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Common fixed point (α_*-ψ-β_i)-contractive set-valued mappings on orthogonal Branciari S_b-metric space

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Abstract

In [24], Khan et al. established some fixed point theorems in complete and compact metric spaces by using altering distance functions. In [16] Gordji et al. described the notion of orthogonal set and orthogonal metric spaces. In [18] Gungor et al. established fixed point theorems on orthogonal metric spaces via altering distance functions. In [25] Lotfy et al, introduced the notion of α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations. In [28] K. Royy et al. described the notion of Branciari S_b-metric space and related fixed point theorems with an application. In this paper, we introduce the notion of the common fixed point (α_*-ψ-β_i)-contractive set-valued mappings on orthogonal Branciari S_b-metric space with the application of the existence of a unique solution to an initial value problem.

Keywords: (α_*-ψ-β_i)-contractive, Branciari S_b-metric space, Common fixed point, Solution to an initial value problem
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1 Introduction

We know, that the fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[34]). Harandi et al. [5] introduced the best proximity pairs for upper semi continuous set-valued maps in hyper convex metric spaces. Samet et al [30] introduced the notion of α-ψ-contractive type mappings. Hassanzadeh Asl et al. [19, 20] introduced the notion of common fixed point theorems for α_*-ψ-contractive multifunction. Farajzadeh et al. [13] introduced the on fixed point theorems for (ξ, α, η)-expansive mappings in complete metric spaces. Gungor et al, established fixed point theorems on orthogonal metric spaces via altering distance functions. Lotfy et al. [25] introduced the notion of α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations. The aim of this paper is to introduce the notion common fixed point (α_*-ψ-β_i)-contractive set-valued mappings on orthogonal Branciari S_b-metric space with application the existence of a unique solution to an initial value problem.

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2 Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis. Let 2^X denote the family of all nonempty subsets of X.

Definition 2.1. ([24]) A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:
(ψ_1) $\psi(0) = 0$ and $\psi(t) > 0$ for all $t \in (0,+\infty)$;
(ψ_2) ψ is continuous and no-decreasing;
(ψ_3) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$;
(ψ_4) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;
for all $t_1, t_2 \in (0,+\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$. We denote Ψ as the set of altering distance function ψ.

Definition 2.2. Let $X \neq \emptyset$ and $\perp \subseteq 2^X \times 2^X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists A, B \subseteq X; (\exists y_0 \in B; \forall x \in A, x \perp y_0) \vee (\exists x_0 \in A; \forall y \in B, x_0 \perp y)$$

it is called $(X, \perp)$ an orthogonal set.

Definition 2.3. [16] Let $(X, \perp)$ be an orthogonal set. Any two subset $A, B \subseteq X$ are said to be orthogonally relation if $A \perp B \vee B \perp A$.

In the following, we give some examples of orthogonal sets.

Example 2.4. Let $X = \mathbb{Z}$, $A = \{x \in \mathbb{Z}/|x| \leq 2\}$ and $B = \{x \in \mathbb{Z}/x = 2k, k \in \mathbb{Z}\}$ define $A \perp B$ if there are $m \in A$, $k \in \mathbb{Z}$ and for all $n \in B$ such that $n = km$. It is easy to see that $A \perp B$. Hence $(\mathbb{Z}, \perp)$ is an orthogonal set.

Example 2.5. Let $X = \mathbb{R}^2$, $A = \{(x, y)/y = ax, a \in \mathbb{R}\}$ and $B = \{(x, y)/x^2 + y^2 = r^2, r \in \mathbb{R}\}$ define $A \perp B$ if there are $(x_0, y_0) \in A$, for all $(x, y) \in B$ such that $y_0' \times y' = -1$ or there are $(x_0, y_0) \in B$, for all $(x, y) \in A$ such that $y' \times y_0' = -1$. It is easy to see that $A \perp B \wedge B \perp A$. Hence $(\mathbb{R}^2, \perp)$ is an O-set.

The extended line is the ordered space $[-\infty; +\infty]$, considering of all points of the number line $\mathbb{R}$ and two points, denoted by $-\infty, +\infty$with the usual order relation for points of $\mathbb{R}$.

Definition 2.6. ([9, 16]) A map $d : X \times X \to [0, \infty]$ is called a generalized metric on the orthogonal set $X, \perp$. If the followig condition are satisfied, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:
$(GMS1)$ $d(x, y) = 0$ if and if $x = y$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
$(GMS2)$ $d(x, y) = d(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
$(GMS3)$ $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for any points x, y, u and $v \in X$ such that $x \perp u$, $u \perp v$, $v \perp y$ and $x \perp y$ considering that if $d(x, u) = \infty$ or $d(u, v) = \infty$ or $d(v, y) = \infty$ then $d(x, u) + d(u, v) + d(v, y) = \infty$.

In this case the orthogonal set X is called generalized orthogonal metric space and is denoted by $(X, d, \perp)$.

In the above definition, if d satisfies only $GMS1$ and $GMS2$, then it is called a semi-metric (see, e.g. [33]).

Sedghi et al.[31] introduced a new type of metric structure consisting of three variables known as S-metric. Subsequently in the year (2016), N. Souayah and N. Mlaiki [32] investigated the notion of S_b-metric spaces which generalized the concept of S-metric spaces.

Definition 2.7. ([29, 31]) A map $S : X^3 \to [0, \infty)$ is called an S-metric on the orthogonal set $(X, \perp)$. If the following conditions are satisfied, for all $x, y, z, t \in X$ such that they are ortogonally to each other:
(i) $S(x, y, z) = 0$ if and if $x = y = z$;
(ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

In this case the orthogonal set $(X, \perp)$ is called orthogonal S-metric space and is denoted by $(X, S, \perp)$.

Example 2.8. ([31]) (1) Let $\mathbb{R}$ be the real line and $X = \mathbb{R}^n$ and $||.||$ a norm on X. Then $S(x,y,z) = ||y+z-2x|| + ||y-z||$ is an S-metric on X.
(2) Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 2.9. ([27, 32]) A map $S_b : X^3 \to [0,\infty)$ is called an S_b-metric on the orthogonal set $(X,\perp)$. If the following conditions are satisfied, for all $x,y,z,t \in X$ and such that they are orthogonally to each other and let $s \geq 1$ be a given real number:
(i) $S_b(x,y,z) = 0$ if and if $x = y = z$;
(ii) $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$.
In this case the orthogonal set $(X,\perp)$ is called orthogonal S_b-metric space and is denoted by $(X,S_b,\perp)$.

Example 2.10. ([32]) Let X be a nonempty set and card$(X) \geq 5$. suppose $X = X_1 \cup X_2$ a partition of X such that card$(X_1) \geq 4$. Let $s \geq 1$, then

$$S_b(x,y,z) = \begin{cases} 0 & \text{if} \quad x = y = z, \\ 5 & \text{if} \quad x = 1 = y \quad \text{and} \quad z = 2, \\ \frac{1}{n+1} & \text{if} \quad x = 1 = y \quad \text{and} \quad z \geq 3, \\ \frac{1}{n+2} & \text{if} \quad x = 2 = y \quad \text{and} \quad z \geq 3, \\ 3 & \quad \text{otherwise.} \end{cases}$$

for all $x,y,z,t \in X$. Then S_b is an S_b-metric on X with coefficient s.

Definition 2.11. ([28]) A map $\lambda : X^3 \to \mathbb{R}_0^+$ is called an Branciari S_b-metric on the orthogonal set $(X,\perp)$. If the following conditions are satisfied, for all $x,y,z \in X$ and for $a,b \in X \setminus \{x,y,z\}$ with $a \neq b$ and such that they are ortogonally to each other and let $k \geq 1$ be a given real number:
(i) $\lambda(x,y,z) = 0$ if and if $x = y = z$;
(ii)

$$\lambda(x,y,z) \leq k[\lambda(x,x,a) + \lambda(y,y,a) + \lambda(z,z,b) + \lambda(a,a,b)]. \tag{2.1}$$

In this case the orthogonal set $(X,\perp)$ is called orthogonal Branciari S_b-metric space and is denoted by $(X,\lambda,\perp)$.

Definition 2.12. ([28]) An orthogonal Branciari S_b-metric on a nonemty set X is said to be symmetric if $\lambda(x,x,y) = \sigma(y,y,x)$ for all $x,y \in X$.

Proposition 2.13. ([28]) (i) Let (X,S,λ) be an orthogonal S-metric spaces (see definition (2.7)). The X is also an orthogonal Branciari S_b-metric space for $k = 2$.
(ii) Let (X,S_b,λ) be an orthogonal S_b-metric space with coefficient $s \geq 1$ (see definition (2.9)). The X is also an orthogonal Branciari S_b-metric space for $k = 2s^2$.

Proposition 2.14. ([28]) Shows that any orthogonal S-metric space or S_b-metric space is also an orthogonal Branciari S_b-metric space but there are several orthogonal Branciari S_b-metric spaces which are neither orthogonal S-metric spaces nor orthogonal S_b-metric spaces.

Example 2.15. ([28]) Let $X = \mathbb{N}$ and $\lambda : X^3 \to \mathbb{R}_0^+$ be defined by

$$\lambda(x,y,z) = \begin{cases} 0 & \text{if} \quad x = y = z, \\ 5 & \text{if} \quad x = 1 = y \quad \text{and} \quad z = 2, \\ \frac{1}{n+1} & \text{if} \quad x = 1 = y \quad \text{and} \quad z \geq 3, \\ \frac{1}{n+2} & \text{if} \quad x = 2 = y \quad \text{and} \quad z \geq 3, \\ 3 & \quad \text{otherwise.} \end{cases}$$

for all $x,y,z,t \in X$. Also we take $\lambda(x,x,y) = \lambda(y,y,x)$ for all $x,y \in X$. Then λ is a symmetric S_b-metric space on X for $k = \frac{5}{3}$ but it is nerther an S-metric nor an S_b-metric for any $k \geq 1$.

Definition 2.16. ([28]) Let $(X, \lambda, \perp)$ be an orthogonal Branciari S_b-metric space. Then
A sequence x_n in an orthogonal Branciari S_b-metric space $(X, \lambda, \perp)$ is called orthogonal Branciari sequence if

$$(\forall n, k \in \mathbb{N}; x_n \perp x_{n+k}) \vee (\forall n, k \in \mathbb{N}; x_{n+k} \perp x_n)$$

(i) An orthogonal Branciari sequence $\{x_n\}$ in $(X, \lambda, \perp)$ is said to be orthogonal Branciari convergent to some $z \in X$ if $\lambda(x_n, x_n, z) \to 0$ as $n \to \infty$.
(ii) An orthogonal Branciari sequence $\{x_n\}$ in $(X, \lambda, \perp)$ is said to be orthogonal Branciari cauchy if $\lambda(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.
(ii) $(X, \lambda, \perp)$ is said to be orthogonal Branciari complete if every orthogonal Branciari cauchy sequence in $(X, \lambda, \perp)$ is orthogonal Branciari convergent to some element in X.

Definition 2.17. We say that $(X, \lambda, \perp)$ has the property $\alpha-$regular orthogonal Branciari S_b-metric space if, either
(i) $\{x_n\}$ is a monotone orthogonal Branciari sequences in X such that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists an orthogonal Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ for all k. Or
(ii) $\{x_n\}$ is a monotone orthogonal Branciari sequences in X such that $\alpha(x_{n+1}, x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists an orthogonal Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x, x, x_{n_k}) \geq 1$ for all k.

Proposition 2.18. [23, 16] Suppose that $\{x_n\}$ is an orthogonal Branciari Cauchy sequence in a $(X, \lambda, \perp)$ be a orthogonal Branciari S_b-metric space with $\lim_{n\to\infty} \lambda(x_n, x_n, u) = 0$ where $u \in X$. Then

$$\lim_{n\to\infty} \lambda(x_n, x_n, z) = \lambda(u, u, z)$$

for all $z \in X$. In particular, the orthogonal Branciari sequence $\{x_n\}$ dose not Branciari converge to z if $z \neq u$.

Definition 2.19. Let $(X, \lambda, \perp)$ be an orthogonal Branciari S_b-metric space. A set-valued mapping $T : X \to 2^X$ is called orthogonal Branciari order closed if for monotone orthogonal Branciari sequences $x_n \in X$ and $y_n \in Tx_n$, with $\lim_{n\to\infty} \lambda(x_n, x_n, x) \to 0$ and $\lim_{n\to\infty} \lambda(y_n, y_n, y) \to 0$, implies $y \in Tx$.

Definition 2.20. Let $(X, \lambda, \perp)$ be an orthogonal Branciari S_b-metric space and $T, S : X \to 2^X$ with given set-valued mappings, $\alpha : X \times X \times X \to [0, +\infty)$, $\alpha_* : 2^X \times 2^X \times 2^X \to [0, +\infty)$, $\alpha_*(A, A, B) = \inf\{\alpha(a, a, b) : a \in A, b \in B\}$, $\psi \in \Psi$, $\Lambda(s, s, Ts) = \inf\{\lambda(s, s, z)/z \in Ts\}$, H_λ is the Hausdorff metric

$$H_\lambda(Tx, Tx, Ty) = \max\{\sup_{a\in Tx} \Lambda(a, a, Ty), \sup_{b\in Ty} \Lambda(Tx, Tx, b)\}.$$

$\beta_i : \mathbb{R}^+ - \{0\} \to [0, 1)$ be four decreasing functions such that $\sum_{i=1}^{4} \beta_i(t) \leq 1$ for every $t > 0$. One says that T, S are α_*-ψ-β_i-orthogonal common contractive set-valued mappings whenever

$$\begin{aligned}&\alpha_*(Ax, Ax, By)\psi(H_\lambda(Ax, Ax, By)) \leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y))\\ &+\beta_2(\Lambda(x, x, Ax))\psi(\Lambda(x, x, Ax)) + \beta_3(\lambda(y, y, By))\psi(\Lambda(y, y, By))\\ &+\beta_4(H_\lambda(Ax, Ax, By)) \min\{\psi(\Lambda(x, x, By), \psi(\Lambda(y, y, Ax))\}.\end{aligned} \tag{2.2}$$

One says that A, B are an α_*- common admissible if

$$\alpha(x, x, y) \geq 1 \Rightarrow \alpha_*(Ax, Ax, By) \geq 1 \tag{2.3}$$

$A, B = T$ or S, $Ax \perp By \vee By \perp Ax$ for all $x, y \in X$ where $x \perp y$ and $x \neq y$. One says that a mapping $A, B : X \to 2^X$ is called common orthogonal preserving ($\perp$-preserving) if $A(x) \perp B(y) \vee A(y) \perp B(x)$ if $x \perp y$.

Example 2.21. ([28]) Let $X = [0, 1)$ and let the metric on X be the Euclidian metric. Define $x \perp y$ if $xy \leq \{\frac{x}{6}, \frac{y}{6}\}$. X is not complete but it is orthogonal complete. Let $x \perp y$ and $xy \leq \frac{x}{6}$. If x_k is an arbitrary Cauchy orthogonal sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0 \vee x_{k_n} \leq \frac{1}{6}$ for all $n \in \mathbb{N}$. it follows that $\{x_{k_n}\}$ converges to a $x \in [0, 1)$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. Let $T, S : X \to 2^X$ be set-valued mapping defined by

$$Tx = \begin{cases} [0, \frac{x}{3}] & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} < x < 1 \end{cases} \quad \text{and} \quad Sx = \begin{cases} [0, \frac{x}{2}] & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

Also, $x \perp y$ and $xy \leq \frac{x}{6}$, so $x = 0$ or $y \leq \frac{1}{6}$. We have the following cases:
case (1) $x = 0$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = \{0\}$ and $Sy = [0, \frac{y}{2}]$;
case (2) $x = 0$ and $\frac{1}{6} < y \leq \frac{1}{2}$, then $Tx = \{0\}$ and $Sy = [0, \frac{y}{2}]$;
case (3) $x = 0$ and $\frac{1}{2} < y$, then $Tx = \{0\}$ and $Sy = \{0\}$;
case (4) $0 \leq x \leq \frac{1}{6}$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
case (5) $\frac{1}{6} < x \leq \frac{1}{3}$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
case (6) $\frac{1}{6} < x \leq \frac{1}{3}$ and $\frac{1}{6} < y \leq \frac{1}{2}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
case (7) $\frac{1}{3} < x$ and $\frac{1}{2} < y$, then $Tx = \{0\}$ and $Sy = \{0\}$.

These cases implies that $TxSy \leq \frac{Tx}{6}$. Hence T and S are common $\perp-$ preserving. Also, one can see that $||Tx - Sy|| \leq \frac{1}{2}||x - y||$. Hence T, S are common $\perp-$contraction.

Definition 2.22. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that $\Lambda(B, B, y) = \lambda(z, z, y)$.

Definition 2.23. A set-valued mapping $T : X \longrightarrow 2^X$ is said to have an approximate values in X if Tx is an approximation for each $x \in X$.

Definition 2.24. Let $(X, \perp, \lambda)$ be an orthogonal Branciari S_b-metric space. If $T : X \to 2^X$ is a set-valued mapping, then $x \in X$ is called fixed point for T if and only if $x \in F(x)$. The set $Fix(T) := \{x \in X/x \in Tx\}$ is called the fixed point set of T.

3 Main result

We should emphasize that throughout this paper we suppose that all set-valued mappings on an orthogonal symmetric S_b-metric space $(X, \lambda, \perp)$ have closed values.

Lemma 3.1. Let $(X, \lambda, \perp)$ be an orthogonal symmetric Branciari S_b-metric space. Suppose that $T, S : X \to 2^X$ are α_*-ψ-β_i-orthogonal common contractive set-valued mappings satisfies the following conditions:
(i) T, S are α_*-orthogonal common admissible;
(ii) there exists $x_0 \in X$ such that,

$$\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0.$$

Then $Fix(T) = Fix(S)$.

Proof . We first show that any fixed point of T is also a fixed point of S and conversely. Since $Fix(T) \neq Fix(S)$, we may assume there exists $x^* \in X$ such that $x^* \in Fix(T)$, but $x^* \notin Fix(S)$, since $\Lambda(x^*, x^*, Sx^*) > 0$. Let $x_0 \in X$ such that $\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0$. Define the orthogonal Branciari sequence $\{x_n\}$ in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a common fixed point for T, S. So, we can assume that $x_{2n} \notin Tx_{2n}$ and $x_{2n+1} \notin Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. Define

$$\alpha(x, x, y) = \begin{cases} 1 & x \perp y \vee y \perp x \\ 0 & otherwise \end{cases}$$

Since T, S are α_*-orthogonal common admissible and

$$\{x_0\} \perp Tx_0 \Rightarrow \alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1,$$

we have

$$\alpha(x_0, x_0, x_1) \geq \alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \alpha_*(Tx_0, Tx_0, Sx_1) \geq 1;$$

$$\alpha(x_1, x_1, x_2) \geq \alpha_*(Tx_0, Tx_0, Sx_1) \geq 1 \Rightarrow \alpha_*(Sx_1, Sx_1, Tx_2) \geq 1;$$

$$\alpha(x_2, x_2, x_3) \geq \alpha_*(Sx_1, Sx_1, Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_2, Tx_2, Sx_3) \geq 1.$$

Inductively, we have

$$\alpha(x_{2n}, x_{2n}, x_{2n+1}) \geq 1 \Rightarrow \alpha_*(Tx_{2n}, Tx_{2n}, Sx_{2n+1}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+1}, x_{2n+2}) \geq 1 \Rightarrow \alpha_*(Sx_{2n+1}, Sx_{2n+1}, Tx_{2n+2}) \geq 1$$

for all $n \in \mathbb{N}_0$. Let

$$\{x_0\} \perp STx_0 \Rightarrow \alpha_*(\{x_0\}, \{x_0\}, STx_0) \geq 1.$$

Similarly, we have

$$\alpha(x_{2n}, x_{2n}, x_{2n+2}) \geq 1 \Rightarrow \alpha_*(Tx_{2n}, Tx_{2n}, STx_{2n}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+1}, x_{2n+3}) \geq 1 \Rightarrow \alpha_*(Sx_{2n+1}, Sx_{2n+1}, TSx_{2n+1}) \geq 1$$

for all $n \in \mathbb{N}_0$. We obtain

$$\begin{aligned}
\psi(\Lambda(x^*, x^*, Sx^*)) \leq& \psi(H_\lambda(Tx^*, Tx^*, Sx^*)) \leq \alpha_*(Tx^*, Tx^*, Sx^*)\psi(H_\lambda(Tx^*, Tx^*, Sx^*)) \\
\leq& \beta_1(\lambda(x^*, x^*, x^*))\psi(\lambda(x^*, x^*, x^*)) + \beta_2(\Lambda(x^*, x^*, Tx^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\
&+ \beta_3(\Lambda(Sx^*, Sx^*, x^*))\psi(\Lambda(Sx^*, Sx^*, x^*)) \\
&+ \beta_4(H_\lambda(Tx^*, Tx^*, Sx^*)) \min\{\psi(\Lambda(x^*, x^*, Sx^*), \psi(\Lambda(x^*, x^*, Tx^*))\} \\
=& \beta_3(\Lambda(Sx^*, Sx^*, x^*))\psi(\Lambda(Sx^*, Sx^*, x^*)) < \psi(\Lambda(Sx^*, Sx^*, x^*)) \\
Symmetric \ =& \psi(\Lambda(x^*, x^*, Sx^*))
\end{aligned}$$

This is contradiction establishes that $Fix(T) \subseteq Fix(S)$. A similar argument establishes the reverse containment, and therefore $Fix(T) = Fix(S)$. □

Theorem 3.2. Let $(X, \lambda, \perp)$ be a complete orthogonal symmetric Branciari S_b-metric space (not necessarily complete metric space). Suppose that $T, S : X \to 2^X$ are α_*-ψ-β_i-orthogonal common contractive set-valued mappings satisfies the following conditions:
(i) T, S are α_*-orthogonal common admissible;
(ii) there exists $x_0 \in X$ such that,

$$\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0$$

(iii) X has the property $\alpha-$regular orthogonal Branciari S_b-metric space,
(iv) T, S are $\perp$-preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated orthogonal Branciari sequences $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof . By lemma (3.1), we have $Fix(T) = Fix(S)$ and we have

$$\alpha(x_n, x_n, x_{n+1}) \geq 1 \vee \alpha(x_n, x_n, x_{n+2}) \geq 1;$$

$$\{x_0\} \perp Tx_0 \perp STx_0 \cdots \vee \{x_0\} \perp STx_0 \perp TSTx_0 \cdots;$$

$$x_0 \perp x_1 \perp x_2 \cdots \vee x_0 \perp x_2 \perp x_3 \cdots;$$

Thus $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}_0$. Without loss of generality, we may assume that $T, S : X \to 2^X$ are α_*-ψ-β_i-orthogonal common contractive set-valued mappings. Consider equation (2.2), with $x = x_{2n+1}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned}
\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \leq& \alpha_*(Tx_{2n}, Tx_{2n}, Sx_{2n+1})\psi(H_\lambda(Tx_{2n}, Tx_{2n}, Sx_{2n+1})) \\
\leq& \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n})) \\
&+ \beta_3(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1}))\psi(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1})) \\
&\beta_4(H_\lambda(Tx_{2n}, Tx_{2n}, Sx_{2n+1})) \min\{\psi(\Lambda(x_{2n}, x_{2n}, Sx_{2n+1}), \psi(\Lambda(x_{2n+1}, x_{2n+1}, Tx_{2n}))\} \\
\leq& \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\
&+ \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \\
&\beta_4(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \min\{\psi(\lambda(x_{2n}, x_{2n}, x_{2n+2}), \psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+1}))\}. \quad (3.1)
\end{aligned}$$

Then

$$\begin{aligned}&(1-\beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})))\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\\ &\leq(\beta_1(\lambda(x_{2n},x_{2n},x_{2n+1}))+\beta_2(\lambda(x_{2n},x_{2n},x_{2n+1})))\psi(\lambda(x_{2n},x_{2n},x_{2n+1}))\end{aligned}\tag{3.2}$$

and

$$\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\leq\frac{(\beta_1(\lambda(x_{2n},x_{2n},x_{2n+1}))+\beta_2(\lambda(x_{2n},x_{2n},x_{2n+1})))}{(1-\beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})))}\psi(\lambda(x_{2n},x_{2n},x_{2n+1}))\tag{3.3}$$

Thus

$$\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\leq\psi(\lambda(x_{2n},x_{2n},x_{2n+1})).\tag{3.4}$$

Similarly,

$$\psi(\lambda(x_{2n},x_{2n},x_{2n+1}))\leq\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n})),\tag{3.5}$$

for all $n\in\mathbb{N}_0$. We have

$$\psi(\lambda(x_{n+1},x_{n+1},x_{n+2}))\leq\psi(\lambda(x_n,x_n,x_{n+1}))\leq\ldots\leq\psi^n(\lambda(x_0,x_0,x_1)),\tag{3.6}$$

for all $n\in\mathbb{N}$. From the property of ψ, we conclude that

$$\lambda(x_n,x_n,x_{n+1})<\lambda(x_{n-1},x_{n-1},x_n),\tag{3.7}$$

for all $n\in\mathbb{N}$, it is clear that

$$\lim_{n\to\infty}\lambda(x_{n+1},x_{n+1},x_{n+2})=0.\tag{3.8}$$

Consider equation (2.2), with $x=x_{2n}$ and $y=x_{2n+2}$. Clearly, we have

$$\begin{aligned}\psi(\lambda(x_{2n},x_{2n},x_{2n+2}))&\leq\alpha_*(Sx_{2n-1},Sx_{2n-1},Sx_{2n+1})\psi(H_\lambda(Sx_{2n-1},Sx_{2n-1},Sx_{2n+1}))\\ &\leq\beta_1(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))\\ &+\beta_2(\Lambda(x_{2n-1},x_{2n-1},Sx_{2n-1}))\psi(\Lambda(x_{2n-1},x_{2n-1},Sx_{2n-1}))\\ &+\beta_3(\Lambda(x_{2n+1},x_{2n+1},Sx_{2n+1}))\psi(\Lambda(x_{2n+1},x_{2n+1},Sx_{2n+1}))\\ &\beta_4(H_\lambda(Sx_{2n-1},Sx_{2n-1},Sx_{2n+1}))\min\{\psi(\Lambda(x_{2n-1},x_{2n-1},Sx_{2n+1}),\psi(\Lambda(x_{2n+1},x_{2n+1},Sx_{2n-1}))\}\\ &\leq\beta_1(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))\\ &+\beta_2(\lambda(x_{2n-1},x_{2n-1},x_{2n}))\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n}))\\ &+\beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\\ &\beta_4(\lambda(x_{2n},x_{2n},x_{2n+2}))\min\{\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n+2}),\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n}))\}.\end{aligned}\tag{3.9}$$

Similarly, consider equation (2.2), with $x=x_{2n-1}$ and $y=x_{2n+1}$. Clearly, we have

$$\begin{aligned}\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))&\leq\alpha_*(Tx_{2n-2},Tx_{2n-2},Tx_{2n})\psi(H_\lambda(Tx_{2n-2},Tx_{2n-2},Tx_{2n}))\\ &\leq\beta_1(\lambda(x_{2n-2},x_{2n-2},x_{2n}))\psi(\lambda(x_{2n-2},x_{2n-2},x_{2n}))\\ &+\beta_2(\Lambda(x_{2n-2},x_{2n-2},Tx_{2n-2}))\psi(\Lambda(x_{2n-2},x_{2n-2},Tx_{2n-2}))\\ &+\beta_3(\Lambda(x_{2n},x_{2n},Tx_{2n}))\psi(\Lambda(x_{2n},x_{2n},Tx_{2n}))\\ &\beta_4(H_\lambda(Tx_{2n-2},Tx_{2n-2},Tx_{2n}))\min\{\psi(\Lambda(x_{2n-2},x_{2n-2},Tx_{2n}),\psi(\Lambda(x_{2n},x_{2n},Tx_{2n-2}))\}\\ &\leq\beta_1(\lambda(x_{2n-2},x_{2n-2},x_{2n}))\psi(\lambda(x_{2n-2},x_{2n-2},x_{2n}))\\ &+\beta_2(\lambda(x_{2n-2},x_{2n-2},x_{2n-1}))\psi(\lambda(x_{2n-2},x_{2n-2},x_{2n-1}))\\ &+\beta_3(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1}))\\ &\beta_4(\lambda(x_{2n-1},x_{2n-1},x_{2n+1}))\min\{\psi(\lambda(x_{2n-2},x_{2n-2},x_{2n+1}),\psi(\lambda(x_{2n-1},x_{2n-1},x_{2n}))\}.\end{aligned}$$

Define $a_{2n}=\lambda(x_{2n-1},x_{2n-1},x_{2n+1})$ and $b_{2n}=\lambda(x_{2n},x_{2n},x_{2n+1})$. Then

$$\begin{aligned}&\psi(a_{2n})\leq\beta_1(a_{2n-1})\psi(a_{2n-1})+\beta_2(b_{2n-1})\psi(b_{2n-1})+\beta_3(b_{2n})\psi(b_{2n})+\\ &\beta_4(a_{2n})\min\{\psi(\lambda(x_{2n-2},x_{2n-2},x_{2n+1}),\psi(b_{2n-1})\}.\end{aligned}\tag{3.10}$$

From the (3.8) $\lim_{n\to\infty}b_{2n}=\lim_{n\to\infty}\lambda(x_{2n},x_{2n},x_{2n+1})=0$. We get

$$\psi(a_{2n})\leq\beta_1(a_{2n-1})\psi(a_{2n-1})\leq\psi(a_{2n-1})\tag{3.11}$$

and hence,

$$\lim_{n\to\infty}a_{2n}=\lim_{n\to\infty}\lambda(x_{2n-1},x_{2n-1},x_{2n+1})=0\Rightarrow\lim_{n\to\infty}a_n=\lim_{n\to\infty}\lambda(x_{n-1},x_{n-1},x_{n+1})=0.$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $\lambda(x_p, x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n+1, m = 2k$ and $n = 2l$ for $k, l \in \mathbb{N}$. Substitute again $x = x_{2l} = x_{2k}$ and $y = x_{2l+1} = x_{2k+1}$ in (2.2), (3.7) which yields

$$\begin{aligned}
\psi(\lambda(x_{2l}, x_{2l}, x_{2l+1})) &= \psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \leq \alpha_*(H_\lambda(Sx_{2k-1}, Sx_{2k-1}, Tx_{2k}))\psi(H(Sx_{2k-1}, Sx_{2k-1}, Tx_{2k}))\\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\
&+\beta_2(\Lambda(x_{2k-1}, x_{2k-1}, Sx_{2k-1}))\psi(\Lambda(x_{2k-1}, x_{2k-1}, Sx_{2k-1}))\\
&+\beta_3(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\\
&\beta_4(H_\lambda(Tx_{2k}, Tx_{2k}, Sx_{2k-1}))\min\{\psi(\Lambda(x_{2k}, x_{2k}, Sx_{2k-1}), \psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k}))\}\\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\
&+\beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\
&+\beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\\
&\beta_4(\lambda(x_{2k+1}, x_{2k+1}, x_{2k}))\min\{\psi(\lambda(x_{2k}, x_{2k}, x_{2k}), \psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k+1}))\}\\
&= (\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\
&+\beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\\
&\leq (\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\
&+\beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1})))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) < \psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))
\end{aligned} \tag{3.12}$$

which is impossible. Now, we shall prove that $\{x_n\}$ is an orthogonal Branciari Cauchy sequence, that is,

$$\lim_{n\to\infty} \lambda(x_n, x_n, x_{n+k}) = 0 \text{ and } x_n \perp x_{n+k}$$

for all $k \in \mathbb{N}$. We have already proved the cases for $k = 1$ and $k = 2$ in (3.7) and (3.10), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case I: Suppose that $S_n = \lambda(x_n, x_n, x_{n+1})$, $\psi(S_n) = \alpha_n S_n$ and $\alpha_n \in (0, \frac{1}{\sqrt{k}})$. Then

$$\begin{aligned}
&S_n = \lambda(x_n, x_n, x_{n+1}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_n)) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_n)\\
&\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_{n-1})) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2}\cdots\alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
&S_n^* = \lambda(x_n, x_n, x_{n+2}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_{n+1})\\
&\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_n)) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2}\cdots\alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0^*
\end{aligned} \tag{3.14}$$

for all $n \geq 1$ and $\alpha = \max_{0\leq i\leq n-1}\{\alpha_i\}$. Now, we shall prove that $\{x_n\}$ is a orthogonal Branciari Cauchy sequence, that is,

$$\lim_{n\to\infty} \lambda(x_n, x_n, x_{n+l}) = 0,$$

for all $l \in \mathbb{N}$. We have already proved the cases for $l = 1$ and $l = 2$ in (3.7) and (3.10), respectively. Now for $l = 2m+1$, where $m \geq 1$. Using the inequality (2.1), we have

$$\begin{aligned}
\lambda(x_n, x_n, x_{n+l}) &\leq k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+l}, x_{n+l}, x_{n+2}) + \lambda(x_{n+1}, x_{n+1}, x_{n+2})]\\
&= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+l}, x_{n+l}, x_{n+2}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2})]\\
\text{Symmetric } &= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k\lambda(x_{n+2}, x_{n+2}, x_{n+l})\\
&\leq 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k(k[\lambda(x_{n+2}, x_{n+2}, x_{n+3})\\
&\quad + \lambda(x_{n+2}, x_{n+2}, x_{n+3}) + \lambda(x_{n+l}, x_{n+l}, x_{n+4}) + \lambda(x_{n+3}, x_{n+3}, x_{n+4})])\\
\text{Symmetric } &= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + 2k^2\lambda(x_{n+2}, x_{n+2}, x_{n+3})\\
&\quad + k^2\lambda(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\lambda(x_{n+4}, x_{n+4}, x_{n+2m+1})\\
&\leq \cdots\\
&\quad \vdots\\
&\leq 2k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] + 2k^2[\lambda(x_{n+2}, x_{n+2}, x_{n+3}) + \lambda(x_{n+3}, x_{n+3}, x_{n+4})]\\
&\quad + \cdots + 2k^m[\lambda(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m-1}) + \lambda(x_{n+2m-1}, x_{n+2m-1}, x_{n+2m})]\\
&\quad + k^m\lambda(x_{n+2m}, x_{n+2m}, x_{n+2m+1})\\
&\leq 2[\{k(\alpha_0^n + \alpha_0^{n+1}) + k^2(\alpha_0^{n+2} + \alpha_0^{n+3}) + \cdots + k^m(\alpha_0^{n+2m-2} + \alpha_0^{n+2m-1})\} + k^m\alpha_0^{n+2m}]S_0\\
&= 2k(1+\alpha_0)\alpha_0^n[1 + k\alpha_0^2 + \cdots + k^m\alpha_0^{2m}]S_0 \frac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0
\end{aligned} \tag{3.15}$$

for all $n \geq 1$. Also for $l = 2m$ we get

$$\lambda(x_n, x_n, x_{n+2m}) \leq \cdots \leq \tfrac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0 + \alpha_0^n (k\alpha^2)^{m-1} S_0^* \tag{3.16}$$

for all $n \geq 1$. Thus we proved that $\{x_n\}$ is a orthogonal Branciari Cauchy sequence in the complete metric space $(X, \lambda, \perp)$, there exists $x^* \in X$ such that $\lim_{n\to\infty} \lambda(x_n, x_n, x^*) = 0$ by $(X, \lambda, \perp)$ has the property $\alpha-$regular Branciari S_b-metric space. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(\{x_{2n_k+1}\}, \{x_{2n_k+1}\}, \{x^*\}) \geq \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*) \geq 1 \text{ for all } k. \tag{3.17}$$

Thus

$$\begin{aligned}
\psi(\Lambda(x^*, x^*, Tx^*)) \leq &\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \psi(\Lambda(x_{2n_k+1}, x_{2n_k+1}, Tx^*)) \\
\leq &\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*)\psi(H_\lambda(Tx_{2n_k}, Tx_{2n_k}, Tx^*)) \\
\leq &\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\
&+ \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx_{2n_k})) \\
&+ \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\
&\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\Lambda(x^*, x^*, Tx_{2n_k}))\} \\
\leq &\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\
&+ \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1})) \\
&+ \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\
&\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\lambda(x^*, x^*, x_{2n_k+1}))\} \\
\leq &\psi(0) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(0) + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(0) \\
&+ \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(0)\} \\
\leq &\beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\
\leq &\psi(\Lambda(x^*, x^*, Tx^*)),
\end{aligned} \tag{3.18}$$

for all k, which is impossible. Hence, $\Lambda(x^*, x^*, Tx^*) = \Lambda(Tx^*, Tx^*, x^*) = 0$ and so $x^* \in Tx^*$. By Lemma (3.1) we have x^* common fixed point of T, S. $\square$

Corollary 3.3. [24] Let $(X, \lambda, \perp)$ be an orthogonal symmetric Branciari complete metric space(not necessarily complete metric space), $f, g : X \to X$ be a self map ,$\psi \in \Psi$ be a sub-additive function and $\alpha, \beta, \gamma : \mathbb{R}^+ - \{0\} \to [0, 1)$ be three decreasing functions such that $(\alpha + 2\beta + \gamma)(t) < 1$ for all $t > 0$. Suppose that f is $\perp$-preserving self mapping satisfying the inequality

$$\begin{aligned}
&\psi(\lambda(fx, fx, gy)) \leq \alpha(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta(\lambda(x, x, y))[\psi(\lambda(x, x, fx)) \\
&+\psi(\lambda(y, y, gy))] + \gamma(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, gy), \psi(\Lambda(y, y, fx))\},
\end{aligned} \tag{3.19}$$

for all $x, y \in X$ where $x \perp y$ and $x \neq y$. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n x_0\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.

Example 3.4. Let $X = \mathbb{Z}$, $A = \{x \in \mathbb{Z} | |x| \leq 2\}$ and $B = \{x \in \mathbb{Z} | x = 2k, k \in \mathbb{N}\}$ define $A \perp B$ if there are $m \in A$, $k \in \mathbb{Z}$ and for all $n \in B$ such that $n = km$. It is easy to see that $A \perp B$. Hence $(\mathbb{Z}, \perp)$ is an O-set.
Let $Y \subseteq X$ be a finite set defined as $Y = \{1, 2, 4, 8\}$. Define $\lambda : Y \times Y \times Y \to [0, \infty)$ as:
$\lambda(1, 1, 1) = \lambda(2, 2, 2) = \lambda(4, 4, 4) = \lambda(8, 8, 8) = 0$,
$\lambda(1, 1, 2) = \lambda(2, 2, 1) = 3$,
$\lambda(2, 2, 8) = \lambda(8, 8, 2) = \lambda(1, 1, 8) = \lambda(8, 8, 1) = 1$ and
$\lambda(1, 1, 4) = \lambda(4, 4, 1) = \lambda(2, 2, 4) = \lambda(4, 4, 2) = \lambda(8, 8, 4) = \lambda(4, 4, 8) = \frac{1}{2}$.

The function λ is not a metric on Y. Indeed, note

$$3 = \lambda(1, 1, 2) \geq \lambda(1, 1, 8) + \lambda(8, 8, 2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, λ is a symmetric Branciari S_b-metric on Y and moreover (Y, λ) is a complete symmetric Branciari S_b-metric space. Define $T, S : Y \to 2^Y$ as: $T1 = T2 = T8 = \{2, 4\}, T4 = \{1, 8\}$

and $S1 = S2 = S4 = \{2, 8\}, S8 = \{1, 2\}$, $\alpha : Y \times Y \times Y \to [0, +\infty), \alpha_* = \inf \alpha$ as

$$\alpha(x, x, y)) = \begin{cases} 1 & x \perp y \vee y \perp x \\ 0 & otherwise \end{cases}$$

$\psi(t) = \frac{2}{3}t$. Clearly, T, S satisfies the conditions of Theorem (3.2) and has a common fixed point $x = 2$.

4 Some consequences

In this section we give some consequences of the main results presented above. Specifically, we apply our results to generalized metric spaces endowed with a partial order.

4.1 Fixed point theorems for weakly increasing on X has the property α-regular orthogonal symmetric Branciari complete metric space

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([14]). Let X be a topological space and $\preceq$ be a partial order on X.

Definition 4.1. ([14]). Let A, B be two nonempty subsets of X, the relations between A and B are definers follows:
(r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(r_2) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 4.2. ([11], [12]). Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing.

Example 4.3. Let $X = \mathbb{R}^+$ endowed with usual ordering. Let $f, g : X \to X$ defined by

$$fx = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty \end{cases} \quad \text{and} \quad gx = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty \end{cases}$$

then it is obvious that $fx \le gfx$ and $gx \le fgx$ for all $x \in X$. Thus f and g are weakly increasing mappings. Note that both f and g are not nondecreasing.

Definition 4.4. ([3]) Let $(X, \preceq)$ be a partially ordered set. Two mapping $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Now we give some examples.

Example 4.5. ([3]) Let $X = [1, \infty)$ and $\le$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \succ_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 4.6. ([3]) Let $X = [1, \infty)$ and $\le$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [0, 1]$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Fx = [0, 1] \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 [0, 1] = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_1$ but $G1 = 1 \succ_2 0, 1 = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Theorem 4.7. Let $(X, \preceq, \perp, \lambda)$ be a partially ordered orthogonal symmetric Branciari complete metric space (not necessarily complete metric space). Suppose that $T, S : X \to 2^X$ are α_*-ψ-β_i-orthogonal common contractive set-valued mappings for all $x, y \in X$ with $x \prec_1 y$ or $x \perp y$ satisfies the following conditions:
(i) T and S be a weakly increasing pair on X w.r.t $\prec_1$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 STx_0$ or $\{x_0\} \perp Tx_0$ and $\{x_0\} \perp STx_0$;
(iii) X has the property α-regular orthogonal symmetric Branciari complete metric space,
(iv) T, S are $\perp$-preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated $O-$sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof . Define the orthogonal sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for T, S. Using that the pair of set-valued mappings T and S is weakly increasing and by define $\alpha : X \times X \times X \to [o, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1, & x \preceq y \vee x \perp y \\ 0, & otherwise. \end{cases}$$

It can be easily shown that the orthogonal sequence x_n is nondecreasing w.r.t, $\preceq$ i.e; and $\alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0$, such that $\alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \preceq x_1 \vee x_0 \perp x_1$. Now since T and S are weakly increasing with respect to $\prec_1$, we have $x_1 \in Tx_0 \prec_1 Sx_1$. Thus there exist some $x_2 \in Sx_1$ such that $x_1 \preceq x_2 \vee x_1 \perp x_2$. Again since T and S are weakly increasing with respect to $\prec_1$, we have $x_2 \in Sx_1 \prec_1 Tx_2$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \preceq x_3 \vee x_2 \perp x_3$. Continue this process, we will get a nondecreasing orthogonal sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n=1}$, $n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{2n} \preceq x_{2n+1}$$

or

$$x_{2n+2} \preceq \cdots$$

or

$$x_0 \perp x_1 \perp x_2 \perp \cdots \perp x_{2n} \perp x_{2n+1} \perp x_{2n+2} \perp \cdots .$$

In particular x_n, x_{n+k} are comparable for all $k \in \mathbb{N}$. $\alpha(x_n, x_n, x_{n+k}) \geq 1$ for all $n \in \mathbb{N}_0$ and by (4) we have $\lim_{n\to\infty} \lambda(x_n, x_n, x_{n+k}) = 0$. Following the proof of Theorem (3.2). Thus we proved that $\{x_n\}$ is a orthogonal Cauchy sequence in the orthogonal symmetric Branciari complete metric space $(X, \perp, \lambda)$, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} \lambda(x_n, x_n, x^*) = 0$$

and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then x^* is a common fixed point of T, S. $\square$

Theorem 4.8. Let $(X, \preceq, \perp, \lambda)$ be a partially ordered orthogonal symmetric Branciari complete metric space (not necessarily complete metric space). Suppose that $T, S : X \to 2^X$ are α_*-ψ-β_i-orthogonal common contractive set-valued mappings for all $x, y \in X$ with $x \prec_2 y$ or $x \perp y$ satisfies the following conditions:
(i) T and S be a weakly increasing pair on X w.r.t $\prec_2$;
(ii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $STx_0\} \prec_2 \{x_0\}$ or $Tx_0 \perp \{x_0\}$ and $STx_0 \perp \{x_0\}$;
(iii) X has the property α-regular orthogonal symmetric Branciari complete metric space,
(iv) T, S are $\perp$-preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated orthogonal sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof . Define the orthogonal sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for T, S. Using that the pair of set-valued mappings T and S is weakly increasing and by define

$$\alpha(x,x,y) = \begin{cases} 1, & x \succeq y \vee x \perp y \\ 0, & otherwise. \end{cases}$$

It can be easily shown that the sequence x_n is non increasing w.r.t, $\preceq$ i.e; and

$$\alpha_*(x_0, x_0, \{Tx_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \succeq x_1.$$

Now since T and S are weakly increasing with respect to $\prec_2$, we have $Sx_1 \prec_2 Tx_0$. Thus there exist some $x_2 \in Sx_1$ such that $x_1 \succeq x_2$. Again since T and S are weakly increasing with respect to $\prec_2$, we have $Tx_2 \preceq_2 Sx_1$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \succeq x_3$. Continue this process, we will get a non increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots \succeq x_{2n} \succeq x_{2n+1} \succeq x_{2n+2} \succeq \cdots$$

or

$$x_0 \perp x_1 \perp x_2 \perp \cdots \perp x_{2n} \perp x_{2n+1} \perp x_{2n+2} \perp \cdots .$$

In particular x_{n+k}, x_n are comparable for all $k \in \mathbb{N}$, $\alpha(x_{n+k}, x_{n+k}, x_n) \geq 1$ for all $n \in \mathbb{N}_0$ and by (4) we have $\lim_{n\to\infty} \lambda(x_{n+k}, x_{n+k}, x_n) = 0$. Following the proof of Theorem (3.2), thus we proved that $\{x_n\}$ is a orthogonal Cauchy sequence in the orthogonal complete metric space $(X, \perp, d)$, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} \lambda(x_n, x_n, x^*) = 0$$

and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then x^* is a common fixed point of T, S. $\square$

4.2 Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to the our common fixed point theorem.
Let us consider Cauchy problem for the first order differential equations system

$$\begin{cases} x' = f(t, x(t), y(t)), & t \in R, \quad x(0) = x_0 \\ y' = g(t, y(t), x(t)), & t \in R, \quad y(0) = y_0 \end{cases} \tag{4.1}$$

Theorem 4.9. Given a point $(t_0, x_0, y_0) \in R \times R^n \times R^n$ and consider the differential equations system (4.1). Let P be a Picard mapping defined by

$$\begin{cases} (Px)(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau), y(\tau))d\tau \\ (Py)(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau), x(\tau))d\tau \end{cases} \tag{4.2}$$

Note that $(Px)(t_0) = x_0$ and $(Py)(t_0) = y_0$ for any x, y. The mappings $x, y : I \to R^n$ are a solution to the differential equations system (4.1) with the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$ if and only if $x = Px$ and $y = Py$, where the functions $f, g : I \times R \times R \to R$ are defined in the domain $D = \{(t, x, y); |t - t_0| \leq a, |x - x_0| \leq b, |y - y_0| \leq c\}$, $x_0, y_0 \in R$ and satisfied the condition

$$|f(t, x_1, y_1) - g(t, x_2, y_2)| \leq \frac{K}{2|t - t_0|}(|x_1 - x_2| + |y_1 - y_2|), \quad 0 < K < 1. \tag{4.3}$$

Let $M = \max_{(t,x(t),y(t))\in D}\{|f(t, x(t), y(t))|, |g(t, x(t), y(t))|\}$. There exists $d = \min\{a, \frac{b}{M}, \frac{c}{M}\}$ such that

$$D_0 = \{(t, x, y)/|t - t_0| \leq d, |x - x_0| \leq M|t - t_0|, |y - y_0| \leq M|t - t_0|\}, \tag{4.4}$$

lies in D. We are trying to find a solution $\varphi(t, x, y)$ and $\varphi(t, y, x)$ for the differential equations system (4.1) with initial condition $\varphi(t_0, x, y) = x_0$ and $\varphi(t_0, y, x) = y_0$ expressed in the form $\varphi(t, x, y) = x_0 + h(t, x, y)$ and $\varphi(t, y, x) =$

$y_0 + h(t, y, x)$. Then the mapping φ defined on the $\{(t, x, y); |t - t_0| \leq d, |x - x_0| \leq b, |y - y_0| \leq c\}$ is the general solution of (4.1). Let

$$X = \{h(t, x, y)/(t, x, y) \in D_0\}.$$

Note that $h(t_0, x, y) = 0$ for any $h \in X$. In space X, we define a relation $\perp$ by

$$h_1 \perp h_2 \iff ||h_1||||h_2|| \leq d(||h_1|| \vee ||h_2||), \tag{4.5}$$

where $||h_1|| \vee ||h_2|| = ||h_1|| or ||h_2||$ which is an orthogonality relation on X. Let $\lambda : X \times X \times X \to [0, \infty]$ be given by

$$\lambda(x, y, z) = ||x - z|| + ||y - z||$$

then

$$\lambda(h_1, h_1, h_2) = ||h_1 - h_2|| + ||h_1 - h_2|| = 2 \sup_{(t,x,y)\in D_0} |h_1(t, x, y) - h_2(t, x, y)|. \tag{4.6}$$

Hence the orthogonal symmetric Branciari metric space $(X, \perp, \lambda)$ is complete. A mappings $A, B : (X, \perp, \lambda) \to (X, \perp, \lambda)$ can be defined by

$$\begin{cases} (Ah)(t, x, y) = \int_{t_0}^{t} f(\tau, x_0 + h(\tau, x, y), y_0 + h(\tau, y, x))d\tau \\ (Bh)(t, y, x) = \int_{t_0}^{t} g(\tau, y_0 + h(\tau, y, x), x_0 + h(\tau, x, y))d\tau \end{cases} \tag{4.7}$$

We now discuss some properties of mappings A and B.
i) A and B are $\perp$-preserving mappings;
ii) $\lambda(Ah_1, Ah_1, Bh_2) \leq \delta\lambda(h_1, h_1, h_2)$ for any h_1 and h_2 in X such that $h_1 \perp h_2$ and $0 \leq \delta < 1$;
iii) A or B is $\perp$-continuous mapping;

Proof . *i)* We recall that A and B are $\perp$-preserving mappings if for h_1, $h_2 \in X, h_1 \perp h_2$, we have $Ah_1 \perp Bh_2$.

$$\begin{aligned} |(Ah_1)(t, x, y)| &= \left| \int_{t_0}^{t} f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x))d\tau \right| \\ &\leq \int_{t_0}^{t} |f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x))|d\tau \\ &\leq \int_{t_0}^{t} M d\tau = M|t - t_0| \\ &\leq M \frac{d}{M} = d. \end{aligned} \tag{4.8}$$

So,

$$||Ah_1||||Bh_2|| \leq d||Bh_2||. \tag{4.9}$$

This means that $||Ah_1|| \perp ||Bh_2||$.

ii) Let h_1, h_2 in X and $h_1 \perp h_2$ we have

$$\begin{aligned} &|(Ah_1)(t, x, y) - (Bh_2)(t, y, x)| \\ &= \left| \int_{t_0}^{t} f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x))d\tau - \int_{t_0}^{t} g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x))d\tau \right| \\ &= \left| \int_{t_0}^{t} (f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x)) - g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x))d\tau \right| \\ &\leq \int_{t_0}^{t} |f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x)) - g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x))|d\tau \\ &\leq \int_{t_0}^{t} (\frac{K}{2|t - t_0|}|x_0 + h_1(\tau, x, y) - x_0 - h_2(\tau, x, y)| + \frac{K}{2|t - t_0|}|y_0 + h_1(\tau, y, x) - y_0 - h_2(\tau, y, x)|)d\tau \\ &= \int_{t_0}^{t} \frac{K}{2|t - t_0|}(2|h_1(\tau, x, y) - h_2(\tau, x, y)|)d\tau = K||h_1 - h_2||. \end{aligned} \tag{4.10}$$

Thus,

$$||Ah_1 - Bh_2|| \leq K||h_1 - h_2||. \tag{4.11}$$

iii) Suppose $\{h_n\}$ is an orthogonal sequence in X such that $\{h_n\}$ converging to $h \in X$. Because A or B is $\perp$-preserving, $\{Ah_n\}$ or $\{Bh_n\}$ is an orthogonal sequence in X. For any $n \in \mathbf{N}$, by ii we have

$$||Ah_n(t,x,y) - Ah(t,x,y)|| \leq K||h_n - h||. \tag{4.12}$$

As n goes to infinity, it follows that A is $\perp$-continuous mapping. The mapping A or B defined above is $\perp$-preserving and $\perp$-continuous on generalized orthogonal metric space $(X, \lambda, \perp)$. Mapping A and B satisfies of Theorem (3.2). Thus, existence and uniqueness of its fixed point $h_0 \in X$ has been guaranteed by Theorem (3.2). We are looking for solutions expressed in the form $\varphi(t,x,y) = x_0 + h(t,x,y)$ and $\varphi(t,y,x) = y_0 + h(t,y,x)$. If h is a common fixed point of A and B then $\psi(t,x,y) = x_0 + Ah(t,x,y)$ and $\varphi(t,y,x) = y_0 + Bh(t,y,x)$ is a common fixed point of our Picard $P(\varphi)$. Hence

$$\begin{aligned} P(\varphi(t,x,y)) =& x_0 + (Ah)(t,x,y) \\ =& x_0 + \int_{t_0}^{t} f(\tau, x_0 + h(\tau,x,y), y_0 + h(\tau,y,x))d\tau \\ =& x_0 + \int_{t_0}^{t} f(\tau, \psi(t,x,y), \varphi(t,y,x))d\tau \\ =& \psi(t,x,y). \end{aligned} \tag{4.13}$$

Similarly $P(\varphi(t,y,x)) = \varphi(t,y,x)$. By Theorem (3.2), $\varphi(t,x,y)$ and $\varphi(t,y,x)$ are a solutions of the differential equations system (4.1) if and only if $P(\varphi(t,y,x)) = \varphi(t,y,x)$ and $P(\varphi(t,x,y)) = \varphi(t,x,y)$. $\square$

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