Awonusika, *Analytical solution of a class of fractional Lane–Emden equation: a power series method*, Int. J. Appl. Comput. Math, **8** (2022), 155.

[4] R.O Awonusika, O.A Mogbojuri, *Approximate analytical solution of fractional Lane-Emden equation by Mittag-Leffler function method*, J. Nigerian Soc. Phys. Sci. **4** (2022), no. 2, 265–280.

[5] F.B.M. Belgacem and A.A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, Int. J. Appl. Math. Stoch. Anal. **2006** (2006), 1–23.

[6] W.E. Boyce and R.C. DiPrima *Elementary differential equations and boundary value problems*, John Wiley & Sons, Inc. New York, 2001.

[7] N. Bildik and S. Deniz *The use of Sumudu decomposition method for solving predator-prey systems*, Math. Sci. Lett. **5** (2016), no. 3, 285–289.

[8] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl. **1** (2015), 73–85.

[9] Y.-M. Chen and H.-H. Liang, *Zero-zero-Hopf bifurcation and ultimate bound estimation of a generalized Lorenz-Stenflo hyperchaotic system*, Math. Meth. Appl. Sci. **40** (2016), no. 10, 3424—3432.

[10] S. Das and P.K. Gupta, *A fractional predator-prey model and its solution*, Int. J. Nonlinear Sci. Numer. Simul. **10** (2009), no. 7, 873–876.

[11] B. Ghanbari, *On approximate solutions for a fractional prey–predator model involving the Atangana–Baleanu derivative*, Adv. Differ. Equ. **2020** (2020), 679.

[12] P. Goswamia and R.T. Alqahtanib, *Solutions of fractional differential equations by Sumudu transform and variational iteration method*, J. Nonlinear Sci. Appl. **9** (2016), 1944–1951.

[13] I. Hashim, M.S.M. Noorani, R. Ahmad, S.A. Bakar, E.S. Ismail and A.M. Zakaria *Accuracy of the Adomian decomposition method applied to the Lorenz system*, Chaos Solitons Fractals **28** (2006), 1149-–1158.

[14] Q.D. Katatbeh, F.B.M. Belgacem, *Applications of the Sumudu transform to fractional differential equations*, Nonlinear Stud. **18** (2011), no. 1, 99–112.

[15] M.M. Khader, N.H. Swetlam and A.M.S. Mahdy, *The Chebyshev collection method for solving fractional order Klein-Gordon equation*, WSEAS Trans. Math. **13** (2014), 32–38.

[16] A. Kiliçman, H. Eltayeb and P.R. Agarwal, *On Sumudu transform and system of differential equations*, Abstr. Appl. Anal. **2010** (2010), Article ID 598702.

[17] D. Kumar, J. Singh and S. Rathore, *Sumudu decomposition method for nonlinear equations*, Int. Math. Forum **7** (2012), no. 11, 515–521.

[18] A.M.S. Mahdy and M. Higazy *Numerical different methods for solving the nonlinear biochemical reaction model*, Int. J. Appl. Comput. Math. **5** (2019), 1–17.

[19] A.M.S. Mahdy, Kh. Lotfy and A.A. El-Bary, *Use of optimal control in studying the dynamical behaviors of fractional financial awareness model*, Soft Comput. **26** (2022), 3401–3409.

[20] A.M.S. Mahdy and G.M.A. Marai, *Sumudu decomposition method for solving fractional Riccati equation*, J. Abstr. Comput Math. **3** (2018), no. 1, 42-50.

[21] A.M.S. Mahdy, A.S. Mohamed and A.A.H. Mtawa, *Sumudu decomposition method for solving fractional-order logistic differential equation*, J. Adv. Math. **10** (2015), no. 7, 3642–3649.

[22] C. Milici, G. Drăgănescu and J.T. Machado, *Introduction to Fractional Differential Equations*, Nonlinear Systems and Complexity, Springer, 2019.

[23] A.S. Mohamed, A.M.S. Mahdy and A.A. H. Mtawa *Aproximate analytical solution to a time-fractional Fokker-Planck equation*, Bothalia **45** (2015), no. 4, 57–69.

[24] M. Mossa Al-Sawalha and M.S.M. Noorani, *On solving the Lorenz system by differential transformation method*, Chin. Phys. Lett. **25** (2008), no. 4, 1217.

[25] I. Podlubny, *Fractional Differential Equation*, Academic Press, San Diego, 1999.

[26] L. Stenflo, *Generalized Lorenz equations for acoustic-gravity waves in the atmosphere*, Phys. Scripta **53** (1996), no. 1, 83–84.

[27] S.H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry and engineering*, CRC Press, Boca Raton, FL, 2018.

[28] G.K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Int. J. Math. Educ. Sci. Technol. **24** (1993) no. 1, 35–43.

[29] A.M. Wazwaz, *A new algorithm for calculating Adomian polynomials for nonlinear operators*, Appl. Math. Comput. **111** (2000), 33–51.

[30] F. Zhang, R. Chen and X. Chen, *Analysis of a generalized Lorenz-Stenflo equation*, Complexity **2017** (2017).