Int. J. Nonlinear Anal. Appl. 14 (2023) 5, 287–301
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29339.4132

Complexity analysis of interior-point methods yielding the best known iteration bound for semidefinite optimization

Derbal Louiza[a], Kebbiche Zakia[a], Bouafia Mousaab[b,*]

[a]*LMFN, Fundamental and Numerical Mathematics Laboratory, Department of Mathematics, Faculty of Science, Ferhat Abbas University, Setif, Algeria*

[b]*LMAH, FR-CNRS-3335, ISCN, 7600 Le Havre, France, University of 8 May 1945 Guelma. BP 401, 24000 Guelma, Algeria*

(Communicated by Haydar Akca)

Abstract

The purpose of this paper is to obtain new complexity results for solving the semidefinite optimization (SDO) problem. We define a new proximity function for the SDO by a new kernel function with an efficient logarithmic barrier term. Furthermore, we formulate an algorithm for the large and small-update primal-dual interior-point method (IPM) for the SDO. It is shown that the best result of iteration bounds for large-update methods and small-update methods can be achieved, namely $\mathcal{O}\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\epsilon}\right)$ for large-update and $\mathcal{O}(q^2\sqrt{n}\log\frac{n}{\epsilon})$ for small-update methods, where $q>1$. The analysis in this paper is new and different from the one using for LO. Several new tools and techniques are derived in this paper. Furthermore, numerical tests to investigate the behavior of the algorithm so as to be compared with other approaches.

Keywords: Kernel function, Proximity function, Semidefinite optimization, Complexity analysis, Primal-dual interior-point methods
2020 MSC: 90C22, 90C51, 65K05

1 Introduction

The semidefinite optimization problem (SDO) is an important class of convex optimization problems, which includes linear optimization (LO) and second-order cone optimization (SOCO) as special cases. To resolve the problem, the interior-point methods (IPMs) gain much more attention. Several IPMs designed for LO have been successfully extended to SDO due to their polynomial complexity and practical efficiency.

Kernel functions play an important role in the design and analysis of IPMs. They are not only used for determining the search directions but also for measuring the distance between the given iterate and the μ-center for algorithms. Currently, algorithms based on kernel functions are the most effective algorithms for solving LO, SOCO, SDO and symmetric optimization (SO) and constitute very active research areas in mathematical programming.

Due to literature, nowadays, it seems that primal-dual IPMs based on kernel functions received a great of interest by the researchers in this field, we refer the interested reader to the works proposed in [2, 3, 5, 6, 7, 9].

*Corresponding author
Email addresses: louiza.derbal@univ-setif.dz (Derbal Louiza), zakia.kebbiche@univ-setif.dz (Kebbiche Zakia), bouafia.mousaab@univ-guelma.dz (Bouafia Mousaab)

Received: December 2022 *Accepted:* February 2023

Motivated by these works, in this paper, we define a new proximity function then we formulate the corresponding algorithm for solving the SDO. So, we propose the following kernel function

$$\psi(t) = \frac{t^2 - 1 - \log t}{2} + \frac{t^{1-q} - 1}{2(q-1)}, \tag{1.1}$$

where $q > 1$ is a parameter. The iteration complexity obtained for large-and small-update methods is, namely,

$$\mathcal{O}\left(qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right), \text{ and } \mathcal{O}(q^2\sqrt{n} \log \frac{n}{\epsilon})$$

respectively. These theoretical results have important theoretical significance in convex quadratic SDO [20, 18] and semidefinite complementarity problems (SDLCP) [1].

The paper is organized as follows : in section 2, we introduce some useful results concerning the cone of symmetric matrices and some concepts of central path for SDO problem. We also describe the generic corresponding algorithm. In section 3, we give the properties of the kernel function. An estimation for the step size is given in section 4. In section 5, the polynomial iteration complexity is provided. Some illustrative and comparative numerical results are reported in section 6. Finally, some remarks are given in the conclusion.

The following notations are used throughout the paper: $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. $\|.\|$ is used as the Frobenius norm for matrices and as the Euclidian norm for vectors. $\mathbb{R}^{m\times n}$ is the space of all $m\times\ n$ matrices whose components are from $\mathbb{R}$. $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ denote the cone of symmetric, symmetric positive semidefinite and symmetric positive definite matrices of the order $n\times\ n$, respectively. E denote the identity matrix. For $A, B \in \mathbb{R}^{n\times n}$, we use the classical lower partial ordering $\succeq$ for symmetric matrices by the mean of $A \succeq B$ ($A \succ B$) if and only if $A - B$ is positive semidefinite matrix (positive definite matrix). Given $A \in \mathbb{R}^{n\times n}$, $Tr(A)$ stands for the trace of the matrix A. For $A, B \in \mathbb{R}^{m\times n}$, the inner product is defined by $A \bullet B = Tr(AB^T)$. For any symmetric definite matrix $Q \in \mathbb{S}^n_{++}$, the expression $Q^{\frac{1}{2}}$ stands for symmetric square root of Q. For $V \in \mathbb{S}^n_{++}$, we assume that the eigenvalues of V are arranged in non-increasing order, that is, $\lambda_1(V) \geq \lambda_2(V) \geq ... \geq \lambda_n(V)$. For a vector $x \in \mathbb{R}^n$, $diag(x)$ is a diagonal matrix whose diagonal entries are component of vector x. Finally, if $g(x) \geq 0$ is a real valued function of real nonnegative variable, the notation $g(x) = \mathcal{O}(x)$ means that $g(x) \leq \bar{c}x$ for some positive constant $\bar{c}$ and $g(x) = \Theta(x)$ that $\bar{c}_1 x \leq g(x) \leq \bar{c}_2 x$ for two positive constants $\bar{c}_1$ and $\bar{c}_2$.

2 Preliminaries for SDO

In this section, we first introduce some useful results regarding the property of the symmetric matrices. Then some concepts of central path for SDO problems are briefly recalled. The structure of generic primal-dual IPMs based on kernel function is also given in this section.

2.1 Special matrix functions

Let us recall some basic facts linear algebra.

Theorem 2.1. (Spectral theorem for symmetric matrices in [19]). The real $n \times n$ matrix A is symmetric if and only if there exists an orthogonal basis with respect to witch Λ is real and diagonal, i.e., a matrix $U \in \mathbb{R}^{\,n\times n}$ such that $U^TU = E$ and $U^TAU = \Lambda$ where Λ is a diagonal matrix.

The columns u_i of the orthogonal matrix U are the eigenvectors of A, satisfying $Au_i = \lambda_i u_i, \quad i = 1, 2, ..., n$, where λ_i is the i-th diagonal entry of Λ.

Definition 2.2. Let $V \in \mathbb{S}^n_{++}$ and $V = Q^T diag\,(\lambda\,(V))\,Q$, where Q is any nonsingular matrix that diagonalizes V. Let $\psi\,(t)$ be defined in (1.1). Then the matrix function $\psi\,(V) : \mathbb{S}^n_{++} \to \mathbb{S}^n_{++}$ is defined by

$$\psi\,(V) = Q^T diag\,(\psi(\lambda_1(V))\,, \psi(\lambda_2(V)), ..., \psi(\lambda_n(V)))Q. \tag{2.1}$$

The real valued matrix function $\Psi(V) : \mathbb{S}^n_{++} \to \mathbb{R}_+$ is defined as follows

Definition 2.3.

$$\Psi(V) = tr\left(\psi\left(V\right)\right) = \sum_{i=1}^{n} \psi(\lambda_i(V)), \tag{2.2}$$

where $\psi\left(V\right)$ is given by (2.1)

Suppose that $\psi\left(t\right)$ is a twice differentiable function for all $t > 0$. So, the first and the second order derivatives of the matrix function $\psi\left(V\right)$, are defined as below[12]:

$$\psi'\left(V\right) = Q^T diag\left(\psi'(\lambda_1(V))\,, \psi'(\lambda_2(V)), ..., \psi'(\lambda_n(V))\right)Q.$$

$$\psi''\left(V\right) = Q^T diag\left(\psi''(\lambda_1(V))\,, \psi''(\lambda_2(V)), ..., \psi''(\lambda_n(V))\right)Q.$$

Definition 2.4. A matrix $X(t)$ is said to be a matrix of function (or a matrix-valued function) if each entry of $X(t)$ is a function of t. i.e., $X(t) = [X_{ij}\left(t\right)]$.

Remark 2.5. In the following, when we apply the function $\psi(.)$ and its derivatives $\psi'(.)$ and $\psi''(.)$, indeed, we face with matrix functions if the argument is a matrix and with a univariate function if the argument is in $\mathbb{R}_{++}$.

In the next lemma we describe some law about the first derivative of matrix of functions. One can find the proof in [11, 13].

Lemma 2.6. Suppose that $X\left(t\right)$ and $Y\left(t\right)$ are two matrices of functions, thus, one has:

$$X'\left(t\right) = \frac{d}{dt} X\left(t\right) = \left(\frac{d}{dt} X_{ij}\left(t\right)\right). \tag{2.3}$$

$$\frac{d}{dt} tr\left(X\left(t\right)\right) = tr\left(\frac{d}{dt} X\left(t\right)\right) = tr\left(X'\left(t\right)\right). \tag{2.4}$$

$$\frac{d}{dt} tr\left(\psi(X\left(t\right))\right) = tr\left(\psi'(X\left(t\right))X'\right). \tag{2.5}$$

$$\begin{aligned} \frac{d}{dt}\left(X\left(t\right)Y\left(t\right)\right) &= \left(\frac{d}{dt} X\left(t\right)\right) Y\left(t\right) + X\left(t\right)\left(\frac{d}{dt} Y\left(t\right)\right) \\ &= X'\left(t\right)Y\left(t\right) + X\left(t\right)Y'\left(t\right). \end{aligned} \tag{2.6}$$

2.2 The central path for SDO

We consider the standard semidefinite optimization problem in the standard form

$$\min_X C \bullet X \quad s.t. \quad A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succeq 0, \tag{P}$$

and its dual:

$$\max_{(y,S)} b^T y \quad s.t. \quad \sum_{i=1}^{m} y_i A_i + S = C, \quad S \geq 0, \tag{D}$$

where each $C, A_i \in \mathbb{S}^n$, for $1 \leq i \leq m$ and $b,\ y \in \mathbb{R}^m$. Moreover, we assume that the matrices A_i are linearly independent. We also assume that both problem (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists $X^0 \succ 0$ and $\left(y^0, S^0\right)$ with $S^0 \succ 0$ in the feasible set of problem (P) and (D), respectively, i.e.,

$$\begin{cases} Ai \bullet X^0 = b_i, & i = 1, 2, ..., m, \\ \sum\limits_{i=1}^{m} y_i^0 A_i + S^0 = C. & \end{cases}$$

It is well know that the IPC can be assumed without loss of generality. In fact we may choose $X^0 = S^0 = E$ as the initial start point. The detailed analysis can be found in [8, 19]. The optimality condition for (P) and (D) are given by the following system

$$\begin{cases} Ai \bullet X = b_i \ , \qquad i = 1,2,...,m, \\ \sum_{i=1}^{m} y_i A_i + S = C, \\ XS = 0, \\ X \succeq 0, S \succeq 0. \end{cases} \tag{2.7}$$

If the IPC hold, the μ-central of (P) and (D) is defined by the solution $(X(\mu), y(\mu), S(\mu))$ of the following system

$$\begin{cases} Ai \bullet X = b_i \ , \qquad i = 1,2,...,m, \\ \sum_{i=1}^{m} y_i A_i + S = C, \\ XS = \mu E, \\ X \succeq 0, S \succeq 0, \end{cases} \tag{2.8}$$

with $\mu > 0$. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (P) and (D). If $\mu \to 0$ then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an ϵ-approximate solution for (P) and (D) [8, 19]. Applying Newton's method to system (2.8) produces the following equations for the search direction ΔX, Δy and ΔS:

$$\begin{cases} Ai \bullet \Delta X = 0, \qquad i = 1,2,...,m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ X\Delta S S^{-1} + \Delta X = \mu S^{-1} - X \ . \end{cases} \tag{2.9}$$

A decisive observation for SDO is that the above Newton system might have no symmetric solution ΔX. Many researchers have proposed different ways of symmetrizing the third equation in the Newton system so the new system has a unique symmetric solution [15, 16]. In this paper, we use the Nesterov-Todd (NT) symmetrization scheme [2]. Let us define

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}} S X^{\frac{1}{2}})^{-\frac{1}{2}} X^{\frac{1}{2}} = S^{-\frac{1}{2}}(S^{\frac{1}{2}} X S^{\frac{1}{2}}) S. \tag{2.10}$$

We replace the term $X\Delta S S^{-1}$in the third equation of (2.9) by $P\Delta S P^T$. The system (2.9) becomes

$$\begin{cases} Ai \bullet \Delta X = 0, \qquad i = 1,2,...,m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ \Delta X + P\Delta S P^T = \mu S^{-1} - X, \qquad X, S \succeq 0. \end{cases} \tag{2.11}$$

Furthermore, we define $D = P^{\frac{1}{2}}$, which can be used to scale X and S to the same matrix V because

$$V = \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} \left(D^{-1} XSD\right)^{\frac{1}{2}}. \tag{2.12}$$

Let us further define

$$\begin{cases} \overline{A}_i = \frac{1}{\sqrt{\mu}} D A_i D, \ i = 1,...,m, \\ D_X = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \\ D_S = \frac{1}{\sqrt{\mu}} D \Delta S D. \end{cases} \tag{2.13}$$

Then NT search direction can be written as the solution of the following system:

$$\begin{cases} \overline{A}_i \bullet D_X = 0, & i = 1, ..., m, \\ \sum\limits_{i=1}^{n} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S \ = V^{-1} - V. \end{cases} \tag{2.14}$$

Define the so-called classical kernel function

$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t.$$

Then $-\psi_c'(V)$ is the same as the right-hand side of the third equation in (2.14). For our IPM, in place of $\psi_c(t)$, we use kernel function defined in (1.1). The new search direction D_X and D_S are derived by solving the following system

$$\begin{cases} \overline{A}_i \bullet D_X = 0, & i = 1, ..., m, \\ \sum\limits_{i=1}^{n} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S \ = -\psi'(V). \end{cases} \tag{2.15}$$

Since D_X and D_S are orthogonal due to the orthogonality of ΔX and ΔS, it is trivial to verify that $D_X \bullet D_S = D_X \bullet D_S = 0$. Then we have

$$D_X = D_S = O_{n\times n} \Leftrightarrow \psi'(V) = O_{n\times n} \quad \Leftrightarrow V = E \quad \Leftrightarrow \Psi(V) = 0,$$

i.e., if and only if $XS = \mu E$, that is, if and only if $X = X(\mu)$ and $S = S(\mu)$ as it should. Otherwise $\Psi(V) > 0$, hence, if $(X, y, S) \neq (X(\mu), y(\mu), S(\mu))$, then $(\Delta X, \Delta y, \Delta S)$ is nonzero. By taking a step along the search direction, with the step size α, one constructs a new triple (X_+, y_+, S_+) according to

$$X_+ = X + \alpha\Delta X,\ y_+ = y + \alpha\Delta y,\ S_+ = S + \alpha\Delta S. \tag{2.16}$$

We can now describe the algorithm in a more formal way. The generic form of this algorithm is shown below

Algorithm 1:Generic primal-dual Algorithm for *SDO*

Input:
a kernel function $\psi(t)$;
a threshold parameter $\tau > 1$;
an accuracy parameter $\epsilon > 0$;
a fixed barrier update parameter $\theta, 0 < \theta < 1$;
begin
  $X := E; S := E; \mu := 1; V := E$;
  while $n\mu \geq \epsilon$ do
  begin (outer iteration)
    $\mu := (1-\theta)\mu$;
    $V := \frac{V}{\sqrt{1-\theta}}$;
    while $\Psi(V) > \tau$ do
    begin (inner iteration)
      Solve system (2.15) and use (2.13) to obtain $(\Delta X, \Delta y, \Delta S)$;
      determine a step size α;
      $X := X + \alpha\Delta X$;
      $S := S + \alpha\Delta S$;
      $y := y + \alpha\Delta y$;
      $V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD$;
    end (inner iteration);
  end (outer iteration);
end.

It is clear from this description that closeness of (X, y, S) to $(X(\mu), y(\mu), S(\mu))$ is measured by the value of $\Psi(V)$, with τ as threshold value: if $\Psi(V) \leq \tau$, then start a new outer iteration by performing a μ-update, otherwise we enter an inner iteration relative to the current value of μ and apply (2.16) to get new iterates.The parameters τ, θ and the step size α should be chosen in such a way that the number of iterations required by the algorithm is as small as possible. Recently in [7] a new kernel function with logarithmic barrier term is proposed. It improves all results of the complexity bound for-update methods based on logarithmic kernel functions for LO. For the case of SDO, we will prove that the corresponding algorithm has $\mathcal{O}\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\epsilon}\right)$ complexity bound for large-update method and $\mathcal{O}\left(q^2\sqrt{n}\log\frac{n}{\epsilon}\right)$ for small-update method.

3 Properties of the kernel (Barrier) function

We start by recalling some properties of $\psi(t)$

$$\begin{cases} \psi'(t) = t - \frac{1}{2t} - \frac{1}{2}t^{-q}, \\ \psi''(t) = 1 + \frac{1}{2t^2} + \frac{q}{2}t^{-q-1} > 1, \\ \psi'''(t) = -\frac{1}{t^3} - \frac{q(q+1)}{2}t^{-q-2} < 0, \\ \psi(1) = 0, \psi'(1) = 0\ , \\ \lim\limits_{t\to 0^+}\psi(t) = \lim\limits_{t\to+\infty}\psi(t) = +\infty. \end{cases} \tag{3.1}$$

Moreover, from (3.1) we have the following lemmas.

Lemma 3.1. [7] For $\psi(t)$, we have

$\psi(t)$ is exponentially convex for all $t > 0$; that is,

$$\psi\left(\sqrt{t_1 t_2}\right) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right). \tag{a}$$

$$\psi''(t) \text{ is monotonically decreasing for all } t > 0. \tag{b}$$

$$t\psi''(t) - \psi'(t) > 0 \text{ for all } t > 0. \tag{c}$$

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, t > 1, \beta > 1. \tag{d}$$

Lemma 3.2. [7] For $\psi(t)$, we have

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\left[\psi'(t)\right]^2, \quad t > 0. \tag{3.2}$$

$$\psi(t) \leq \frac{q+3}{4}(t-1)^2, \quad t > 1. \tag{3.3}$$

Let $\sigma : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for all $t \in]0, 1]$. Then we have the following lemma.

Lemma 3.3. [7] For $\psi(t)$, we have

$$1 + \sqrt{\frac{4}{q+3}s} \leq \sigma(s) \leq 1 + \sqrt{2s}, \quad s \geq 0. \tag{3.4}$$

$$\rho(s) > \frac{1}{(4s+2)^{\frac{1}{q}}}, \quad s \geq 0. \tag{3.5}$$

At some place below we apply the function Ψ to a positive vector v. The interpretation of $\Psi(v)$ is compatible with Definition 2.2, when identifying the vector v with its diagonal matrix $V = diag(v)$. When applying Ψ to this matrix we obtain

$$\Psi(V) = \sum_{i=1}^{n}\psi(v_i), \quad v \in int\ \mathbb{R}^n_+.$$

Theorem 3.4. (Theorem 3 in [18]) Let σ be as defined in Lemma 3.3. Then for any positive definite matrix V, and any $\beta \geq 1$, one has

$$\Psi(\beta V) \leq n\psi\left(\beta\sigma\left(\frac{\Psi(V)}{n}\right)\right).$$

In the next theorem, we obtain an estimate for the effect of a $\mu-$update on the value of $\Psi(V)$.

Theorem 3.5. Let $0 \leq \theta < 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then

$$\Psi(V_+) \leq n\psi\left(\frac{\sigma\left(\frac{\tau)}{n}\right)}{\sqrt{1-\theta}}\right) \leq \frac{\left(n\theta + 2\tau + 2\sqrt{2\tau n}\right)}{2(1-\theta)}.$$

Proof . Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\sigma\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi(V)}{n}\right) \geq 1$. Using Theorem 3.4 with $\beta = \frac{1}{\sqrt{1-\theta}}$ and the function σ is monotonically increasing since $\psi(t)$ is that for $t \geq 1$, we have

$$\Psi(V_+) \leq n\psi\left(\frac{\sigma\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{\sigma\left(\frac{\tau)}{n}\right)}{\sqrt{1-\theta}}\right).$$

This prove the first inequality. The second inequality follows from: for $t \geq 1$, we have $\psi(t) \leq \frac{t^2-1}{2}$. Then

$$\begin{aligned}
\Psi(V_+) &\leq n\psi\left(\frac{\sigma\left(\frac{\tau)}{n}\right)}{\sqrt{1-\theta}}\right) \leq \frac{n}{2}\left(\frac{\sigma^2\left(\frac{\tau)}{n}\right)}{1-\theta} - 1\right) \\
&= \frac{n}{2\left(1-\theta\right)}\left(\sigma^2\left(\frac{\tau)}{n}\right) - (1-\theta)\right) \\
&\leq \frac{n}{2\left(1-\theta\right)}\left(\left(1+\sqrt{\frac{2\tau}{n}}\right)^2 - (1-\theta)\right) \\
&= \frac{\left(n\theta + 2\tau + 2\sqrt{2\tau n}\right)}{2(1-\theta)}.
\end{aligned}$$

□

Denote

$$\Psi_0 = \frac{\left(n\theta + 2\tau + 2\sqrt{2\tau n}\right)}{2(1-\theta)}, \tag{3.6}$$

then Ψ_0 is an upper bound for $\Psi(V)$ during the Newton process of the algorithm.

3.1 Properties of $\Psi(V)$ and $\delta\left(V\right)$

In the analysis of the algorithm, we also use the norm- based proximity measure $\delta\left(V\right)$ defined by

$$\delta\left(V\right) := \frac{1}{2}\left\|\Psi'(V)\right\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\psi'\left(\lambda_i\left(V\right)\right)^2} = \frac{1}{2}\left\|D_X + D_S\right\|, \tag{3.7}$$

in term of $\Psi(V)$. Since $\Psi(V)$ is strictly convex and attains its minimal value zero at $V = E$, we have

$$\Psi'(V) = 0 \Longleftrightarrow \delta\left(V\right) = 0 \Longleftrightarrow V = E.$$

Lemma 3.6. Let $\delta(V)$ be as defined in (3.7). Then we have

$$\delta(V) \geq \sqrt{\frac{\Psi(V)}{2}}. \tag{3.8}$$

Proof . From (3.2) and (3.7), we have

$$\Psi(V) = \sum_{i=1}^{n} \psi(\lambda_i(V)) \leq \sum_{i=1}^{n} \frac{1}{2}\psi'(\lambda_i(V))^2 = \frac{1}{2}\|\nabla\Psi(V)\|^2 = 2\delta(V)^2,$$

Hence, we have $\delta(V) \geq \sqrt{\frac{\Psi(V)}{2}}$. This completes the proof. □

Remark 3.7. Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 3.6 and the assumption that $\Psi(V) \geq \tau$, we have $\delta(V) \geq \frac{1}{\sqrt{2}}$.

4 An estimation for the step size

At the start of each outer iteration, just before the update of μ with the factor $1-\theta$, we have $\Psi(V) \leq \tau$. Due to the update of μ, the matrix V defined by (2.12) is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, wich leads to an increasing in the value of $\Psi(V)$. Then, during the inner iterations, $\Psi(V)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm the largest values of $\Psi(V)$ occur just after the updates of μ. After that , we derive an estimate for the effect of μ-update on the value of $\Psi(V)$.

4.1 Decrease of $\Psi(V)$ in the inner iteration

In each inner iteration the search directions ΔX, Δy and ΔS are obtained by solving the system (2.15), and using (2.13). After a step with size α, the new iterate is given by

$$X_+ = X + \alpha\Delta X,\ y_+ = y + \alpha\Delta y,\ S_+ = S + \alpha\Delta S.$$

We may write

$$X_+ = X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D,$$

and

$$S_+ = S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V + \alpha D_S)D,$$

denoting the matrix V after the step as V_+, we have

$$V_+ = \frac{1}{\sqrt{\mu}}\left(D^{-1}X_+S_+D\right)^{\frac{1}{2}},$$

note that V_+^2 is unitarily similar to the matrix $\frac{1}{\mu}X_+^{\frac{1}{2}}S_+X_+^{\frac{1}{2}}$ and thus to

$$(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}.$$

Consequently, the eigenvalues of the matrix V_+ are precisely the same as those of the matrix

$$\tilde{V}_+ = \left((V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

By the definition of $\Psi(V)$, we obtain

$$\Psi(V_+) = \Psi(\tilde{V}_+).$$

Defining

$$f(\alpha) = \Psi(V_+) - \Psi(V) = \Psi(\tilde{V}_+) - \Psi(V). \tag{4.1}$$

Lemma 4.1. ([Lemma 9 in [10]) Let $V_1$and V_2 be two symmetric positive definite matrices, then

$$\Psi\left((V_1^{\frac{1}{2}}V_2V_1^{\frac{1}{2}})^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi\left(V+\alpha D_X\right)+\Psi\left(V+\alpha D_S\right)\right).$$

Using Lemma 4.1, it follows that

$$\Psi(V_+) = \Psi(\tilde{V}_+) \leq \frac{1}{2}\left(\Psi\left(V_1\right)+\Psi\left(V_2\right)\right).$$

From the definition (4.1) of $f(\alpha)$, we now have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}[\Psi\left(V+\alpha D_X\right)+\Psi(V+\alpha D_S)] - \Psi(V),$$

which means that $f_1(\alpha)$ gives an upper bound for the decrease of the barrier function $\Psi(V)$. Furthermore, we can easily verify that

$$f(0) = f_1(0) = 0.$$

Taking the derivative with respect to α, we get

$$f_1'(\alpha) = \frac{1}{2}Tr\left(\psi'\left(V+\alpha D_X\right)D_X+\psi'\left(V+\alpha D_S\right)D_S\right).$$

Using the last equation of (2.15) and (3.7), we obtain

$$f_1'(0) = \frac{1}{2}tr\left(\psi'\left(V\right)\left(D_X+D_S\right)\right) = \frac{1}{2}tr\left(-\psi'(V)^2\right) = -2\delta(V)^2.$$

Differentiating once more, we obtain

$$f_1''(\alpha) = \frac{1}{2}Tr\left(\psi''\left(V+\alpha D_X\right)D_X^2+\psi''\left(V+\alpha D_S\right)D_S^2\right). \tag{4.2}$$

In what follows, we use the short notation: $\delta := \delta(V)$, and state some important results.

Lemma 4.2. (Lemma 10 in [10]). One has

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_n(V)-2\alpha\delta).$$

Next we will choose a suitable step size for the algorithm. This should be chosen such that X_+ and S_+ are feasible and another one is to make $\Psi(V_+) - \Psi(V)$ decreases sufficiently. Putting $v_i = \lambda_i(V), \quad 1 \leq i \leq n$, and $\nu_n := \min(v)$, we have:

$$f_1''(\alpha) \leq 2\delta^2\psi''(v_n-2\alpha\delta),$$

which is the same inequality as in Lemma 4.1 in [3]. From this stage on we can apply similar arguments as in [3, 9] for the LO case to obtain the following results which require no further proof.

Lemma 4.3. (Lemma. 4.2 in [3]) One has $f_1'(\alpha) \leq 0$, certainly holds if α satisfies the inequality

$$-\psi'(v_n-2\alpha\delta)+\psi'(v_n) \leq 2\delta. \tag{4.3}$$

Lemma 4.4. (Lemma. 4.3 in [3]) The largest step size α that satisfies (4.3) is given by

$$\bar{\alpha} := \frac{1}{2\delta}\left(\rho\left(\delta\right)-\rho\left(2\delta\right)\right). \tag{4.4}$$

Lemma 4.5. (Lemma. 4.4 in [3]) Let $\bar{\alpha}$ is defined in Lemma 4.4. Then

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho\left(2\delta\right))}.$$

As in the LO case, we use a default step size $\tilde{\alpha}$ that is the lower bound of the $\bar{\alpha}$ and consists of δ.

$$\tilde{\alpha} := \frac{1}{\psi''(\rho\,(2\delta))}. \tag{4.5}$$

Lemma 4.6. Let ρ and $\bar{\alpha}$ be as defined in Lemma 4.5. If $\Psi\,(V) \geq \tau \geq 1$, then we have

$$\bar{\alpha} \geq \frac{2}{2 + (q+1)(8\delta + 2)^{\frac{q+1}{q}}}.$$

Proof . Using (3.1), (3.5) and Lemma 4.5, we have

$$\begin{aligned}
\bar{\alpha} &\geq \frac{1}{\psi''(\rho\,(2\delta))} = \frac{1}{1 + \frac{1}{2}(\frac{1}{\rho(2\delta)})^2 + \frac{q}{2}(\frac{1}{\rho(2\delta)})^{q+1}} \\
&\geq \frac{1}{1 + \frac{1}{2}(4(2\delta)+2)^{\frac{2}{q}} + \frac{q}{2}(4(2\delta)+2)^{\frac{q+1}{q}}} \\
&\geq \frac{1}{1 + \frac{1}{2}(8\delta+2)^{\frac{q+1}{q}} + \frac{q}{2}(8\delta+2)^{\frac{q+1}{q}}} \\
&= \frac{2}{2 + (q+1)(8\delta+2)^{\frac{q+1}{q}}}.
\end{aligned}$$

This completes the proof. □

Denoting

$$\tilde{\alpha} := \frac{2}{2 + (q+1)(8\delta + 2)^{\frac{q+1}{q}}}. \tag{4.6}$$

Lemma 4.7. (Lemma 1.3.3 in [12]) Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, one has

$$h(t) \leq \frac{th'(0)}{2}, 0 \leq t \leq t^*.$$

Since $f_1(\alpha)$ holds the condition of the above lemma, $f(\alpha) \leq f_1(\alpha) \leq \dfrac{f_1'(0)}{2}\alpha$ for all $0 \leq \alpha \leq \bar{\alpha}$, then we have the following lemma to obtain the upper bound for the decreasing value of the proximity in the inner iteration.

Lemma 4.8. (Lemma 4.5 in [3]) if the step size α satisfies $\alpha \leq \bar{\alpha}$, then

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 4.9. Let $\tilde{\alpha}$ be the default step size as defined in (4.6) and $\Psi(V) \geq 1$, then

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{2}}{(q+1)\,148}\,[\Psi(V)]^{\frac{q-1}{2q}}.$$

Proof . Using Lemma 4.8, Remark 3.7, and substituting the value of $\tilde{\alpha}$, we have

$$\begin{aligned}
f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2 \\
&= -\frac{2\delta^2}{2+(q+1)(8\delta+2)^{\frac{q+1}{q}}} \\
&\leq -\frac{2\delta^2}{2(2\delta)+(q+1)(4(2\delta)+2(2\delta))^{\frac{q+1}{q}}} \\
&\leq -\frac{2\delta^2}{2(2\delta)^{\frac{q+1}{q}}+(q+1)(12\delta))^{\frac{q+1}{q}}} \\
&= -\frac{2\delta^2}{\left(2(2)^{\frac{q+1}{q}}+(q+1)(12)^{\frac{q+1}{q}}\right)(\delta)^{\frac{q+1}{q}}} \\
&\leq -\frac{2\delta^{\frac{q-1}{q}}}{2(2)^2+(q+1)(12)^2} \\
&\leq -\frac{2\delta^{\frac{q-1}{q}}}{(q+1)148} \\
&\leq -\frac{\sqrt{2}}{(q+1)148}\left[\Psi(V)\right]^{\frac{q-1}{2q}}.
\end{aligned}$$

This completes the proof. □

5 Iteration complexity

In this section, we derive the complexity bounds for large and small-update methods.

5.1 Upper bound for the number of inner iterations

We denote the value of $\Psi(V)$ after μ-update by Ψ_0

$$\Psi(V_+) \leq \Psi_0 = \frac{\left(n\theta+2\tau+2\sqrt{2\tau n}\right)}{2(1-\theta)} = L\left(n,\theta,\tau\right).$$

We need to count how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$. the subsequent values in the same outer iteration are denoted as Ψ_k , $k = 1, 2, ..., K$, where K denotes the total number of inner iterations in the outer iteration. According to decrease of $f(\tilde{\alpha})$, for $k = 1, 2, ..., K-1$, we obtain

$$\Psi_{k+1} \leq \Psi_k - \frac{\sqrt{2}}{(q+1)\,148}\left[\Psi(V)\right]^{\frac{q-1}{2q}}. \tag{5.1}$$

Lemma 5.1. (Lemma 14 in [13]) Let $\{t_k > 0, k = 0, 1, ..., K\}$ be a sequence of positive numbers satisfying $t_{k+1} \leq t_k - \beta t_k^{1-\gamma}$, $k = 0, 1, ..., K-1$, where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \left[\frac{t_0^{\gamma}}{\beta\gamma}\right]$.

Letting $t_k = \Psi_k$, $\beta = \frac{\sqrt{2}}{(q+1)\,148}$ and $\gamma = \frac{q+1}{2q}$.

Theorem 5.2. Let K be the total number of inner iterations in the outer iterations. Then we have

$$K \leq \left(148\sqrt{2q}\right)\left[\Psi_0\right]^{\frac{q+1}{2q}}.$$

Proof . By Lemma 5.1, we have $K \leq \left[\frac{t_0^{\gamma}}{\beta\gamma}\right] = \left(148\sqrt{2q}\right)\left[\Psi_0\right]^{\frac{q+1}{2q}}$. This completes the proof. □

5.2 Upper bound for total number of iterations

The number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{n}{\epsilon}$ (cf. Lemma II. 17, p.116, in [14] Multiplication of this number by the bound for the number of inner iterations in Theorem 5.2 yields the total number of iterations, namely,

$$\left(148\sqrt{2q}\right)[\Psi_0]^{\frac{q+1}{2q}}\frac{\log\frac{n}{\epsilon}}{\theta}. \tag{5.2}$$

For large-update methods, with $\tau=\mathcal{O}(n)$ and $\theta=\Theta(1)$. After some elementary reductions, the iteration bound becomes

$$\mathcal{O}\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\epsilon}\right).$$

In the case of small-update methods, we have $\tau=\mathcal{O}(1)$, $\theta=\Theta(\frac{1}{\sqrt{n}})$ and by (3.3), with

$$\psi(t)\le\frac{q+3}{4}(t-1)^2,\ t>1,$$

we obtain

$$\begin{aligned}
\Psi(V_+) &\le n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi(V)}{n}\right)\right)\\
&\le n\frac{q+3}{4}\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi(V)}{n}\right)-1\right)^2\\
&= \frac{n(q+3)}{4(1-\theta)}\left(\sigma\left(\frac{\Psi(V)}{n}\right)-\sqrt{1-\theta}\right)^2.
\end{aligned}$$

Using (3.4), we have

$$\begin{aligned}
\frac{n(q+3)}{4(1-\theta)}\left(\sigma\left(\frac{\Psi(V)}{n}\right)-\sqrt{1-\theta}\right)^2 &\le \frac{n(q+3)}{4(1-\theta)}\left(1+\sqrt{2\frac{\Psi(V)}{n}}-\sqrt{1-\theta}\right)^2\\
&= \frac{n(q+3)}{4(1-\theta)}\left(\left(1-\sqrt{1-\theta}\right)+\sqrt{2\frac{\Psi(V)}{n}}\right)^2\\
&\le \frac{n(q+3)}{4(1-\theta)}\left(\theta+\sqrt{2\frac{\tau}{n}}\right)^2\\
&= \frac{n(q+3)}{4(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2=\Psi_0.
\end{aligned}$$

where we also used that $1-\sqrt{1-\theta}=\frac{\theta}{1+\sqrt{1-\theta}}\le\theta$ and $\Psi(V)\le\tau$, using this upper for Ψ_0, we get the following iteration bound:

$$\left(148\sqrt{2q}\right)[\Psi_0]^{\frac{q+1}{2q}}\frac{\log\frac{n}{\epsilon}}{\theta}.$$

Note now $\Psi_0=\mathcal{O}(q)$, and the iteration bound becomes $\mathcal{O}(q^2\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity.

6 Numerical results

In this section, our main focus is to provide a numerical experiences regarding the practical performance of the new proposed kernel function in comparison with some other kernel functions which have been proposed in the literature.

The Algorithm is coded in MATLAB (R2014a) and our experiments were performed on PC with Processor Genuine Intel (R) CPR T2080 @ 1,73GHZ installed memory (RAM) 2,00GO. Let us consider the following special case of SDO problem, whose primal-dual pair have the following data [17]

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}.$$

The optimal solution of the primal problem is given by

$$X^* = \begin{bmatrix} 0.0714 & -0.0718 & 0.0169 & 0.0649 & -0.1583 \\ -0.0718 & 0.0724 & -0.0183 & -0.0602 & 0.1676 \\ 0.0169 & -0.0183 & 0.0103 & -0.0084 & -0.0772 \\ 0.0649 & -0.0602 & -0.0084 & 0.1481 & 0.0056 \\ -0.1583 & 0.1676 & -0.0772 & 0.0056 & 0.6022 \end{bmatrix},$$

and for the dual problem an optimal solution is given by

$$S^* = \begin{bmatrix} 1.4338 & 0.5754 & -0.0295 & -0.4043 & 0.2169 \\ 0.5754 & 1.0956 & 0.3401 & 0.2169 & -0.1120 \\ -0.0295 & 0.3401 & 1.1874 & 0.2169 & 0.0478 \\ -0.4043 & 0.2169 & 0.2169 & 0.2831 & -0.1415 \\ 0.2169 & -0.1120 & 0.0478 & -0.1415 & 0.0957 \end{bmatrix},$$

$$y^* = \begin{bmatrix} 0.8585 \\ 1.0937 \\ 0.7831 \end{bmatrix}.$$

the optimal value of both problem is equal to -1.0957.

Starting by the strictly feasible primal and dual solution $X = E$, $S = E$ and $y = (1, 1, 1)$, we apply algorithm 1 on the above mentioned SDO problem with the proposed kernel function $\psi(t)$ and the kernel functions listed in Table 1

Table 1: Considered kernel functions

$\psi_1(t)=\frac{t^2-1}{2} - \log(t)$	[14]
$\psi_2(t)=\frac{t^2-1}{2} - (t-1)e^{\frac{1}{t}-1}$	[20]
$\psi_3(t) = \frac{t^2-1}{2} + \frac{\frac{1}{t}-1}{2} - \frac{t-1}{2}$	[12]
$\psi_4(t) = \frac{t^2-1}{2} + \frac{q^{\frac{1}{t}-1}-1}{q\log q} - \frac{q-1}{q}(t-1),\ q > 1$	[1]
$\psi_5(t) = t - 1 + \frac{t^{1-q}-1}{q-1},\ q > 1$	[2]

The threshold parameter $\tau = 3$, and the accuracy parameter $\epsilon = 10^{-8}$ in all experiments. We compute at each inner iteration a practical step size $\alpha_{\max}$ such that $X + \alpha_{\max}\,\Delta X > 0$ and $S + \alpha_{\max}\,\Delta S > 0$ with $\alpha_{\max} = \rho$ min (α_X, α_S) and $\rho \in (0, 1)$, where

$$\alpha_X = \begin{cases} -\frac{1}{\lambda_{\min}(X^{-1}\Delta X)} & \text{if } \lambda_{\min}(X^{-1}\Delta X) < 0 \\ 1 & \text{if } \lambda_{\min}(X^{-1}\Delta X) \geq 0 \end{cases},$$

and

$$\alpha_S = \begin{cases} -\frac{1}{\lambda_{\min}(S^{-1}\Delta S)} & \text{if } \lambda_{\min}(S^{-1}\Delta S) < 0 \\ 1 & \text{if } \lambda_{\min}(S^{-1}\Delta S) \geq 0 \end{cases}.$$

The number of iterations and the time produced by Algorithm 1 are denoted by 'It' and 'T', respectively. The results of applying Algorithm 1 on our kernel function with ($q = 2$) and different values of θ and different values of the step size α are given in Table 2.

Table 2: Numerical results of ψ

	0.1		0.3		0.5		0.7		0.9	
$\alpha \setminus \theta$	It	T	It	T	It	T	It	T	It	T
0.3	79	0.5612	74	0.3862	71	0.3292	68	0.3157	60	0.4755
0.7	28	0.4453	25	0.2849	26	0.1925	23	0.0998	19	0.2442
0.9	19	0.4513	17	0.3161	16	0.1640	16	0.1925	12	0.0618
$\alpha_{\max}$	15	0.3077	15	0.1651	15	0.1749	16	0.1455	15	0.0710

It is clear from Table 2, that the performance of our kernel function with the practical step size $\alpha_{\max}$ is well promising in solving SDO problems. It provides less number of iterations and reduced calculation time.

To illustrate the numerical behavior of the proposed function against some other kernel functions in the literature, we have implemented Algorithm 1 with these kernel functions and we have resumed the results in Table 3.

Table 3: Numerical results of $\psi_i(t)$ $(1 \leq i \leq 5)$ and $\psi(t)$

	0.1		0.3		0.5		0.7		0.9	
θ	It	T	It	T	It	T	It	T	It	T
$\psi_1(t)$	15	0.4000	15	0.2689	15	0.1479	16	0.1340	15	0.2429
$\psi_2(t)$	15	0.3771	15	0.1182	15	0.2279	16	0.1187	15	0.2024
$\psi_3(t)$	15	0.4007	15	0.1908	15	0.2875	16	0.1412	16	0.1330
$\psi_4(t)$	15	0.3243	15	0.2939	15	0.1970	16	0.2637	16	0.2079
$\psi_5(t)$	15	0.2738	15	0.4743	15	0.4191	16	1.1946	16	0.1319
$\psi(t)$	15	0.3077	15	0.1651	15	0.1749	16	0.1455	15	0.0710

7 Conclusion

In this paper, we have defined a new primal-dual path-following interior-point algorithm for solving the semidefinite optimization problem with full NT-symmetrization scheme and derived the currently best know iteration bound for the algorithm with large-and small-update methods, namely,

$$\mathcal{O}\left(qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right), \text{ and } \mathcal{O}(q^2\sqrt{n} \log \frac{n}{\epsilon}), \text{ where } q > 1 \text{ is a parameter.}$$

The analysis in this paper is new and different from the one using for LO. Several new tools and techniques are derived in this paper. Numerical results shows that the proposed kernel function is well promising and outperforms some existing kernel functions in the literature.

Some interesting topics remain for further research. Firstly, find new kernel functions and insist on their analytical behavior in order to reduce the number of iterations. Secondly, the search directions used in this paper are all based on the NT-symmetrization scheme. It may be possible to design similar algorithms using other symmetrization schemes and to obtain polynomial time iteration bounds. Finally we would like to point out that our result can also be extended in other direction. For instance, the local convergence properties of these algorithms deserves to be investigated. It is also worthwhile to build similar algorithms for convex quadratic, linear complementarity problems and general convex optimization problems

References

[1] M. Achache and N.Tabchouche, *Complexity analysis and numerical implementation of large-update interior-point methods for SDLCP based on a new parametric barrier kernel function,* Optimization **67** (2018), no. 8, 1–20.

[2] Y.Q. Bai, M. El Ghami and C. Roos, *A primal-dual interior-point method for linear optimization based on a new proximity function*, Optim. Meth. Software **17** (2002), no. 6, 985–1008.

[3] Y.Q. Bai, M. El Ghami and C. Roos, *A comparative study of kernel function for primal-dual interior point algorithms in linear optimization*, SIAM J. Optim. **15** (2004), no. 1, 101–128.

[4] H. Barsam and H. Mohebi, *Characterizations of upward and downward sets in semimodules by using topical functions*, Numer. Funct. Anal. Optim. **37** (2016), no. 11, 1354-1377.

[5] A. Benhadid, K. Saoudi and F. Merahi, *An interior point approach for linear complementarity problem using new parametrized kernel function*, Optimization **71** (2022), no. 15, 4403–4422.

[6] A. Benhadid and F. Merahi, *Complexity analysis of an interior-point algorithm for linear optimization based on a new parametric kernel function with a double barrier term*, Numer. Algebra Control Optim. **13** (2023), no. 2, 224–238.

[7] M. Bouafia, D. Benterki and A. Yassine, *An efficient parameterized logarithmic kernel function for linear optimization,* Optim. Lett. **12** (2018), no. 5, 1079-1097.

[8] E. De Klerk, *Aspects of Semidefinite Programming*, Applied Optimization. vol. 65. Kluwer Academic, Dordrecht, 2002.

[9] L. Derbal and Z Kebbiche, *Theoretical and numerical result for linear optimization problem based on a new kernel function*, J. Sib. Federal Univer. Math. Phys. **12** (2019), no. 2, 160–172.

[10] M. El Ghami, *Primal-dual algorithms for semidefinite optimization problems based on generalized trigonometric barrier function*, Int. J. Pure Appl. Math. **114** (2017), no. 4, 797–818.

[11] R.A. Horn and R.J. Charles, *Matrix Analysis*, Compridge University Press, UK, 1986.

[12] J. Peng, C. Roos and T. Terlaky, *Self-regularity: A New Paradigm for Primal-dual Interior-point Algorithms*, New Jersey, Princeton University Press, 2002.

[13] J. Peng, C. Roos and T. Terlaky, *Self-regular proximities and new search directions for linear and semidefinite optimization*, Math. Program.. **93** (2002), no. A, 129–171.

[14] C. Roos, T. Terlaky and J.P. Vial, *Theory and algorithms for linear optimization: An interior point approach*, John Wiley & Sons, Chichester, UK, 1997.

[15] M.J. Todd, K.C. Toh and R.H. Tütüncü, *The Nesterov-Todd direction in semidefinite programming, mathematical programming*, SIAM J. Optim. **8** (1998), no. 3, 769–796.

[16] M.J. Todd, *A study of search directions in primal-dual interior-point methods for semidefinite programming*, Optim. Meth. Softw. **11** (1999), 1–46.

[17] G.Q. Wang, Y.Q. Bai and C. Roos, *Primal-dual interior point algorithms for semidefinite optimization based on a simple kernel function*, J. Math. Model. Algorithms **4** (2005), no. 4, 409–433.

[18] G.Q. Wang and Y.Q. Bai, *A class of polynomial primal-dual interior-point algorithms for semidefinite optimization*, J. Shanghai University (English Edition) **10** (2006), no. 3, 198–207.

[19] H. Wolkowicz, R. Saigal and L. Vandenberghe, *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, Dordrecht, Kluwer Academic Publishers, 2000.

[20] M.W. Zhang, *A large-update interior-point algorithm for convex quadratic semi-definite optimization based on a new kernel function*, Acta Math. Sinica English Ser. **28** (2012), no. 11, 2313–2328.