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Starlikeness of an integral operator associated with Mittag-Leffler functions

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Abstract

In the present paper, we introduce a new integral operator involving with Mittag-Leffler function and the Salagean operator. Further, we obtain some sufficient conditions for this integral operator belonging to certain classes of starlike functions.

Keywords: Analytic function, Univalent function, Mittag-Leffler function, Starlike function, Salagean derivative, Integral operator
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1 Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we represent $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) which are also univalent in Δ.

In 1936, Robertson [22] (see also [24]) introduced two most important and widely used classes of univalent functions as follows:
A function $f(z) \in \mathcal{A}$ is said to be starlike of order ϵ if it satisfies the following analytic criteria

$$\Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \epsilon, \quad z \in \Delta, \quad \text{for some } \ \epsilon(0 \le \epsilon < 1).$$

Also, a function $f(z) \in \mathcal{A}$ is said to be convex of order ϵ if it satisfies the following analytic criteria

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \epsilon, \quad z \in \Delta, \text{for some} \quad \epsilon(0 \le \epsilon < 1).$$

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The classes of all starlike functions and convex functions of order ϵ are denoted by $\mathcal{S}^*(\epsilon)$ and $\mathcal{C}(\epsilon)$, respectively. For $\epsilon = 0$, these classes reduce to the classes $\mathcal{S}^*$ and $\mathcal{C}$, respectively. In 1983, Salagean [23] introduced an interesting derivative operator D^p known as Salagean derivative operator. Using this operator he generalized and unified the classes of starlike and convex functions, by investigating a new class $\mathcal{S}(p, \epsilon)$ consisting of functions f of the form (1.1) and satisfying the following analytic criteria

$$\Re\left\{\frac{D^{p+1}f(z)}{D^p f(z)}\right\} > \epsilon, \quad z \in \Delta, \quad p \in \mathbb{N} \cup \{0\}, \quad \text{for some } \epsilon (0 \leq \epsilon < 1).$$

It is worthy to note that for $p = 0$ and $p = 1$, the class $\mathcal{S}(p, \epsilon)$ reduce to the classes $\mathcal{S}^*(\epsilon)$ and $\mathcal{C}(\epsilon)$, respectively. The class $\mathcal{S}(p, \epsilon)$ was further studied by Kadioğlu [9]. By using Salagean operator several researchers introduced various subclasses of analytic and harmonic univalent functions. Recent work on Salagean operator may be find in [7, 8, 10]. Analogues to the class $\mathcal{S}(p, \epsilon)$, Porwal and Kumar [19] introduced a new class $\mathcal{N}(p, \omega)$ consisting of functions f of the form (1.1) and satisfying the following analytic criteria

$$\Re\left\{\frac{D^{p+1}f(z)}{D^p f(z)}\right\} < \omega, \quad z \in \Delta, , \quad p \in \mathbb{N} \cup \{0\}, \quad \text{for some } \omega \left(1 < \omega \leq \frac{2^p + 1}{2^{p-1} + 1}\right).$$

The applications of various special functions on integral operator is a current and interesting topic of research in geometric function theory. From time-to-time various integral operators associated with several special functions like Bessel functions, Mittag-Leffler functions, Dini functions, Struve function and Lommel functions are introduced and extensively studied by several researchers. Noteworthy, contribution in this direction may be found in [1, 5, 6, 11, 12, 15, 17, 18, 19, 20, 21, 26].

Now, we recall the definition of Mittag-Leffler function $E_\alpha(z)$ which was introduced by Mittag-Leffler[13] and defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \ \Re(\alpha) > 0).$$

In 1905, Wiman[27, 28] generalized the Mittag-Leffler function in $E_{\alpha,\beta}(z)$ by the relation

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $z, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$. It should be easy to see that the function $E_{\alpha,\beta}(z)$ defined by (1.2) is not in class $\mathcal{A}$. Thus, first we normalize the Mittag-Leffler function as follows

$$\begin{aligned} \mathbb{E}_{\alpha,\beta}(z) &= \Gamma(\beta) z \, E_{\alpha,\beta}(z) \\ \mathbb{E}_{\alpha,\beta}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \, z^n, \end{aligned} \tag{1.3}$$

where $z, \alpha, \beta \in \mathbb{C}$, $\beta \neq 0, -1, -2, \cdots, \Re(\alpha) > 0$. In the present work, we shall restrict our attention to the case for real-valued α, β and $z \in \Delta$. For specific values of α and β, the function $\mathbb{E}_{\alpha,\beta}(z)$ reduces to many well-known functions

$$\begin{aligned} \mathbb{E}_{2,1}(z) &= z \cosh \sqrt{z} \\ \mathbb{E}_{2,2}(z) &= \sqrt{z} \sinh \sqrt{z} \\ \mathbb{E}_{2,3}(z) &= 2[\cosh \sqrt{z} - 1] \quad \text{and} \\ \mathbb{E}_{2,4}(z) &= \frac{6[\sinh \sqrt{z} - \sqrt{z}]}{\sqrt{z}}. \end{aligned}$$

For further study of Mittag-Leffler function and generalized Mittag-Leffler function, interesting reader may refer to [2, 3].

Motivated with the above mentioned work, Srivastava *et al.* [26] investigated a new integral operator associated with Mittag-Leffler functions and obtain various interesting results.

In the present work, we introduce a new integral operator involving Mittag-Leffler function in the following way

$$F_{\alpha_i,\,\beta_i,\,\gamma_i,\,\lambda_j}(p,z) = \int_0^z \prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(t)}{t}\right)^{\gamma_i}\prod_{j=1}^{m}\left(\frac{D^p f_j(t)}{t}\right)^{\lambda_j} dt \tag{1.4}$$

where the functions $\mathbb{E}_{\alpha_i,\,\beta_i}(z)$ is normalized Mittag-Leffler functions defined by (1.3), parameters γ_i, λ_j are positive real numbers such that the integral in (1.4) exists. The integral operator defined by (1.4) reduces to various integral operators for specific values of parameters $\alpha_i, \beta_i, \gamma_i, \lambda_j, p$, studied earlier by various researchers.

1. For $\gamma_i = 0 (i = 1, 2, \ldots, n)$, the integral operator studied by Porwal [16].
2. For $\gamma_i = 0 (i = 1, 2, \ldots, n)$, $p = 0, 1$ the integral operator studied by Breaz [4].
3. For $\gamma_i = 0 (i = 1, 2, \ldots, n)$, $p = 1, m = 1$ the integral operator studied by Passai and Pescar [14].
4. For $\lambda_j = 0 (j = 1, 2, \ldots, m)$ the integral operator studied by Srivastava *et al.* [26].

In the present paper, we obtain some sufficient conditions for the integral operator defined by (1.4) is in the class $\mathcal{S}^*$. For simplicity we can write $\mathbb{F}(p,z) = F_{\alpha_i,\,\beta_i,\,\gamma_i,\,\lambda_j}(p,z)$.

2 Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. ([26]) Let $\alpha \geq 1, \beta \geq 1$. Then

$$\left|\frac{z\,\mathbb{E}'_{\alpha,\,\beta}(z)}{\mathbb{E}_{\alpha,\,\beta}(z)} - 1\right| \leq \frac{2\beta+1}{\beta^2-\beta-1}, \quad z \in \Delta.$$

Lemma 2.2. ([25]) If $f \in \mathcal{A}$ satisfies

$$\Re\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} < \frac{\delta+1}{2(\delta-1)}, \quad z \in \Delta, \text{ for some } \quad 2 \leq \delta < 3,$$

$$\text{or} \quad \Re\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} < \frac{5\delta-1}{2(\delta+1)}, \quad z \in \Delta, \text{ for some } \quad 1 < \delta \leq 2, \; \textit{then} \; f \in \mathcal{S}^*.$$

Lemma 2.3. ([25]) If $f \in \mathcal{A}$ satisfies

$$\Re\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} > -\frac{\delta+1}{2\delta(\delta-1)}, \quad z \in \Delta, \text{ for some } \quad \delta \leq -1,$$

$$\text{or} \quad \Re\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} > \frac{3\delta+1}{2\delta(\delta+1)}, \quad z \in \Delta, \text{ for some } \quad \delta > 1, \; \textit{then} \; f \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right).$$

3 Main Results

Theorem 3.1. Let n, m be natural numbers and $\alpha_1, \alpha_2, \cdots, \alpha_n \geq 1$, $\beta_1, \beta_2, \cdots, \beta_n \geq \frac{1}{2}(1+\sqrt{5})$ and suppose that $\beta = \min\{\beta_1, \beta_2, \cdots, \beta_n\}$ and suppose that $\gamma_i (i = 1, 2, \ldots, n)$, λ_j $(j = 1, 2, \ldots, m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $N(p, \omega_j)$ for $(j = 1, 2, \ldots, m)$, also let $\omega = \max\{\omega_1, \omega_2, \ldots, \omega_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i + (\omega-1)\sum_{j=1}^{m}\lambda_j \leq \frac{3-\delta}{2(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}^*$ for some $2 \leq \delta < 3$.

Proof . Differentiating equation (1.4) we have

$$\mathbb{F}'(p,z)=\prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}{z}\right)^{\gamma_i}\prod_{j=1}^{m}\left(\frac{D^p f_j(z)}{z}\right)^{\lambda_j}.$$

Taking logarithmic differentiation, we have

$$\frac{\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}=\sum_{i=1}^{n}\ \gamma_i\left(\frac{\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-\frac{1}{z}\right)+\sum_{j=1}^{m}\lambda_j\left(\frac{(D^p f_j(z))'}{D^p f_j(z)}-\frac{1}{z}\right)$$

or equivalently

$$1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}=\sum_{i=1}^{n}\ \gamma_i\left(\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right)+\sum_{j=1}^{m}\lambda_j\left(\frac{D^{p+1} f_j(z)}{D^p f_j(z)}-1\right)+1. \tag{3.1}$$

Taking the real part of both side of (3.1) we have

$$\begin{aligned}\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}&=\sum_{i=1}^{n}\ \gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)-1}\right\}+\sum_{j=1}^{m}\lambda_j\Re\left(\frac{D^{p+1} f_j(z)}{D^p f_j(z)}-1\right)+1\\ &\le 1+\sum_{i=1}^{n}\ \gamma_i\left|\frac{z\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right|+\sum_{j=1}^{m}\lambda_j\,(\omega_j-1) \qquad (3.2)\\ &\le 1+\sum_{i=1}^{n}\ \gamma_i\left(\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right)+\sum_{j=1}^{m}\lambda_j\,(\omega_j-1)\,. \qquad (3.3)\end{aligned}$$

For all $z\in\Delta$ and $(\beta_1,\ \beta_2,\ \cdots,\ \beta_n)\ge\frac{1}{2}(1+\sqrt{5})$. Since the function $\phi:\ \left(\frac{1}{2}(1+\sqrt{5}),\infty\right)\to\mathbb{R}$, defined by $\phi(x)=\dfrac{2x+1}{x^2-x-1}$ is decreasing. Therefore, for all $i\in\{1,\ 2,\ \cdots,\ n\}$, we obtain

$$\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\le\frac{2\beta+1}{\beta^2-\beta-1}.$$

Using this result, inequality (3.2) can be written as

$$\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}\le 1+\frac{2\beta+1}{\beta^2-\beta-1}\ \sum_{i=1}^{n}\ \gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j.$$

Since

$$\begin{aligned}1+\frac{2\beta+1}{\beta^2-\beta-1}\ \sum_{i=1}^{n}\ \gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j&<\frac{\delta+1}{2(\delta-1)}\\ \frac{2\beta+1}{\beta^2-\beta-1}\ \sum_{i=1}^{n}\ \gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j&<\frac{\delta+1}{2(\delta-1)}-1\\ &=\frac{3-\delta}{2(\delta-1)}.\end{aligned}$$

Therefore, from Lemma 2.2, $\mathbb{F}(p,z)\in\mathcal{S}^*$ for some $2\le\delta<3$. Thus, the proof of Theorem 3.1 is established. □

Theorem 3.2. Let n,m be natural numbers and $\alpha_i\ge 1,\ \beta_i\ge\frac{1}{2}(1+\sqrt{5})$ for $i=1,2,\ldots,n$; $\beta=\min\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ and suppose that $\gamma_i(i=1,2,\ldots,n)$, $\lambda_j\ (j=1,2,\ldots,m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $N(p,\omega_j)$ for $(j=1,2,\ldots,m)$, $p\in N_0, 1<\omega_j\leq \dfrac{2^p+1}{2^{p-1}+1}$, also let $\omega=\max\{\omega_1,\omega_2,\ldots,\omega_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j\leq\frac{3\,(\delta-1)}{2\delta+1}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}^*$ for some $1<\delta\leq 2$.

Proof . The proof of above theorem is much similar to that Theorem 3.1. Therefore, we omit the detail. □

Theorem 3.3. Let n,m be natural numbers and $\alpha_i\geq 1$, $\beta_i\geq\dfrac{1}{2}(1+\sqrt{5})$ for $i=1,2,\ldots,n$; $\beta=\max\{\beta_1,\,\beta_2,\,\cdots,\,\beta_n\}$ and suppose that $\gamma_i(i=1,2,\ldots,n)$, λ_j $(j=1,2,\ldots,m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $\mathcal{S}(p,\epsilon_j)$ for $(j=1,2,\ldots,m)$, $p\in N_0, 0\leq\epsilon_j<1$, also let $\epsilon=\min\{\epsilon_1,\epsilon_2,\ldots,\epsilon_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(1-\epsilon)\sum_{j=1}^{m}\lambda_j\leq\frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}\left(\dfrac{\delta+1}{2\delta}\right)$ for some $\delta\leq -1$.

Proof . The equation (3.1) can be re-written as

$$\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}=\sum_{i=1}^{n}\,\gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\}+\sum_{j=1}^{m}\lambda_j\Re\left(\frac{D^{p+1}f_j(z)}{D^pf_j(z)}\right)+1-\sum_{i=1}^{n}\gamma_i-\sum_{j=1}^{m}\lambda_j. \tag{3.4}$$

From Lemma 2.1, we have

$$\left|\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right|\leq\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using the identity $\Re\{z\}\leq|z|$, we have

$$\Re\left\{1-\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\}\leq\frac{2\beta_i+1}{\beta_i^2-\beta_i-1},\quad\text{or}\quad\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\}\geq 1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using the above result in (3.4), we have

$$\begin{aligned}\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}&\geq\sum_{i=1}^{n}\,\gamma_i\left(1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right)+\sum_{j=1}^{m}\lambda_j\epsilon_j+1-\sum_{i=1}^{n}\gamma_i-\sum_{j=1}^{m}\lambda_j\\&\geq 1-\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\,\gamma_i-(1-\epsilon)\sum_{j=1}^{m}\lambda_j\\&\geq-\frac{\delta+1}{2\delta(\delta-1)},\qquad\text{(by the given hypothesis).}\end{aligned}$$

□

Theorem 3.4. Let n,m be natural numbers and $\alpha_i\geq 1$, $\beta_i\geq\dfrac{1}{2}(1+\sqrt{5})$ for $i=1,2,\ldots,n$; $\beta=\max\{\beta_1,\,\beta_2,\,\cdots,\,\beta_n\}$ and suppose that $\gamma_i(i=1,2,\ldots,n)$, λ_j $(j=1,2,\ldots,m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $\mathcal{S}(p,\epsilon_j)$ for $(j=1,2,\ldots,m)$, $p\in N_0, 0\leq\epsilon_j<1$, also let $\epsilon=\min\epsilon_1,\epsilon_2,\ldots,\epsilon_m$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(1-\epsilon)\sum_{j=1}^{m}\lambda_j\leq\frac{\delta+1-2\delta^2}{2\delta(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}\left(\dfrac{\delta+1}{2\delta}\right)$ for some $\delta>1$.

Proof . The proof of above theorem runs parallel to that of Theorem 3.3. Therefore, we omit the details involved. □

Remark 3.5.
1. If we put $\gamma_i = 0$ $(i = 1, 2, \ldots, n)$ in Theorem 3.1–3.4 then we obtain the corresponding results for the integral operator introduced by Porwal [16].
2. If we put $\lambda_j = 0$ $(j = 1, 2, \ldots, m$ in Theorem 3.1–3.4 then we obtain the corresponding results for the integral operator studied by Srivastava et al. [26]

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References

[1] M. Arif and M. Raza, *Some properties of an integral operator defined by Bessel functions*, Acta Univ. Apulensis **26** (2011), 69–74.

[2] A.A. Attiya, *Some applications of Mittag-Leffler function in the unit disk*, Filomat 30 (2016), no. 7, 2075–2081.

[3] D. Bansal and J.K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ. **61** (2016), no. 3, 338–350.

[4] D. Breaz, *Certain integral operators on the classes $M(\beta_i)$ and $N(\beta_i)$*, J. Inequal. Appl. **2008** (2008), Art. ID 719354, 1–4.

[5] E. Deniz, H. Orhan and H.M. Srivastava, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math. **15** (2011), 883–917.

[6] B.A. Frasin, *Sufficient condition for integral operator defined by Bessel functions*, J. Math. Inequal. **4** (2010), no. 3, 301–306.

[7] H.Ö. Güney, G.I. Oros and S. Owa, *An application of Salagean operator concerning starlike functions*, Axioms **11** (2022), no. 2, 50. https://doi.org/10.3390/axioms11020050

[8] A.R.S. Juma and L.I. Cotirla, *On harmonic univalent function defined by generalized Salagean derivatives*, Acta Univ. Apulensis **23** (2010), 179–188.

[9] E. Kadioğlu, *On subclass of univalent functions with negative coefficients*, Appl. Math. Comput. **146** (2003), 351–358.

[10] A.A. Lupaş, *On special fuzzy differential subordinations obtained for Riemann–Liouville fractional integral of Ruscheweyh and Sălăgean operators*, Axioms **11** (2022), no. 9, 428.

[11] N. Magesh, S. Porwal and S.P. Singh, *Some geometric properties of an integral operator involving Bessel functions*, Novi Sad J. Math. **47** (2017), no. 2, 149–156.

[12] S. Mahmood, H.M. Srivastava, S.N. Malik, M. Raza, N. Shahzadi and S. Zainab, *A certain family of integral operators associated with the Struve functions*, Symmetry **11** (2019), Art. ID 463, 1–16.

[13] G.M. Mittag-Leffler, *Sur la nouvelle function $E(x)$*, C. R. Acad. Sci. Paris **137** (1903), 554–558.

[14] N.N. Pasai and V. Pescar, *On the integral operators of Kim-merkes and Pfaltzgraff*, Mathematica **32** (1990), no. 2, 185–192.

[15] G.H. Park, H.M. Srivastava and N.E. Cho, *Univalence and convexity conditions for certain integral operators associated with the Lommel function of the first kind*, AIMS Math. **6** (2021), no. 10, 11380—11402.

[16] Saurabh Porwal, *Mapping properties of an integral operator*, Acta Univ. Apulensis **27** (2011), 151–155.

[17] Saurabh Porwal, *Geometric properties of an integral operator associated with Bessel functions*, Electronic J. Math. Anal. Appl. **8** (2020), no. 2, 75–80.

[18] S. Porwal and D. Breaz, *Mapping properties of an integral operator involving Bessel functions*, Analytic Number Theory, Approximation Theory and Spect. Funct., 821-826, Springer, New York, 2014.

[19] S. Porwal and M. Kumar, *Mapping properties of an integral operator involving Bessel functions on some subclasses of univalent functions*, Afr. Mat. **28** (2017), no. 1-2, 165–170

[20] S. Porwal, A. Gupta and G. Murugasundaramoorthy, *New sufficient conditions for starlikeness of certain integral operators involving Bessel functions*, Acta Univ. Math. Belii Ser. Math. **27** (2017), 10–18.

[21] M. Raza, S. Noreen and S.N. Malik, *Geometric properties of integral operators defined by Bessel functions*, J. Ineq. Spec. Funct., **7**(2016), 34-48.

[22] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. **37** (1936), no. 2, 374–408.

[23] G.S. Salagean, *Subclasses of univalent functions*, Complex Anal. Fifth Roman. Finish Seminar, Bucharest, 1983, pp. 362–372.

[24] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[25] H. Shiraishi and S. Owa, *Starlikeness and convexity for analytic functions concerned with Jack's Lemma*, Int. J. Open Problem Comput. Math. **2** (2009), no. 1, 37–47.

[26] H.M. Srivastava, B.A. Frasin and V. Pescar, *Univalance of integral operators involving Mittag-Leffler functions*, Appl. Math. Inf. Sci. **11** (2017), no. 3, 635–641.

[27] A. Wiman, *Ŭber den fundamental satz in der Theorie der Funcktionen* $E(x)$, Acta Math. **29** (1905), 191–201.

[28] A. Wiman, *Ŭber die Nullstellum der Funcktionen* $E(x)$, Acta Math. **29** (1905), 271–234.