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Differential subordinations and superordinations results of analytic univalent functions using the El-Deeb-Lupas operator

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Abstract

In the present paper, we discuss some differential subordinations and superordinations results for a subclass of analytic univalent functions in the open unit disk U using El-Deeb –Lupa's operator $\mathcal{H}^n_{\lambda,\tau}$. Also, we study some sandwich theorems.

Keywords: analytic function, subordination, superordination, sandwich, El-Deeb-Lupas operator
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1 Introduction

Let $S = S(U)$ be the class of all functions that are analytic in U where $U = \{z \in C : |z| < 1\}$ is the open unit disk. Let $S[a, n]$ be a subclass of the functions $f \in S$, which is given by

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \qquad n \in N, a \in C. \tag{1.1}$$

We also assume $\acute{S} \subset S$ where $\acute{S}$ is said to be the subclass of analytic and univalent functions in U, of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Now, we assume that $f, g \in S$, so that the function f is subordinate to function g, or the function g is superordinate to the function f, if there exists the Schwarz function $\mathfrak{W}$ such that $f(z) = g(\mathfrak{W}(z))$, where $\mathfrak{W}(z)$ is analytic function in U with $|\mathfrak{W}(z)| < 1$ and $\mathfrak{W}(0) = 0$, $z \in U$,then one can say that $f \prec g$ or $f(z) \prec g(z)$ for $z \in U$ [13].

In addition, if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $(U) \subset g(U)$ [13, 17, 18].

Definition 1.1. [17] Let $\varphi : C^3 \times U \to C$ and let $\hbar(z)$ is univalent in U. If $\mathcal{P}(z)$ is analytic function in U and fulfills the second-order differential subordination:

$$\varphi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec \hbar(z), \tag{1.3}$$

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then $\mathcal{P}(z)$ is said to be a solution of the differential subordination (1.3), and the univalent function $\mathfrak{U}(z)$ say it a dominant of the solution of differential subordination (1.3), or more simply a dominant, if $\mathcal{P}(z) \prec \mathfrak{U}(z)$ for each $\mathcal{P}(z)$ satisfying (1.3). A dominant function $\widetilde{\mathfrak{U}}(z)$ that satisfies $\widetilde{\mathfrak{U}}(z) \prec \mathfrak{U}(z)$ for each dominant $\mathfrak{U}(z)$ of (1.3) is called the best dominant of (1.3).

Definition 1.2. [17] Let $\mathcal{P}, \hbar \in \acute{S}$ and $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)$ are univalent functions in U and if p satisfies the second-orde differential superordination:

$$\hbar \prec (z)\varphi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z), \tag{1.4}$$

then p is said to be a differential superordination solution (1.4). An analytic function $\mathfrak{U}(z)$ which is known as a subordinant of the solutions of differential superordination (1.4), or more simply a subordinant if $\mathcal{P} \prec \mathfrak{U}$ for each the functions $\mathcal{P}$ satisfying (1.4). If $\widetilde{\mathfrak{U}}$ is univalent subordinant and that satisfy $\mathfrak{U} \prec \widetilde{\mathfrak{U}}$ for each the subordinats $\mathfrak{U}$ of (1.4), then is the best subordinat.

Many authors [1, 2, 3, 10, 17, 20] obtained the necessary and sufficient conditions on the functions $\hbar, \mathcal{P}$ and φ where by the following implication is true

$$\hbar \prec (z)\varphi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z),$$

then

$$\mathfrak{U}(z) \prec \mathcal{P}(z) \tag{1.5}$$

Utilizing the outcomes Look [4, 5, 6, 7, 11, 12, 15, 16, 18, 19, 21] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$\mathfrak{U}_1(z) \prec \frac{zf'(z)}{f(z)} \prec \mathfrak{U}_2(z)$$

where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are given univalent functions in U with $\mathfrak{U}_1(0) = \mathfrak{U}_2(0) = 1$. Also a number of authors Look [2, 4, 6, 7, 8, 9] they found some differential subordination and superordination results and sandwich theorems.

Let $f \in \acute{S}$, El-Deeb and Lupas [14] defined the following generalized integral operator:

$$\mathcal{H}^n_{\lambda,\tau} f(z) = \frac{1+\lambda}{z^\lambda} \int_0^z t^{\lambda-1} \mathcal{H}^{n-1}_{\lambda,\tau} f(t) dt, \tag{1.6}$$

where $(\tau > 0, \lambda \geq 0, n \in N_0 = N_0 \cup \{0\})$.

For $f(z) \in \acute{S}$ given by (1.2), we have

$$\mathcal{H}^n_{\lambda,\tau} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+\lambda}{k+\lambda}\right)^n \frac{\tau^{k-1}}{(k-1)!} e^{-\tau} a_n z^n. \tag{1.7}$$

From (1.7), we note that

$$z(\mathcal{H}^n_{\lambda,\tau} f(z))' = (\lambda+1)\mathcal{H}^{n-1}_{\lambda,\tau} f(z) - \lambda \mathcal{H}^n_{\lambda,\tau} f(z) \tag{1.8}$$

The specific goal of this research to find sufficient conditions for certain normalized analytic function f to satisfy:

$$\mathfrak{U}_1(z) \prec \left[\frac{\mathcal{H}^n_{\lambda,\tau} f(z)}{z}\right]^\gamma \partial \prec \mathfrak{U}_2(z),$$

and

$$\mathfrak{U}_1(z) \prec \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau} f(z)}{\mathcal{H}^n_{\lambda,\tau} f(z)}\right]^\gamma \prec \mathfrak{U}_2(z),$$

wherever $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are provided univalent functions in U with $\mathfrak{U}_1(0) = \mathfrak{U}_2(0) = 1$.

In this paper, we will derive Some sandwich theorems with the operator $\mathcal{H}^n_{\lambda,\tau} f(z)$.

2 Preliminaries

We need the following lemmas and definitions, to prove our results.

Definition 2.1. [17] Denote by Q the class of all functions q that are analytic and injective on $\overline{U}|E(\mathfrak{U})$, where $\overline{U} = U \cup \{z \in \partial U\}$, and

$$E(\mathfrak{U}) = \{\varepsilon \in \partial U : \mathfrak{U}(z) = \infty\}$$

and are such that $\mathfrak{U}'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U \backslash E(\mathfrak{U})$. Further, let the subclass of Q as to which $\mathfrak{U}(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1 = \{\mathfrak{U} \in Q : \mathfrak{U}(0) = 1\}$.

Lemma 2.2. [18] Suppose that the function $\mathfrak{U}$ is a convex univalent in U, let $\lambda \in C, \mathcal{B} \in C|\{0\}$ and Suppose that

$$Re\left\{1 + \frac{z\mathfrak{U}''(z)}{\mathfrak{U}'(z)}\right\} > \left\{0, -Re\left(\frac{\lambda}{\mathcal{B}}\right)\right\}. \tag{2.1}$$

If $\mathcal{P}$ is analytic in U and

$$\lambda\mathcal{P}(z) + \mathcal{B}z\mathcal{P}'(z) \prec \lambda\mathfrak{U}(z) + \mathcal{B}z\mathfrak{U}'(z) \tag{2.2}$$

then $\mathcal{P} \prec \mathfrak{U}$ and $\mathfrak{U}$ is the best dominant of (2.2).

Lemma 2.3. [5] Let $\mathfrak{U}$ be univalent in U. and let φ and θ be analytic in the domain D containing $\mathfrak{U}(U)$ with $\varphi(\mathfrak{W}) \neq 0$, when $\mathfrak{W} \in \mathfrak{U}(U)$. Set $Q(z) = z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z))$ and $\hbar(z) = \theta(\mathfrak{U}(z)) + Q(z)$. Suppose that

a. $Q(z)$ is starlike univalent in U.

b. $Re\left\{\frac{\hbar'(z)}{Q(z)}\right\} > 0,\ z \in U$.

If $\mathcal{P}$ is analytic in U, with $\mathcal{P}(0) = \mathfrak{U}(0),\ \mathcal{P}(U) \subseteq D$ and

$$\theta(\mathcal{P}(z)) + z\mathcal{P}'(z)\varphi(\mathcal{P}(z)) \prec \theta(\mathfrak{U}(z)) + z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z)), \tag{2.3}$$

then $\mathcal{P} \prec \mathfrak{U}$ and $\mathfrak{U}$ is the best dominant of (2.3).

Lemma 2.4. [18] Suppose that $\mathfrak{U}$ is convex univalent in U and let $\mathcal{B} \in C$, that $Re(\mathcal{B}) > 0$. If $\mathcal{P} \in \mathcal{H}[\mathfrak{U}(0), 1] \cap Q$ and $\mathcal{P}(z) + \mathcal{B}z\mathcal{P}'(z)$ is univalent in U, then

$$\mathfrak{U}(z) + \mathcal{B}z\mathfrak{U}'(z) \prec \mathcal{P}(z) + \mathcal{B}z\mathcal{P}'(z), \tag{2.4}$$

then $\mathfrak{U} \prec \mathcal{P}$ and $mathfrakU$ is the best subordinant of (2.4).

Lemma 2.5. [18] Let $\mathfrak{U}(z)$ be a convex univalent function in the unit disk U and let φ andθ be analytic in the domain D containing $\mathfrak{U}(U)$. Suppose that:

a. $Re\left\{\frac{\theta'(\mathfrak{U}(z))}{\varphi(\mathfrak{U}(z))}\right\} > 0,\ z \in U$.

b. $Q(z) = z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z))$ is starlike univalent in U.

If $\mathcal{P} \in S[\mathfrak{U}(0), 1] \cap Q$, with $\mathcal{P}(U) \subset D, \theta(\mathcal{P}(z)) + z\mathcal{P}'(z)\varphi(\mathcal{P}(z))$ is univalent in U and

$$\theta(\mathfrak{U}(z)) + z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z)) \prec \theta(\mathcal{P}(z)) + z\mathcal{P}'(z)\varphi(\mathcal{P}(z)), \tag{2.5}$$

then $\mathfrak{U} \prec \mathcal{P}$ and q is the best subordinant of (2.5).

3 Differential Subordination Results

We present a few differential subordination results by using the El-Deeb-Lupas operator.

Theorem 3.1. Suppose that $\mathfrak{U}$ be a convex univalent function in U with $\mathfrak{U}(0) = 1, \gamma > 0, 0 \neq \varepsilon \in C$, and suppose that $\mathfrak{U}$ satisfies:

$$Re\left\{1 + \frac{z\mathfrak{U}''(z)}{\mathfrak{U}'(z)}\right\} > \left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\}. \tag{3.1}$$

If $f \in \acute{S}$ satisfies the subordination condition:

$$(\lambda+1)\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)} - 1\right) + \left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \mathfrak{U}(z) + \frac{\varepsilon}{\gamma}z\mathfrak{U}'(z), \tag{3.2}$$

then

$$\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \mathfrak{U}(z), \tag{3.3}$$

and $\mathfrak{U}$ is the best dominant of (3.2).

Proof . We shall define the function $\mathcal{P}$ by

$$\mathcal{P}(z) = \left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma, \tag{3.4}$$

then the function $\mathcal{P}(z)$ is analytic and $\mathcal{P}(0) = 1$, therefore, differentiating (3.4) with respect to (z) and using the identity (1.8), we obtain

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} = \gamma\left[\frac{z(\mathcal{H}^n_{\lambda,\tau}f(z))'}{\mathcal{H}^n_{\lambda,\tau}f(z)} - 1\right]. \tag{3.5}$$

Hence

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} = \gamma\left[(\lambda+1)\left(\frac{P^{\mu,\mathcal{B},l}_{\lambda,\lambda-1,\theta,K}f(z)}{P^{\mu,\mathcal{B},l}_{\lambda,\lambda,\theta,K}f(z)} - 1\right)\right].$$

Therefore,

$$\frac{z\mathcal{P}'(z)}{\gamma} = \left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma\left[(\lambda+1)\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)} - 1\right)\right].$$

The subordination (3.2) from the hypothesis becomes

$$\mathcal{P}(z) + \frac{\varepsilon}{\gamma}z\mathcal{P}'(z) \prec \mathfrak{U}(z) + \frac{\varepsilon}{\gamma}z\mathfrak{U}'(z).$$

An application of lemma 2.2 with $\mathcal{B} = \frac{\varepsilon}{\gamma}$ and $\varepsilon = 1$, we obtain (3.3). □

Putting $\mathfrak{U}(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. Let $\gamma > 0, 0 \neq \varepsilon \in C\backslash\{0\}$ and

$$Re\left\{1 + \frac{2z}{1-z}\right\} > \left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\}.$$

If $f \in \acute{S}$ satisfies the subordination condition:

$$(\lambda+1)\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)} - 1\right) + \left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \left(\frac{1 - z^2 + 2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right),$$

then

$$\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \left(\frac{1+z}{1-z}\right)$$

and $\mathfrak{U}(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.3. Let $\mathfrak{U}$ be a convex univalent function in U with $\mathfrak{U}(0) = 1, \mathfrak{U}'(z) \neq 0 (z \in U)$ and assume that $\mathfrak{U}$ satisfies:

$$Re\left\{1 + \frac{k}{\varepsilon}(\mathfrak{U}(z))^k + \frac{k-1}{\varepsilon}(\mathfrak{U}(z))^{k-1} - z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)} + z\frac{\mathfrak{U}''(z)}{\mathfrak{U}'(z)}\right\} > 0, \tag{3.6}$$

where $k \in C, \varepsilon \in C\backslash\{0\}$ and $z \in U$. Suppose that $z\frac{\mathfrak{U}'(z))}{\mathfrak{U}(z)}$ is starlike univalent in U. If $f \in \acute{S}$ satisfies

$$\Psi(n, \lambda, \tau, k, \varepsilon; z) \prec (1 + \mathfrak{U}(z))\mathfrak{U}(z)^{k-1} + \varepsilon z\frac{\mathfrak{U}'(z))}{\mathfrak{U}(z)}, \tag{3.7}$$

where

$$\Psi(n, \lambda, \tau, k, \varepsilon; z) = \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma k} + \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma(k-1)} + \varepsilon\gamma(\lambda+1)\left(\frac{\mathcal{H}^{n-2}_{\lambda,\tau}f(z)}{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)} - \frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right), \tag{3.8}$$

then

$$\left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma} \prec \mathfrak{U}(z), \tag{3.9}$$

and $\mathfrak{U}$ is the best dominant of (3.7).

Proof . Consider a function $\mathcal{P}$ by

$$\mathcal{P}(z) = \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma}. \tag{3.10}$$

Then the function $\mathcal{P}(z)$ is analytic in U and $\mathcal{P}(0) = 1$, differenitating (3.10), with respect to (z) and using the identity (1.8), we obtain

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} = \gamma\left[(\lambda+1)\left(\frac{\mathcal{H}^{n-2}_{\lambda,\tau}f(z)}{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)} - \frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right)\right].$$

By setting

$$\varphi(\mathfrak{W}) = \frac{\varepsilon}{\mathfrak{W}}, \mathfrak{W} \neq 0, \text{ and } \theta(\mathfrak{W}) = (1 + \mathfrak{W})\mathfrak{W}^{k-1}.$$

we see that $\theta(\mathfrak{W})$ is analytic in C and $\varphi(\mathfrak{W})$ is analytic in $C\backslash\{0\}$ and that $\varphi(\mathfrak{W}) \neq 0, \mathfrak{W} \in C|\{0\}$. Also, we obtain

$$Q(z) = z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z)) = \varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)},$$

and

$$\hbar(z) = \theta(\mathfrak{U}(z)) + Q(z) = (1 + \mathfrak{U}(z))\mathfrak{U}(z)^{k-1} + \varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)}.$$

It is obvious that $Q(z)$ is starlike univalent in U, we have

$$Re\left\{\frac{z\hbar'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{k}{\varepsilon}(\mathfrak{U}(z))^k + \frac{k-1}{\varepsilon}(\mathfrak{U}(z))^{k-1} - z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)} + z\frac{\mathfrak{U}''(z)}{\mathfrak{U}'(z)}\right\} > 0.$$

Using a simple calculation, we get

$$\Psi(n, \lambda, \tau, k, \varepsilon; z) = (1 + \mathcal{P}(z))(\mathcal{P}(z))^{k-1} + \varepsilon z\frac{\mathcal{P}'(z)}{\mathcal{P}(z)}, \tag{3.11}$$

where $\Psi(n, \lambda, \tau, k, \varepsilon; z)$ is given by (3.8).

From (3.7) and (3.11), we have

$$(1 + \mathcal{P}(z))(\mathcal{P}(z))^{k-1} + \varepsilon z\frac{\mathcal{P}'(z)}{\mathcal{P}(z)} \prec (1 + \mathfrak{U}(z))(\mathfrak{U}(z))^{k-1} + \varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)}. \tag{3.12}$$

Therefore, by Lemma 2.3, we get $\mathcal{P}(z) \prec \mathfrak{U}(z)$. By using (3.10), we get the result. □

make up $\mathfrak{U}(z) = \left(\frac{1+Az}{1+Bz}\right)$, $-1 \leq B < A \leq 1$ in Theorem 3.3, we obtain the following:

Corollary 3.4. Let $-1 \leq B < A \leq 1$ and

$$Re\left\{\frac{k}{\varepsilon}\left(\frac{1+Az}{1+Bz}\right)^{k}+\frac{k-1}{\varepsilon}\left(\frac{1+Az}{1+Bz}\right)^{k-1}+\frac{1+Bz(4+3Az)}{(1+Bz)(1+Az)}\right\}>0,$$

where $\varepsilon \in C\backslash\{0\}$ and $z \in U$, if $f \in \acute{S}$ satisfies:

$$\Psi(n,\lambda,\tau,k,\varepsilon;z) \prec \left[\left[1+\left(\frac{1+Az}{1+Bz}\right)\right]\left(\frac{1+Az}{1+Bz}\right)^{k-1}+\varepsilon z\frac{A-B}{(1+Az)(z+Bz)}\right],$$

and $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is given by (3.8),

then

$$\left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma} \prec \left(\frac{1+Az}{1+Bz}\right)$$

and $\mathfrak{U}(z)=\left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

4 Differential Superordination Results

Theorem 4.1. Let $\mathfrak{U}$ be a convex univalent function in U with $\mathfrak{U}(0)=1, \gamma>0$ and $Re\{\varepsilon\}>0$. Let $f \in \acute{S}$ satisfies:

$$\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma} \in S[\mathfrak{U}(0),1]\cap Q \tag{4.1}$$

and

$$(\lambda+1)\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}, \tag{4.2}$$

Be univalent in U. If

$$\mathfrak{U}(z)+\frac{\varepsilon}{\gamma}z\mathfrak{U}'(z) \prec (\lambda+1)\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}, \tag{4.3}$$

then

$$\mathfrak{U}(z) \prec \left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}, \tag{4.4}$$

and $\mathfrak{U}$ is the best subordinant of (4.3).

Proof . Define the function $\mathcal{P}$ by

$$\mathcal{P}(z)=\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}. \tag{4.5}$$

Differentiating (4.5) with respect to z, we get

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)}=\gamma\left[\frac{z(\mathcal{H}^{n}_{\lambda,\tau}f(z))'}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right]. \tag{4.6}$$

We using (1.8) with some simplification from (4.6), we get

$$(\lambda+1)\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}=\mathcal{P}(z)+\frac{\varepsilon}{\gamma}z\mathcal{P}'(z).$$

by using Lemma 2.4, we get the desired result. □

Putting $\mathfrak{U}(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem 4.1, we obtain the subsequent corollary:

Corollary 4.2. Let $\gamma > 0$ and $Re\{\varepsilon\} > 0$. If $f \in \acute{S}$ satisfies

$$\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma} \in S[\mathfrak{U}(0),1] \cap Q$$

and $(\lambda+1)\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}$ be univalent in U. If

$$\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right) \prec (\lambda+1)\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma}\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma},$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{\mathcal{H}^{n}_{\lambda,\tau}f(z)}{z}\right]^{\gamma},$$

and $\mathfrak{U}(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4.3. Let $\mathfrak{U}$ be a convex univalent function in U with $\mathfrak{U}(0) = 1, \mathfrak{U}'(0) \neq 0$ and Suppose that $\mathfrak{U}$ satisfies:

$$Re\left\{\frac{k}{\varepsilon}(\mathfrak{U}(z))^{k}\mathfrak{U}'(z)+\frac{k-1}{\varepsilon}(\mathfrak{U}(z))^{k-1}\mathfrak{U}'(z)\right\} > 0 \tag{4.7}$$

where $k \in C, \varepsilon \in C\backslash\{0\}$ and $z \in U$.

Let $z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)}$ is starlike univalent function in U. Let $f \in \acute{S}$ satisfies:

$$\left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma} \in S[\mathfrak{U}(0),1] \cap Q,$$

and $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is univalent function in U, where $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is given by (3.8). If

$$(1+\mathfrak{U}(z))(\mathfrak{U}(z))^{k-1}+\varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)} \prec \Psi(n,\lambda,\tau,k,\varepsilon;z), \tag{4.8}$$

then

$$\mathfrak{U}(z) \prec \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma}, \tag{4.9}$$

and $\mathfrak{U}$ is the best subordinant of (4.8).

Proof . Consider a function P by

$$P(z) = \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right]^{\gamma}. \tag{4.10}$$

Differentiating (4.10) with respect to z, we obtain

$$\frac{zP'(z)}{P(z)} = \gamma\left[(\lambda+1)\left(\frac{\mathcal{H}^{n-2}_{\lambda,\tau}f(z)}{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}-\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^{n}_{\lambda,\tau}f(z)}\right)\right].$$

By setting $\varphi(\mathfrak{W}) = \frac{\varepsilon}{\mathfrak{W}}, \mathfrak{W} \neq 0$, and $\theta(\mathfrak{W}) = (1+\mathfrak{W})\mathfrak{W}^{k-1}$.
we see that $\theta(\mathfrak{W})$ is analytic in C and $\varphi(\mathfrak{W})$ is analytic in $C\backslash\{0\}$ and that $\varphi(\mathfrak{W}) \neq 0, \mathfrak{W} \in C\backslash\{0\}$. Also, we get

$$Q(z) = z\mathfrak{U}'(z)\varphi(\mathfrak{U}(z)) = \varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)},$$

we see that $Q(z)$ is starlike univalent function in U,

$$Re\left\{\frac{\theta'(\mathfrak{U}(z))}{\varphi(\mathfrak{U}(z))}\right\} = Re\left\{\frac{k}{\varepsilon}(\mathfrak{U}(z))^k\mathfrak{U}'(z) + \frac{k-1}{\varepsilon}(\mathfrak{U}(z))^{k-1}\mathfrak{U}'(z)\right\} > 0.$$

Using a simple calculation, we obtain

$$\Psi(n,\lambda,\tau,k,\varepsilon;z) = (1+\mathcal{P}(z))(\mathcal{P}(z))^{k-1} + \varepsilon z\frac{\mathcal{P}'(z)}{\mathcal{P}(z)}, \tag{4.11}$$

where $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is given by (3.8).

We have from (4.8) and (4.11)

$$(1+\mathfrak{U}(z))(\mathfrak{U}(z))^{k-1} + \varepsilon z\frac{\mathfrak{U}'(z)}{\mathfrak{U}(z)} \prec (1+\mathcal{P}(z))(\mathcal{P}(z))^{k-1} + \varepsilon z\frac{\mathcal{P}'(z)}{\mathcal{P}(z)}. \tag{4.12}$$

Therefore, by Lemma 2.5, we get $\mathfrak{U}(z) \prec \mathcal{P}(z)$. □

5 Sandwich Results

Theorem 5.1. Let $\mathfrak{U}_1$ be a convex univalent function in U with $\mathfrak{U}_1(0) = 1, \gamma > 0$ and $Re\{\varepsilon\} > 0$ and let $\mathfrak{U}_2$ be univalent function in U, $\mathfrak{U}_2(0) = 1$ and satisfies (3.1). Let $f \in \acute{S}$ satisfies:

$$\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \in S[1,1]\cap Q$$

and $(\lambda+1)\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma$ be univalent in U. If

$$\mathfrak{U}_1(z) + \frac{\varepsilon}{\gamma}z\mathfrak{U}_1'(z) \prec (\lambda+1)\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma\left(\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)}-1\right)+\left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \mathfrak{U}_2(z) + \frac{\varepsilon}{\gamma}z\mathfrak{U}_2'(z),$$

then

$$\mathfrak{U}_1(z) \prec \left[\frac{\mathcal{H}^n_{\lambda,\tau}f(z)}{z}\right]^\gamma \prec \mathfrak{U}_2(z),$$

and $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are respectively the best subordinant and the best dominant.

Theorem 5.2. Let $\mathfrak{U}_1$ be a convex univalent function in U with $\mathfrak{U}_1(0) = 1$ and satisfies (4.7). Let $\mathfrak{U}_2$ be univalent function in U with $\mathfrak{U}_2(0) = 1$ and satisfies (3.6). Let $f \in \acute{S}$ satisfies:

$$\left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)}\right]^\gamma \in S[1,1]\cap Q,$$

and $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is univalent in U, where $\Psi(n,\lambda,\tau,k,\varepsilon;z)$ is given by (3.8). If

$$(1+\mathfrak{U}_1(z))(\mathfrak{U}_1(z))^{k-1} + \varepsilon z\frac{\mathfrak{U}_1'(z)}{\mathfrak{U}_1(z)} \prec \Psi(\gamma,\mu,\mathcal{B},l,\lambda,\theta,k,\tau,\varepsilon;z) \prec (1+\mathfrak{U}_2(z))(\mathfrak{U}_2(z))^{k-1} + \varepsilon z\frac{\mathfrak{U}_2'(z)}{\mathfrak{U}_2(z)},$$

then

$$\mathfrak{U}_1(z) \prec \left[\frac{\mathcal{H}^{n-1}_{\lambda,\tau}f(z)}{\mathcal{H}^n_{\lambda,\tau}f(z)}\right]^\gamma \prec \mathfrak{U}_2(z)$$

and $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are respectively the best subordinant and the best dominant.

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