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Quantum dual Simpson type inequalities for q-differentiable convex functions

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Abstract

This work introduces the quantum analogue of the dual Simpson type integral inequalities for the class of q-differentiable convex functions through a new identity. The results are also accompanied by their applications.

Keywords: Dual Simpson inequality, q-derivatives, q-integrals, convex functions
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1 Introduction

The dual Simpson's formula, as indicated in [8], is a well-known inequality in scientific literature and can be expressed as follows:

$$\left| \frac{1}{b-a} \int_a^b f(u)\, du - \frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right) \right| \le \frac{7(b-a)^4}{23040} \left\| f^{(4)} \right\|_\infty , \tag{1.1}$$

where f is a function that is four-times continuously differentiable on the interval (a,b), and $\left\| f^{(4)} \right\|_\infty = \sup_{x \in (a,b)} \left| f^{(4)}(x) \right|$.

Convexity is well known to have a vital and central role in numerous domains, including economics, fnance, optimization, and engineering. This concept has a strong relationship to inequalities and the combination of the two has become a widely researched topic. It is worth noting that a function is considered convex on I if it meets the following inequality:

$$f(\varkappa x + (1-\varkappa) y) \le \varkappa f(x) + (1-\varkappa) f(y)$$

for all $\varkappa \in [0,1]$, $x, y \in I$ (see [17]).

Quantum calculus, commonly referred to as calculus without limits, is a calculus theory in which regularity is not essential. In [9], Jackson presented the foundations of this theory as well as its entire mathematical presentation,

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introducing the q-integral and q-derivative in a systematic manner. According to certain mathematical historians, its origins date back to Euler and Jacobi. This theory is highly intriguing and has a wide range of applications in various branches of mathematics and physics. Following the publication of the Tariboon and Ntouyas studies [21, 22], quantum calculus gained unquestioned popularity in the subject of integral inequalities, with several novel inequalities and quantum analogues appearing in the literature, see [1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 18, 20, 23, 24] Inspired by ongoing advancements in this constantly evolving field, this paper introduces some quantum dual Simpson-type integral inequalities for functions with convex q-derivatives, which are derived from a novel quantum identity. Applications of our results are given.

2 Preliminaries

Definition 2.1 ([10, 22]). Consider a continuous function $\varphi : J = [a, b] \to \mathbb{R}$. The definition of the q-derivative of φ at $u \in J$ is as follows:

$$ {}_aD_q\varphi(u) = \frac{\varphi(u) - \varphi(qu + (1-q)a)}{(1-q)(u-a)}, u \neq a, $$

and

$$ {}_aD_q\varphi(a) = \lim_{u\to a} {}_aD_q\varphi(u). $$

Definition 2.2 ([10, 22]). Consider a continuous function $\varphi : J = [a, b] \to \mathbb{R}$. The definition of the q-integral over the interval J is as follows:

$$ \int_a^u \varphi(\varkappa)\ {}_ad_q\varkappa = (1-q)(u-a)\sum_{n=0}^{\infty} q^n \varphi(q^n u + (1-q^n)a), \quad u \in J, $$

Additionally, if $c \in (a, u)$, the definite q-integral over the interval J is defined as:

$$ \int_c^u \varphi(\varkappa)\ {}_ad_q\varkappa = \int_a^u \varphi(\varkappa)\ {}_ad_q\varkappa - \int_a^c \varphi(\varkappa)\ {}_ad_q\varkappa. $$

Theorem 2.3 ([21]). If $\varphi : J \to \mathbb{R}$ is a continuous function, then it holds that:

- ${}_aD_q\int_a^u \varphi(\varkappa)\ {}_ad_q\varkappa = \varphi(u)$.
- $\int_c^u {}_aD_q\varphi(\varkappa)\ {}_ad_q\varkappa = \varphi(u) - \varphi(c), \quad c \in (a, u)$.

Theorem 2.4 ([21]). Assuming φ and $\psi : J \to \mathbb{R}$ are continuous functions, and $\nu \in \mathbb{R}$, it follows that for $u \in J$:

- $\int_a^u [\varphi(\varkappa) + \psi(\varkappa)]\ {}_ad_q\varkappa = \int_a^u \varphi(\varkappa)\ {}_ad_q\varkappa + \int_a^u \psi(\varkappa)\ {}_ad_q\varkappa$,
- $\int_a^u (\nu\varphi)(\varkappa)\ {}_ad_q\varkappa = \nu\int_a^u \varphi(\varkappa)\ {}_ad_q\varkappa$,
- $\int_c^u \varphi(\varkappa)\ {}_aD_q\psi(\varkappa)\ {}_ad_q\varkappa = \varphi(\varkappa)\psi(\varkappa)|_c^u - \int_c^u \psi(q\varkappa + (1-q)a)\ {}_aD_q\varphi(\varkappa)\ {}_ad_q\varkappa, \quad c \in (a, u)$.

Lemma 2.5 ([22]). For $p \in \mathbb{R}$ with $p \neq -1$, it can be shown that the following formula holds:

$$ \int_a^x (\varkappa - a)^p\ {}_ad_q\varkappa = \frac{1-q}{1-q^{p+1}}(x-a)^{p+1}. \tag{2.1} $$

Lemma 2.6 ([19]). Let φ and ψ be continuous functions defined on the interval $[a,b]$, then for $0 < c \leq 1$, the following equality holds true:

$$\int_0^c \psi(\varkappa)\ {}_aD_q\varphi(\varkappa b+(1-\varkappa)a)\ {}_0d_q\varkappa = \psi(c)\varphi(cb+(1-c)a) - \psi(0)\varphi(0) - \int_0^c {}_aD_q\psi(\varkappa).\varphi(q\varkappa b+(1-q\varkappa)a)\ {}_0d_q\varkappa.$$

3 Auxiliary results

To demonstrate our findings, it is necessary to consider the following auxiliary results.

Lemma 3.1. Let $f : I = [a,b] \to \mathbb{R}$ be a function that is q-differentiable on interval I. Under the assumption that $aDqf$ is integrable on I and that $0 < q < 1$, it follows that

$$\begin{aligned}
&\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right) - \frac{1}{b-a}\int_a^b f(\varkappa)\ {}_ad_q\varkappa \\
&= (b-a)\left(\int_0^{\frac14} q\varkappa\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa + \int_{\frac14}^{\frac12}\left(q\varkappa-\frac{2}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa\right. \\
&\left. + \int_{\frac12}^{\frac34}\left(q\varkappa-\frac{1}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa + \int_{\frac34}^{1}(q\varkappa-1)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa\right).
\end{aligned}$$

Proof . Using Lemma 2.6, we have

$$\int_0^{\frac14} q\varkappa\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa = \frac{q}{4(b-a)}f\left(\frac{3a+b}{4}\right) - \frac{q}{b-a}\int_0^{\frac14} f((1-q\varkappa)a+q\varkappa b)\ {}_0d_q\varkappa. \tag{3.1}$$

In an analogous manner, we get

$$\begin{aligned}
&\int_{\frac14}^{\frac12}\left(q\varkappa-\frac{2}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa \\
&= \int_0^{\frac12}\left(q\varkappa-\frac{2}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa - \int_0^{\frac14}\left(q\varkappa-\frac{2}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa \\
&= \frac{3q-4}{6(b-a)}f\left(\frac{a+b}{2}\right) + \frac{2}{3(b-a)}f(a) - \frac{q}{b-a}\int_0^{\frac12} f((1-q\varkappa)a+q\varkappa b)\ {}_0d_q\varkappa \\
&\quad - \frac{3q-8}{12(b-a)}f\left(\frac{3a+b}{4}\right) - \frac{2}{3(b-a)}f(a) + \frac{q}{b-a}\int_0^{\frac14} f((1-q\varkappa)a+q\varkappa b)\ {}_0d_q\varkappa,
\end{aligned} \tag{3.2}$$

$$\int_{\frac12}^{\frac34}\left(q\varkappa-\frac{1}{3}\right)\ {}_aD_qf((1-\varkappa)a+\varkappa b)\ {}_0d_q\varkappa$$

$$
\begin{aligned}
&= \int_0^{\frac{3}{4}} \left(q\varkappa - \frac{1}{3}\right) \ {}_aD_q f\left((1-\varkappa)a + \varkappa b\right) \ {}_0d_q\varkappa - \int_0^{\frac{1}{2}} \left(q\varkappa - \frac{1}{3}\right) \ {}_aD_q f\left((1-\varkappa)a + \varkappa b\right) \ {}_0d_q\varkappa \\
&= \frac{9q-4}{12(b-a)} f\left(\frac{a+3b}{4}\right) + \frac{1}{3(b-a)} f(a) - \frac{q}{b-a} \int_0^{\frac{3}{4}} f\left((1-q\varkappa)a + q\varkappa b\right) \ {}_0d_q\varkappa \\
&\quad - \frac{3q-2}{6(b-a)} f\left(\frac{a+b}{2}\right) - \frac{1}{3(b-a)} f(a) + \frac{q}{b-a} \int_0^{\frac{1}{2}} f\left((1-q\varkappa)a + q\varkappa b\right) \ {}_0d_q\varkappa \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{3}{4}}^{1} (q\varkappa - 1) \ {}_aD_q f\left((1-\varkappa)a + \varkappa b\right) \ {}_0d_q\varkappa \\
&= \int_0^{1} (q\varkappa - 1) \ {}_aD_q f\left((1-\varkappa)a + \varkappa b\right) \ {}_0d_q\varkappa - \int_0^{\frac{3}{4}} (q\varkappa - 1) \ {}_aD_q f\left((1-\varkappa)a + \varkappa b\right) \ {}_0d_q\varkappa \\
&= \frac{q-1}{b-a} f(b) + \frac{1}{b-a} f(a) - \frac{q}{b-a} \int_0^{1} f\left((1-q\varkappa)a + q\varkappa b\right) \ {}_0d_q\varkappa \\
&\quad - \frac{3q-4}{4(b-a)} f\left(\frac{a+3b}{4}\right) - \frac{1}{b-a} f(a) + \frac{q}{b-a} \int_0^{\frac{3}{4}} f\left((1-q\varkappa)a + q\varkappa b\right) \ {}_0d_q\varkappa. \qquad (3.4)
\end{aligned}
$$

According to Definition 2.2, we have

$$
\begin{aligned}
\frac{q}{b-a} \int_0^{1} f\left((1-q\varkappa)a + q\varkappa b\right) \ {}_0d_q\varkappa &= \frac{q}{b-a} \left((1-q) \sum_{n=0}^{\infty} q^n f\left(q^{n+1}b + \left(1-q^{n+1}\right)a\right) \right) \\
&= \frac{1}{b-a} \left((1-q) \sum_{n=0}^{\infty} q^{n+1} f\left(q^{n+1}b + \left(1-q^{n+1}\right)a\right) \right) \\
&= \frac{1}{b-a} (1-q) \left(\sum_{n=0}^{\infty} q^n f\left(q^n b + (1-q^n)a\right) - f(b) \right) \\
&= \frac{1}{b-a} (1-q) \sum_{n=0}^{\infty} q^n f\left(q^n b + (1-q^n)a\right) - \frac{1-q}{b-a} f(b) \\
&= \frac{1}{(b-a)^2} \left((1-q)(b-a) \sum_{n=0}^{\infty} q^n f\left(q^n b + (1-q^n)a\right) \right) - \frac{1-q}{b-a} f(b) \\
&= \frac{1}{(b-a)^2} \left(\int_a^b f(u) \ {}_ad_q u \right) - \frac{1-q}{b-a} f(b). \qquad (3.5)
\end{aligned}
$$

The desired result is produced by summing (3.1)-(3.4), using (3.5) and then multiplying the obtained equality by $(b-a)$. □

The following lemmas can be shown by simple calculations, so we have omitted them.

Lemma 3.2. The following equalities hold for a constant q such that $0 < q < 1$.

$$
\int_0^{\frac{1}{4}} q\varkappa \ {}_0d_q\varkappa = \frac{q}{16(1+q)}, \qquad (3.6)
$$

$$\int_{0}^{\frac{1}{4}} q\varkappa^{2} \ {}_{0}d_{q}\varkappa = \frac{q}{64\left(1+q+q^{2}\right)}, \tag{3.7}$$

$$\int_{0}^{\frac{1}{4}} q\varkappa\left(1-\varkappa\right) \ {}_{0}d_{q}\varkappa = \frac{3q+3q^{2}+4q^{3}}{64\left(1+q\right)\left(1+q+q^{2}\right)}, \tag{3.8}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left|q\varkappa-\frac{2}{3}\right| \ {}_{0}d_{q}\varkappa = \frac{8-q}{48\left(1+q\right)}, \tag{3.9}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left|q\varkappa-\frac{2}{3}\right| \varkappa \ {}_{0}d_{q}\varkappa = \frac{8+q+q^{2}}{64\left(1+q\right)\left(1+q+q^{2}\right)}, \tag{3.10}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left|q\varkappa-\frac{2}{3}\right| \left(1-\varkappa\right) \ {}_{0}d_{q}\varkappa = \frac{8+25q+25q^{2}-4q^{3}}{192\left(1+q\right)\left(1+q+q^{2}\right)}, \tag{3.11}$$

$$\mathcal{D}_{1} = \int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varkappa-\frac{1}{3}\right| \ {}_{0}d_{q}\varkappa = \begin{cases} \frac{4-11q}{48(1+q)} & \text{if } 0<q<\frac{4}{9} \\ \frac{-28+57q}{144(1+q)} & \text{if } \frac{4}{9}\leq q\leq\frac{2}{3} \\ \frac{-4+11q}{48(1+q)} & \text{if } \frac{2}{3}<q<1, \end{cases} \tag{3.12}$$

$$\mathcal{D}_{2} = \int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varkappa-\frac{1}{3}\right| \varkappa \ {}_{0}d_{q}\varkappa = \begin{cases} \frac{20-37q-37q^{2}}{192(1+q)(1+q+q^{2})} & \text{if } 0<q<\frac{4}{9} \\ \frac{-340+477q+477q^{2}}{1728(1+q)(1+q+q^{2})} & \text{if } \frac{4}{9}\leq q\leq\frac{2}{3} \\ \frac{-20+37q+37q^{2}}{192(1+q)(1+q+q^{2})} & \text{if } \frac{2}{3}<q<1, \end{cases} \tag{3.13}$$

$$\mathcal{D}_{3} = \int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varkappa-\frac{1}{3}\right| \left(1-\varkappa\right) \ {}_{0}d_{q}\varkappa = \begin{cases} \frac{-4+9q+9q^{2}-44q^{3}}{192(1+q)(1+q+q^{2})} & \text{if } 0<q<\frac{4}{9} \\ \frac{4-195q-195q^{2}+648q^{3}}{1728(1+q)(1+q+q^{2})} & \text{if } \frac{4}{9}\leq q\leq\frac{2}{3} \\ \frac{4-9q-9q^{2}+44q^{3}}{192(1+q)(1+q+q^{2})} & \text{if } \frac{2}{3}<q<1, \end{cases} \tag{3.14}$$

$$\int_{\frac{3}{4}}^{1} \left(1-q\varkappa\right) \ {}_{0}d_{q}\varkappa = \frac{4-3q}{16\left(1+q\right)}, \tag{3.15}$$

$$\int_{\frac{3}{4}}^{1} \left(1-q\varkappa\right)\varkappa \ {}_{0}d_{q}\varkappa = \frac{28-9q-9q^{2}}{64\left(1+q\right)\left(1+q+q^{2}\right)}, \tag{3.16}$$

$$\int_{\frac{3}{4}}^{1} \left(1-q\varkappa\right)\left(1-\varkappa\right) \ {}_{0}d_{q}\varkappa = \frac{-12+13q+13q^{2}-12q^{3}}{64\left(1+q\right)\left(1+q+q^{2}\right)}, \tag{3.17}$$

$$\mathcal{C} = \int_{\frac{1}{2}}^{\frac{3}{4}} \left|q\varkappa-\frac{1}{3}\right|^{p} \ {}_{0}d_{q}\varkappa = \begin{cases} \frac{1-q}{q(1-q^{p+1})}\left(\frac{(4-6q)^{p+1}-(4-9q)^{p+1}}{(12)^{p+1}}\right) & \text{if } 0<q<\frac{4}{9} \\ \frac{1-q}{q(1-q^{p+1})}\left(\frac{(4-6q)^{p+1}+(9q-4)^{p+1}}{(12)^{p+1}}\right) & \text{if } \frac{4}{9}\leq q\leq\frac{2}{3} \\ \frac{1-q}{q(1-q^{p+1})}\left(\frac{(9q-4)^{p+1}-(6q-4)^{p+1}}{(12)^{p+1}}\right) & \text{if } \frac{2}{3}<q<1, \end{cases} \tag{3.18}$$

and for $0 \leq \alpha < \beta \leq 1$, we have

$$\int_{\alpha}^{\beta} 1 \ {}_{0}d_{q}\varkappa = \beta - \alpha, \tag{3.19}$$

$$\int_{\alpha}^{\beta} \varkappa \ {}_{0}d_{q}\varkappa = \frac{\beta^2 - \alpha^2}{1+q}, \tag{3.20}$$

and

$$\int_{\alpha}^{\beta} (1-\varkappa) \ {}_{0}d_{q}\varkappa = \frac{(\beta-\alpha)(1-(\beta+\alpha)+q)}{1+q}. \tag{3.21}$$

4 Main results

Theorem 4.1. Consider a function $f : I = [a,b] \to \mathbb{R}$ that is q-differentiable on I and ${}_{a}D_{q}f$ is integrable on I with $0 < q < 1$. If the absolute value of ${}_{a}D_{q}f$ is convex, then it follows that:

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_{a}^{b} f(\varkappa) \ {}_{a}d_{q}\varkappa - \frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right) \right) \right| \\
&\leq (b-a)\left(\left(\mathcal{D}_3 + \frac{-28+73q+73q^2-28q^3}{192(1+q)(1+q+q^2)} \right) |{}_{a}D_{q}f(a)| + \left(\mathcal{D}_2 + \frac{36-7q-7q^2}{64(1+q)(1+q+q^2)} \right) |{}_{a}D_{q}f(b)| \right),
\end{aligned}$$

where $\mathcal{D}_2$ and $\mathcal{D}_3$ are defined by (3.13) and (3.14) respectively.

Proof . By invoking Lemma 3.1, the principles of modulus, and the convexity of $|{}_{a}D_{q}f|$, we obtain:

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_{a}^{b} f(\varkappa) \ {}_{a}d_{q}\varkappa - \frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right) \right) \right| \\
&\leq (b-a)\left(\int_{0}^{\frac{1}{4}} q\varkappa \left|{}_{a}D_{q}f((1-\varkappa)a+\varkappa b)\right| \ {}_{0}d_{q}\varkappa + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varkappa - \frac{2}{3} \right| \left|{}_{a}D_{q}f((1-\varkappa)a+\varkappa b)\right| \ {}_{0}d_{q}\varkappa \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varkappa - \frac{1}{3} \right| \left|{}_{a}D_{q}f((1-\varkappa)a+\varkappa b)\right| \ {}_{0}d_{q}\varkappa + \int_{\frac{3}{4}}^{1} (1-q\varkappa) \left|{}_{a}D_{q}f((1-\varkappa)a+\varkappa b)\right| \ {}_{0}d_{q}\varkappa \right) \\
&\leq (b-a)\left(\int_{0}^{\frac{1}{4}} q\varkappa \left((1-\varkappa)|{}_{a}D_{q}f(a)| + \varkappa|{}_{a}D_{q}f(b)|\right) \ {}_{0}d_{q}\varkappa + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varkappa - \frac{2}{3} \right| \left((1-\varkappa)|{}_{a}D_{q}f(a)| + \varkappa|{}_{a}D_{q}f(b)|\right) \ {}_{0}d_{q}\varkappa \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| q\varkappa - \frac{1}{3} \right| \left((1-\varkappa)|{}_{a}D_{q}f(a)| + \varkappa|{}_{a}D_{q}f(b)|\right) \ {}_{0}d_{q}\varkappa + \int_{\frac{3}{4}}^{1} (1-q\varkappa)\left((1-\varkappa)|{}_{a}D_{q}f(a)| + \varkappa|{}_{a}D_{q}f(b)|\right) \ {}_{0}d_{q}\varkappa \right) \\
&= (b-a)\left(|{}_{a}D_{q}f(a)| \int_{0}^{\frac{1}{4}} q\varkappa(1-\varkappa) \ {}_{0}d_{q}\varkappa + |{}_{a}D_{q}f(b)| \int_{0}^{\frac{1}{4}} q\varkappa^2 \ {}_{0}d_{q}\varkappa \right. \\
&\quad + |{}_{a}D_{q}f(a)| \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varkappa - \frac{2}{3} \right| (1-\varkappa) \ {}_{0}d_{q}\varkappa + |{}_{a}D_{q}f(b)| \int_{\frac{1}{4}}^{\frac{1}{2}} \left| q\varkappa - \frac{2}{3} \right| \varkappa \ {}_{0}d_{q}\varkappa
\end{aligned}$$

$$+\left|{}_aD_qf(a)\right|\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|(1-\varkappa)\ {}_0d_q\varkappa+\left|{}_aD_qf(b)\right|\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|\varkappa\ {}_0d_q\varkappa$$

$$\left.+\left|{}_aD_qf(a)\right|\int_{\frac34}^{1}(1-q\varkappa)(1-\varkappa)\ {}_0d_q\varkappa+\left|{}_aD_qf(b)\right|\int_{\frac34}^{1}(1-q\varkappa)\varkappa\ {}_0d_q\varkappa\right)$$

$$=(b-a)\left(\left|{}_aD_qf(a)\right|\left(\int_{0}^{\frac14}q\varkappa(1-\varkappa)\ {}_0d_q\varkappa+\int_{\frac14}^{\frac12}\left|q\varkappa-\frac23\right|(1-\varkappa)\ {}_0d_q\varkappa\right.\right.$$

$$\left.+\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|(1-\varkappa)\ {}_0d_q\varkappa+\int_{\frac34}^{1}(1-q\varkappa)(1-\varkappa)\ {}_0d_q\varkappa\right)$$

$$\left.\left.+\left|{}_aD_qf(b)\right|\left(\int_{0}^{\frac14}q\varkappa^2\ {}_0d_q\varkappa+\int_{\frac14}^{\frac12}\left|q\varkappa-\frac23\right|\varkappa\ {}_0d_q\varkappa+\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|\varkappa\ {}_0d_q\varkappa+\int_{\frac34}^{1}(1-q\varkappa)\varkappa\ {}_0d_q\varkappa\right)\right)\right)$$

$$=(b-a)\left(\left(\frac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}+\frac{8+25q+25q^2-4q^3}{192(1+q)(1+q+q^2)}+\mathcal{D}_3+\frac{-12+13q+13q^2-12q^3}{64(1+q)(1+q+q^2)}\right)\left|{}_aD_qf(a)\right|\right.$$

$$\left.+\left(\frac{q}{64(1+q+q^2)}+\frac{8+q+q^2}{64(1+q)(1+q+q^2)}+\mathcal{D}_2+\frac{28-9q-9q^2}{64(1+q)(1+q+q^2)}\right)\left|{}_aD_qf(b)\right|\right)$$

$$=(b-a)\left(\left(\mathcal{D}_3+\frac{-28+73q+73q^2-28q^3}{192(1+q)(1+q+q^2)}\right)\left|{}_aD_qf(a)\right|+\left(\mathcal{D}_2+\frac{36-7q-7q^2}{64(1+q)(1+q+q^2)}\right)\left|{}_aD_qf(b)\right|\right),$$

The above results have been derived using equations (3.8) to (3.19). This concludes the proof. □

Corollary 4.2. Let $q\to1^-$. Then Theorem 4.1 gives

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac13\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|\le\frac{5(b-a)}{48}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Theorem 4.3. Consider a function $f:I=[a,b]\to\mathbb{R}$ that is q-differentiable on I and ${}_aD_qf$ is integrable on I with $0<q<1$. If $\left|{}_aD_qf\right|^\zeta$ is convex, with $\zeta>1$ and $\frac1p+\frac1\zeta=1$, then it follows that:

$$\left|\frac{1}{b-a}\int_a^b f(\varkappa)\ {}_ad_q\varkappa-\frac13\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$

$$\le(b-a)\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac1p}\left(\left(\frac{1}{4^{p+1}}\right)^{\frac1p}\left(\frac{3+4q}{16(1+q)}\left|{}_aD_qf(a)\right|^\zeta+\frac{1}{16(1+q)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac1\zeta}\right.$$

$$+\left(\frac{(8-3q)^{p+1}-(8-6q)^{p+1}}{(12)^{p+1}}\right)^{\frac1p}\left(\frac{1+4q}{16(1+q)}\left|{}_aD_qf(a)\right|^\zeta+\frac{3}{16(1+q)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac1\zeta}$$

$$+(\mathcal{W})^{\frac1p}\left(\left(\frac{4q-1}{16(1+q)}\left|{}_aD_qf(a)\right|^\zeta+\frac{5}{16(1+q)}\left|{}_aD_qf(b)\right|^\zeta\right)\right)^{\frac1\zeta}$$

$$\left.+\left(\frac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac1p}\left(\frac{4q-3}{16(1+q)}\left|{}_aD_qf(a)\right|^\zeta+\frac{7}{16(1+q)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac1\zeta}\right),$$

where $\mathcal{C}=\frac{1-q}{q(1-q^{p+1})}\mathcal{W}$ with $\mathcal{C}$ defined as in (3.18)

Proof . By invoking Lemma 3.1, the principles of modulus, Hölder's inequality, and the convexity of $|{}_aD_qf|^\zeta$, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(\varkappa)\ {}_ad_q\varkappa-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|\\
&\le(b-a)\left(\left(\int_0^{\frac14}(q\varkappa)^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_0^{\frac14}|{}_aD_qf((1-\varkappa)a+\varkappa b)|^\zeta\ {}_0d_q\varkappa\right)^{\frac1\zeta}\right.\\
&\quad+\left(\int_{\frac14}^{\frac12}\left|q\varkappa-\frac23\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac14}^{\frac12}|{}_aD_qf((1-\varkappa)a+\varkappa b)|^\zeta\ {}_0d_q\varkappa\right)^{\frac1\zeta}\\
&\quad+\left(\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac12}^{\frac34}|{}_aD_qf((1-\varkappa)a+\varkappa b)|^\zeta\ {}_0d_q\varkappa\right)^{\frac1\zeta}\\
&\quad\left.+\left(\int_{\frac34}^{1}(1-q\varkappa)^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac34}^{1}|{}_aD_qf((1-\varkappa)a+\varkappa b)|^\zeta\ {}_0d_q\varkappa\right)^{\frac1\zeta}\right)\\
&\le(b-a)\left(\left(\int_0^{\frac14}(q\varkappa)^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_0^{\frac14}\left((1-\varkappa)|{}_aD_qf(a)|^\zeta+\varkappa|{}_aD_qf(b)|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac1\zeta}\right.\\
&\quad+\left(\int_{\frac14}^{\frac12}\left|q\varkappa-\frac23\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac14}^{\frac12}\left((1-\varkappa)|{}_aD_qf(a)|^\zeta+\varkappa|{}_aD_qf(b)|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac1\zeta}\\
&\quad+\left(\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac12}^{\frac34}\left((1-\varkappa)|{}_aD_qf(a)|^\zeta+\varkappa|{}_aD_qf(b)|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac1\zeta}\\
&\quad\left.+\left(\int_{\frac34}^{1}(1-q\varkappa)^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac34}^{1}\left((1-\varkappa)|{}_aD_qf(a)|^\zeta+\varkappa|{}_aD_qf(b)|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac1\zeta}\right)\\
&\le(b-a)\left(\left(\int_0^{\frac14}(q\varkappa)^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_0^{\frac14}(1-\varkappa)\ {}_0d_q\varkappa|{}_aD_qf(a)|^\zeta+\int_0^{\frac14}\varkappa\ {}_0d_q\varkappa|{}_aD_qf(b)|^\zeta\right)^{\frac1\zeta}\right.\\
&\quad+\left(\int_{\frac14}^{\frac12}\left|q\varkappa-\frac23\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\int_{\frac14}^{\frac12}(1-\varkappa)\ {}_0d_q\varkappa|{}_aD_qf(a)|^\zeta+\int_{\frac14}^{\frac12}\varkappa\ {}_0d_q\varkappa|{}_aD_qf(b)|^\zeta\right)^{\frac1\zeta}\\
&\quad+\left(\int_{\frac12}^{\frac34}\left|q\varkappa-\frac13\right|^p\ {}_0d_q\varkappa\right)^{\frac1p}\left(\left(\int_{\frac12}^{\frac34}(1-\varkappa)\ {}_0d_q\varkappa|{}_aD_qf(a)|^\zeta+\int_{\frac12}^{\frac34}\varkappa\ {}_0d_q\varkappa|{}_aD_qf(b)|^\zeta\right)\right)^{\frac1\zeta}
\end{aligned}$$

$$+\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)^{p}\ {}_{0}d_{q}\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}(1-\varkappa)\ {}_{0}d_{q}\varkappa\left|{}_{a}D_{q}f\left(a\right)\right|^{\zeta}+\int_{\frac{3}{4}}^{1}\varkappa\ {}_{0}d_{q}\varkappa\left|{}_{a}D_{q}f\left(b\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\Bigg)$$

$$=(b-a)\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{1}{4^{p+1}}\right)^{\frac{1}{p}}\left(\frac{3+4q}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(a\right)\right|^{\zeta}+\frac{1}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(b\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right.$$

$$+\left(\frac{(8-3q)^{p+1}-(8-6q)^{p+1}}{(12)^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1+4q}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(a\right)\right|^{\zeta}+\frac{3}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(b\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$+(\mathcal{W})^{\frac{1}{p}}\left(\left(\frac{4q-1}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(a\right)\right|^{\zeta}+\frac{5}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(b\right)\right|^{\zeta}\right)\right)^{\frac{1}{\zeta}}$$

$$\left.+\left(\frac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac{1}{p}}\left(\frac{4q-3}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(a\right)\right|^{\zeta}+\frac{7}{16\left(1+q\right)}\left|{}_{a}D_{q}f\left(b\right)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right),$$

where $\mathcal{C}=\frac{1-q}{q(1-q^{p+1})}\mathcal{W}$, and we have used

$$\int_{0}^{\frac{1}{4}}(q\varkappa)^{p}\ {}_{0}d_{q}\varkappa=\frac{1-q}{4^{p+1}\left(1-q^{p+1}\right)},$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|^{p}\ {}_{0}d_{q}\varkappa=\frac{1-q}{q\left(1-q^{p+1}\right)}\left(\frac{(8-3q)^{p+1}-(8-6q)^{p+1}}{(12)^{p+1}}\right),$$

$$\int_{\frac{3}{4}}^{1}(1-q\varkappa)^{p}\ {}_{0}d_{q}\varkappa=\frac{1-q}{q\left(1-q^{p+1}\right)}\left(\frac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)$$

and (3.8)-(3.21). This concludes the proof. □

Corollary 4.4. Let $q\to 1^{-}$. Then Theorem 4.3 gives

$$\left|\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$

$$\leq\frac{b-a}{16\left(1+p\right)^{\frac{1}{p}}}\left(\left(\left(\frac{7\left|f'\left(a\right)\right|^{\zeta}+\left|f'\left(b\right)\right|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}+\left(\frac{\left|f'\left(a\right)\right|^{\zeta}+7\left|f'\left(b\right)\right|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}\right.\right.$$

$$\left.\left.+\left(\frac{5^{p+1}-2^{p+1}}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5\left|f'\left(a\right)\right|^{\zeta}+3\left|f'\left(b\right)\right|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}+\left(\frac{3\left|f'\left(a\right)\right|^{\zeta}+5\left|f'\left(b\right)\right|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}\right)\right)\right).$$

Furthermore, by invoking the discrete power mean inequality, we deduce:

$$\left|\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$

$$\leq\frac{b-a}{8\left(1+p\right)^{\frac{1}{p}}}\left(1+\frac{1}{3}\left(\frac{5^{p+1}-2^{p+1}}{3}\right)^{\frac{1}{p}}\right)\left(\frac{\left|f'\left(a\right)\right|^{\zeta}+\left|f'\left(b\right)\right|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

Theorem 4.5. Given a function $f:I=[a,b]\to\mathbb{R}$ that is q-differentiable on I and with an integrable ${}_{a}D_{q}f$ for $0<q<1$, if the convexity of $\left|{}_{a}D_{q}f\right|^{\zeta}$ is satisfied for $\zeta\geq 1$, then we have:

$$\left|\frac{1}{b-a}\int_a^b f(\varkappa)\ {}_ad_q\varkappa-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$
$$\leq(b-a)\left(\left(\frac{q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^\zeta+\frac{q}{64(1+q+q^2)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac{1}{\zeta}}\right.$$
$$+\left(\frac{8-q}{48(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{8+25q+25q^2-4q^3}{192(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^\zeta+\frac{8+q+q^2}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac{1}{\zeta}}$$
$$+(\mathcal{D}_1)^{1-\frac{1}{\zeta}}\left(\mathcal{D}_3\left|{}_aD_qf(a)\right|^\zeta+\mathcal{D}_2\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac{1}{\zeta}}$$
$$\left.+\left(\frac{4-3q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{-12+13q+13q^2-12q^3}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^\zeta+\frac{28-9q-9q^2}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^\zeta\right)^{\frac{1}{\zeta}}\right),$$

where $\mathcal{D}_1,\mathcal{D}_2$ and $\mathcal{D}_3$ are given by (3.12), (3.13) and (3.14) respectively.

Proof . By invoking Lemma 3.1, the principles of modulus, the convexity of $|{}_aD_qf|^\zeta$, and power mean inequality, we get

$$\left|\frac{1}{b-a}\int_a^b f(\varkappa)\ {}_ad_q\varkappa-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$
$$\leq(b-a)\left(\left(\int_0^{\frac{1}{4}}q\varkappa\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_0^{\frac{1}{4}}q\varkappa\left|{}_aD_qf((1-\varkappa)a+\varkappa b)\right|^\zeta\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}\right.$$
$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\left|{}_aD_qf((1-\varkappa)a+\varkappa b)\right|^\zeta\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\left|{}_aD_qf((1-\varkappa)a+\varkappa b)\right|^\zeta\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)\left|{}_aD_qf((1-\varkappa)a+\varkappa b)\right|^\zeta\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}\right)$$
$$\leq(b-a)\left(\left(\int_0^{\frac{1}{4}}q\varkappa\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_0^{\frac{1}{4}}q\varkappa\left((1-\varkappa)\left|{}_aD_qf(a)\right|^\zeta+\varkappa\left|{}_aD_qf(b)\right|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}\right.$$
$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\ {}_0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\left((1-\varkappa)\left|{}_aD_qf(a)\right|^\zeta+\varkappa\left|{}_aD_qf(b)\right|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\ {}_0d_q\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\left((1-\varkappa)\left|{}_aD_qf(a)\right|^\zeta+\varkappa\left|{}_aD_qf(b)\right|^\zeta\right)\ {}_0d_q\varkappa\right)^{\frac{1}{\zeta}}$$

$$+\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)\ _0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)\left((1-\varkappa)\left|{}_aD_qf(a)\right|^{\zeta}+\varkappa\left|{}_aD_qf(b)\right|^{\zeta}\right)\ _0d_q\varkappa\right)^{\frac{1}{\zeta}}\Bigg)$$

$$=(b-a)\left(\left(\int_{0}^{\frac{1}{4}}q\varkappa\ _0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\left|{}_aD_qf(a)\right|^{\zeta}\int_{0}^{\frac{1}{4}}q\varkappa(1-\varkappa)\ _0d_q\varkappa+\left|{}_aD_qf(b)\right|^{\zeta}\int_{0}^{\frac{1}{4}}q\varkappa^2\ _0d_q\varkappa\right)^{\frac{1}{\zeta}}\right.$$

$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\ _0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\left|{}_aD_qf(a)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|(1-\varkappa)\ _0d_q\varkappa+\left|{}_aD_qf(b)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|q\varkappa-\frac{2}{3}\right|\varkappa\ _0d_q\varkappa\right)^{\frac{1}{\zeta}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\ _0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\left|{}_aD_qf(a)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|(1-\varkappa)\ _0d_q\varkappa+\left|{}_aD_qf(b)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|q\varkappa-\frac{1}{3}\right|\varkappa\ _0d_q\varkappa\right)^{\frac{1}{\zeta}}$$

$$\left.+\left(\int_{\frac{3}{4}}^{1}(1-q\varkappa)\ _0d_q\varkappa\right)^{1-\frac{1}{\zeta}}\left(\left|{}_aD_qf(a)\right|^{\zeta}\int_{\frac{3}{4}}^{1}(1-q\varkappa)(1-\varkappa)\ _0d_q\varkappa+\left|{}_aD_qf(b)\right|^{\zeta}\int_{\frac{3}{4}}^{1}(1-q\varkappa)\varkappa\ _0d_q\varkappa\right)^{\frac{1}{\zeta}}\right)$$

$$=(b-a)\left(\left(\frac{q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{q}{64(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right.$$

$$+\left(\frac{8-q}{48(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{8+25q+25q^2-4q^3}{192(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{8+q+q^2}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$+(\mathcal{D}_1)^{1-\frac{1}{\zeta}}\left(\mathcal{D}_3\left|{}_aD_qf(a)\right|^{\zeta}+\mathcal{D}_2\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$\left.+\left(\frac{4-3q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{-12+13q+13q^2-12q^3}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{28-9q-9q^2}{64(1+q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right),$$

where we have used (3.6)-(3.17). This concludes the proof. □

Corollary 4.6. Let $q\to 1^-$. Then Theorem 4.5 gives

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|$$
$$\leq(b-a)\left(\frac{1}{32}\left(\left(\frac{5\left|f'(a)\right|^{\zeta}+\left|f'(b)\right|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}+\left(\frac{\left|f'(a)\right|^{\zeta}+5\left|f'(b)\right|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}\right)\right.$$
$$\left.+\frac{7}{96}\left(\left(\frac{9\left|f'(a)\right|^{\zeta}+5\left|f'(b)\right|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}+\left(\frac{5\left|f'(a)\right|^{\zeta}+9\left|f'(b)\right|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}\right)\right).$$

Furthermore, by invoking the discrete power mean inequality, we deduce:

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)\right|\leq\frac{5(b-a)}{24}\left(\frac{\left|f'(a)\right|^{\zeta}+\left|f'(b)\right|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

5 Applications

The dual Simpson quadrature rule

Consider the quadrature formula that utilizes the partition Υ of points $a = u_0 < u_1 < ... < u_n = b$ in the interval $[a, b]$,

$$\int_a^b f(u)\,du = \lambda(f, \Upsilon) + R(f, \Upsilon),$$

where

$$\lambda(f, \Upsilon) = \sum_{i=0}^{n-1} \frac{u_{i+1} - u_i}{3} \left(2f\left(\frac{3u_i + u_{i+1}}{4}\right) - f\left(\frac{u_i + u_{i+1}}{2}\right) + 2f\left(\frac{u_i + 3u_{i+1}}{4}\right)\right),$$

and $R(f, \Upsilon)$ represents the associated error.

Proposition 5.1. Consider a differentiable function $f : [a, b] \to \mathbb{R}$, with $0 \le a < b$ and $f' \in L^1[a, b]$. If for $\zeta \ge 1$,, $|f'|^\zeta$ is convex, then we have

$$|R(f, \Upsilon)| \le \sum_{i=0}^{n-1} \frac{5(u_{i+1} - u_i)^2}{24} \left(\frac{|f'(u_i)|^\zeta + |f'(u_{i+1})|^\zeta}{2}\right)^{\frac{1}{\zeta}}.$$

Proof . By using Corollary 4.6 on the subintervals $[u_i, u_{i+1}]$ of the partition Υ, $(i = 0, 1, ..., n - 1)$, we get

$$\left|\frac{1}{u_{i+1} - u_i} \int_{u_i}^{u_{i+1}} f(u)\,du - \frac{1}{3}\left(2f\left(\frac{3u_i + u_{i+1}}{4}\right) - f\left(\frac{u_i + u_{i+1}}{2}\right) + 2f\left(\frac{u_i + 3u_{i+1}}{4}\right)\right)\right|$$
$$\le \frac{5(u_{i+1} - u_i)}{24} \left(\frac{|f'(u_i)|^\zeta + |f'(u_{i+1})|^\zeta}{2}\right)^{\frac{1}{\zeta}}.$$

Multiplying the above inequality by $(u_{i+1} - u_i)$, summing the inequalities for $i = 0, 1, ..., n - 1$ and then using the triangular inequality, we achieve the intended outcome. □

Application to special means

Consider the following means for arbitrary real numbers a, b:

The Arithmetic mean: $A(a, b) = \frac{a+b}{2}$.

The p-Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \ne b$ and $p \in \mathbb{R} \setminus \{0, -1\}$.

Proposition 5.2. Let $\frac{2}{3} < q < 1$ and $a, b \in \mathbb{R}$ $(0 < a < b)$, then we have for $n \ge 2$

$$\left|A^n(a, b) + \frac{3(n+1)(1-q)}{1 - q^{n+1}} L_n^n(a, b) - 4A\left(\left(\frac{3a+b}{4}\right)^n, \left(\frac{a+3b}{4}\right)^n\right)\right|$$
$$\le \frac{b-a}{64}\left(\frac{-24 + 64q + 64q^2 + 16q^3}{(1+q)(1+q+q^2)}\left|na^{n-1}\right| + \frac{88 + 16q + 16q^2}{(1+q)(1+q+q^2)}\left|\frac{b^n - (q(b-a)+a)^n}{(1-q)(b-a)}\right|\right).$$

Proof . The assumption arises from the application of Theorem 4.1 to the function $f(u) = u^n$. □

Proposition 5.3. Let $0 < q < \frac{4}{9}$ and $a, b \in \mathbb{R}$ $(0 < a < b)$, then we have for $n \ge 2$

$$\left|A^n(a, b) + \frac{3(n+1)(1-q)}{1 - q^{n+1}} L_n^n(a, b) - 4A\left(\left(\frac{3a+b}{4}\right)^n, \left(\frac{a+3b}{4}\right)^n\right)\right|$$
$$\le \frac{3(b-a)}{16}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{3+4q}{4(1+q)}\left|{}_aD_q f(a)\right|^\zeta + \frac{1}{4(1+q)}\left|{}_aD_q f(b)\right|^\zeta\right)^{\frac{1}{\zeta}}\right.$$

$$+\frac{1}{3}\left(\frac{(8-3q)^{p+1}-(8-6q)^{p+1}}{3}\right)^{\frac{1}{p}}\left(\frac{1+4q}{4(1+q)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{3}{4(1+q)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$
$$+\frac{1}{3}\left(\frac{(4-6q)^{p+1}-(4-9q)^{p+1}}{3}\right)^{\frac{1}{p}}\left(\left(\frac{4q-1}{4(1+q)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{5}{4(1+q)}\left|{}_aD_qf(b)\right|^{\zeta}\right)\right)^{\frac{1}{\zeta}}$$
$$+\left((4-3q)^{p+1}-(4-4q)^{p+1}\right)^{\frac{1}{p}}\left(\frac{4q-3}{4(1+q)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{7}{4(1+q)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\Bigg).$$

Proof . The assumption arises from the application of Theorem 4.3 to the function $f(u)=u^n$. □

Proposition 5.4. Let $\frac{4}{9}\leq q\leq\frac{2}{3}$ and $a,b\in\mathbb{R}$ $(0<a<b)$, then we have for $n\geq 2$

$$\left|A^n(a,b)+\frac{3(n+1)(1-q)}{1-q^{n+1}}L_n^n(a,b)-4A\left(\left(\frac{3a+b}{4}\right)^n,\left(\frac{a+3b}{4}\right)^n\right)\right|$$
$$\leq\frac{3(b-a)}{16(1+q)}\left(q\left(\frac{3+3q+4q^2}{4(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{1+q}{4(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right.$$
$$+\frac{8-q}{3}\left(\frac{8+25q+25q^2-4q^3}{4(8-q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{24+3q+3q^2}{4(8-q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$
$$+\frac{-28+57q}{9}\left(\frac{4-195q-195q^2+648q^3}{12(-28+57q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{-340+477q+477q^2}{12(-28+57q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$
$$\left.+(4-3q)\left(\frac{-12+13q+13q^2-12q^3}{4(4-3q)(1+q+q^2)}\left|{}_aD_qf(a)\right|^{\zeta}+\frac{28-9q-9q^2}{4(4-3q)(1+q+q^2)}\left|{}_aD_qf(b)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).$$

Proof . The assumption arises from the application of Theorem 4.5 to the function $f(u)=u^n$. □

6 Conclusion

In this work, we established quantum dual Simpson type integral inequalities for the class of convex q-derivative functions using a new identity. We also presented applications of the obtained results. The quantum integral inequalities make a significant contribution to the understanding of quantum number theory, quantum information theory, and quantum algorithm theory. The results presented in this paper can be used for future research in these fields. This work also opens up new horizons for research in the area of quantum inequalities for other types of generalized convexity.

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