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Asymptotic behavior of a radical quadratic functional equation in quasi-β-Banach spaces

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Abstract

Let $\mathbb{R}$ be the set of real numbers and $\big(Y, \|\cdot\|\big)$ be a real quasi-β-Banach space. In this paper, we prove the Hyers-Ulam stability on a restricted domain in quasi-β-spaces for the following two radical functional equations

$$f\big(\sqrt{x^2+y^2}\big) = f(x) + f(y)$$

and

$$f\big(\sqrt{x^2+y^2}\big) = g(x) + f(y)$$

where $f, g : \mathbb{R} \to Y$. Also, we discuss an asymptotic behavior for these equations.

Keywords: radical functional equation, Hyers-Ulam stability, quasi-β-normed spaces, restricted domain
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1 Introduction

When defining, in some way, a class of approximate solutions of a given functional equation, one can ask if each mapping from this class can be approximated in some way by an exact solution of the considered equation. Specifically, when a functional equation is replaced with an inequality that serves as a perturbation of the considered equation. S. M. Ulam proposed the first functional equation stability problem in 1940 [23].

Ulam's problem:

Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \varepsilon$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \delta$$

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for all $x \in G_1$*?*

We say that the homomorphism equation $h(x *_1 y) = h(x) *_2 h(y)$ is stable if the answer is affirmative. Many researchers have interested in this issue since then. In 1941, D. H. Hyers [9] offered a first partial response to Ulam's problem, presenting the stability result as follows:

Theorem 1.1. [9] Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a function such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

T. Aoki [1] and D. G. Bourgin [2] investigated the stability problem with unbounded Cauchy variations. Th. M. Rassias [16] used a direct method to prove a generalization of Theorem 1.1 by weakening the condition for the bound of the norm of Cauchy difference.

Theorem 1.2. [16] Let E_1 and E_2 be two Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $0 \leq p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Theorem 1.2 was then modified and improved by Th. M. Rassias [17],[18] as follows:

Theorem 1.3. [17],[18] Let E_1 be a normed space, E_2 be a Banach space, and $f : E_1 \to E_2$ be a function. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

When $p = 0$, Theorem 1.3 is reduced to Theorem 1.1. The equivalent result is not valid for $p = 1$. A number of authors have studied the stability problems of many functional equations in-depth, and there are many interesting findings to be found (see, for instance, [5, 10, 19, 20] and references therein).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is referred to as *a quadratic functional equation*. A quadratic mapping is defined as a solution of the quadratic functional equation. In 1983, F. Skof [21] proved a generalized Hyers-Ulam stability problem for the quadratic functional equation for mappings $f : E \to F$, where E is a normed space and F is a Banach space.

P. W. Cholewa [3] proved that the Skof's result is still true if the relevant domain E is replaced by an abelian group. There are various interesting results which deal with the stability of functional equations in restricted domains

[4, 5, 11, 13, 14, 15]. In 2004, J. Tabor [22] presented and proved a version of the Hyers-Rassias-Gajda stability in quasi-Banach spaces.

In this paper, we discuss the Hyers-Ulam stability on restricted domain in quasi-β-normed spaces for the following two equations of these equations.

$$f\big(\sqrt{x^2+y^2}\big) = f(x) + f(y) \tag{1.4}$$

and

$$f\big(\sqrt{x^2+y^2}\big) = g(x) + f(y) \tag{1.5}$$

where $f, g : \mathbb{R} \to Y$ are functions such that Y is a quasi-β-Banach space, also we obtain an asymptotic behavior for them. Some basic facts about quasi-β-normed spaces must be remembered.

Definition 1.4. Let β be a fixed real number with $0 < \beta \leq 1$, and $\mathbb{K}$ be either $\mathbb{R}$ *or* $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
2. $\|\lambda x\| = |\lambda^\beta| \cdot \|x\|$, for all $x \in X$ and $\lambda \in \mathbb{K}$,
3. There is a constant $\mathcal{K} \geq 1$ such that $\|x + y\| \leq \mathcal{K}(\|x\| + \|y\|)$, for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible $\mathcal{K}$ is called *the module of concavity* of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

2 Stability results for Eq. (1.4)

Let $\big(Y, \|\cdot\|\big)$ be a quasi-β-Banach space. In 2012, Kim et al. [12] gave the Hyers-Ulam stability for Eq. (1.4) in quasi-β-normed spaces as follows:

Theorem 2.1. [12] Let $\varepsilon \geq 0$. If a function $f : \mathbb{R} \to Y$ such that $f(0) = 0$ and satisfies the following inequality

$$\|f(\sqrt{x^2+y^2}) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in \mathbb{R}$, then there exists a unique quadratic function $F : \mathbb{R} \to Y$ satisfying Eq. (1.4) and the following inequality

$$\|f(x) - F(x)\| \leq \frac{2\mathcal{K}\varepsilon}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta$$

for all $x \in \mathbb{R}$.

In the following theorem, we present an investigation of the Hyers-Ulam stability for Eq. (1.4) on restricted domain in quasi-β-normed spaces.

Theorem 2.2. Let $d > 0$ and $\varepsilon \geq 0$ be fixed. If a mapping $f : \mathbb{R} \to Y$, such that $f(0) = 0$, satisfies the following functional inequality

$$\left\| f\left(\sqrt{x^2+y^2}\right) - f(x) - f(y) \right\| \leq \varepsilon \tag{2.1}$$

for all $(x, y) \in \mathbb{R}^2$ with $|x| + |y| \geq d$, then there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.4) satisfying the following inequality

$$\|f(x) - F(x)\| \leq \frac{2\mathcal{K}^2(2\mathcal{K}+1)\,\varepsilon}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta, \tag{2.2}$$

for all $x \in \mathbb{R}$.

Proof . We consider the difference operator $D_f : \mathbb{R}^2 \to Y$ defined as:

$$D_f(x, y) := f\left(\sqrt{x^2+y^2}\right) - f(x) - f(y), \quad x, y \in \mathbb{R}.$$

We observe that

$$\begin{aligned} D_f(x,y) =& f\big(\sqrt{x^2+y^2}\big) + f(t) - f\big(\sqrt{x^2+y^2+t^2}\big) + f\big(\sqrt{x^2+y^2+t^2}\big) - f\big(\sqrt{x^2+t^2}\big) - f(y) \\ &+ f\big(\sqrt{x^2+t^2}\big) - f(t) - f(x) \\ =& -D_f\big(\sqrt{x^2+y^2}, t\big) + D_f\big(\sqrt{x^2+t^2}, y\big) + D_f(x,t), \end{aligned}$$

for all $x, y, t \in \mathbb{R}$. Assume that $|x| + |y| < d$ and let $t \in \mathbb{R}$ such that $|t| = d$. Therefore, we note

$$\sqrt{x^2+y^2} + |t| \geq d,$$

$$\sqrt{x^2+t^2} + |y| \geq d$$

and

$$|x| + |t| \geq d.$$

Using the definition of D_f, we obtain

$$\|D_f\big(\sqrt{x^2+y^2}, t\big)\| \leq \varepsilon, \;\; \|D_f(\sqrt{x^2+t^2}, y)\| \leq \varepsilon, \;\; \|D_f(x,t)\| \leq \varepsilon,$$

for all $x, y \in \mathbb{R}$. Thus, using the triangle inequality, we get

$$\big\|D_f(x,y)\big\| \leq \mathcal{K}(2\mathcal{K}+1)\,\varepsilon \tag{2.3}$$

for all $x, y \in \mathbb{R}$. According to Theorem 2.1, there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.4) and the following inequality

$$\|f(x) - F(x)\| \leq \frac{2\mathcal{K}^2(2\mathcal{K}+1)\,\varepsilon}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta, \tag{2.4}$$

for all $x \in \mathbb{R}$. □

In view of Theorem 2.2, we get the following corollary.

Corollary 2.3. Suppose that $f : \mathbb{R} \to Y$ such that $f(0) = 0$ and satisfying the equation

$$f\left(\sqrt{x^2+y^2}\right) - f(x) - f(y) = 0 \tag{2.5}$$

for all $(x, y) \in \mathbb{R}^2$ with $|x| + |y| \geq d$. Then, the equation (2.5) holds for all $x, y \in \mathbb{R}$.

Let us define a set B as $B := \big\{(x,y) \in \mathbb{R}^2 : |x| < d \text{ and } |y| < d\big\}$ for some $d > 0$. In view of the fact that

$$\big\{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq 2d\big\} \subset \mathbb{R}^2 - B,$$

we deduce that the following corollary is a direct consequence of Theorem 2.2.

Corollary 2.4. Assume that a mapping $f : \mathbb{R} \to Y$ with $f(0) = 0$ satisfies the inequality (2.1) for all $(x, y) \in \mathbb{R}^2 - B$ and some $\varepsilon \geq 0$. Then there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.4) that satisfies the inequality (2.2).

In the following corollary, we give the asymptotic behavior of Eq. (1.4).

Corollary 2.5. Suppose that $f : \mathbb{R} \to Y$ with $f(0) = 0$ satisfies the condition

$$\big\|f\big(\sqrt{x^2+y^2}\big) - f(x) - f(y)\big\| \to 0, \quad as \quad |x| + |y| \to \infty. \tag{2.6}$$

Then f is a solution of Eq. (1.4).

Proof . Due to the asymptotic condition (2.6), there exists a strictly positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ monotonically decreasing to 0 such that

$$\left\|f\left(\sqrt{x^2+y^2}\right)-f(x)-f(y)\right\|\leq\varepsilon_n, \tag{2.7}$$

for all $x,y\in\mathbb{R}$ with $|x|+|y|>n$. Hence, it follows from (2.7) and Theorem 2.2 that there exists a unique solution $F_n:\mathbb{R}\to Y$ of Eq. (1.4) such that

$$\left\|f(x)-F_n(x)\right\|\leq\frac{2\mathcal{K}^2(2\mathcal{K}+1)\,\varepsilon_n}{2^\beta-\mathcal{K}},\quad \mathcal{K}<2^\beta, \tag{2.8}$$

for all $x\in\mathbb{R}$. Let $l,m\in\mathbb{N}$ such that $m\geq l$. Since $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a monotonically decreasing to 0 and in view of (2.8), we obtain

$$\begin{aligned}\left\|f(x)-F_m(x)\right\|&\leq\frac{2\mathcal{K}^2(2\mathcal{K}+1)\,\varepsilon_m}{2^\beta-\mathcal{K}}\\&\leq\frac{2\mathcal{K}^2(2\mathcal{K}+1)\varepsilon_l}{2^\beta-\mathcal{K}},\quad \mathcal{K}<2^\beta,\end{aligned}$$

for all $x\in\mathbb{R}$. Then the uniqueness of F_n implies that $F_m=F_l$. Hence, letting $n\to\infty$ in (2.8), we deduce that $f=F_m$ which satisfies Eq. (1.4). □

3 Stability results for Eq. (1.5)

In this section, we give the Hyers-Ulam stability for the functional equation (1.5) on restricted domain in quasi-β-normed spaces.

Theorem 3.1. Let $\varepsilon\geq 0$. If the functions $f,g:\mathbb{R}\to Y$, with $f(0)=0$, satisfy the following inequality

$$\left\|f\left(\sqrt{x^2+y^2}\right)-g(x)-f(y)\right\|\leq\varepsilon, \tag{3.1}$$

for all $x,y\in\mathbb{R}$, then there exists a unique solution $F:\mathbb{R}\to Y$ of Eq. (1.4) such that satisfies the following two inequalities

$$\left\|f(x)-F(x)\right\|\leq\frac{2\mathcal{K}^2(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon}{2^\beta-\mathcal{K}},\quad \mathcal{K}<2^\beta$$

and

$$\left\|g(x)-F(x)\right\|\leq\frac{2\mathcal{K}^3(2\mathcal{K}^2+\mathcal{K}+1)\varepsilon}{2^\beta-\mathcal{K}}+2\mathcal{K}^2(\mathcal{K}+1)\,\varepsilon,\quad \mathcal{K}<2^\beta$$

for all $x\in\mathbb{R}$.

Proof . Letting $x=y=0$ in (3.1), we get

$$\left\|g(0)\right\|\leq\varepsilon. \tag{3.2}$$

Setting $x=0$ and $y=x$ in (3.1), we have

$$\left\|f\left(|x|\right)-g(0)-f(x)\right\|\leq\varepsilon,\quad x\in\mathbb{R}. \tag{3.3}$$

Putting $y=0$ in (3.1), we obtain

$$\left\|f\left(|x|\right)-g(x)\right\|\leq\varepsilon,\quad x\in\mathbb{R}. \tag{3.4}$$

So, it follows from (3.1), (3.2), (3.3) and (3.4) that

$$\left\|f\left(\sqrt{x^2+y^2}\right)-f(x)-f(y)\right\|\leq\mathcal{K}(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon,\quad x,y\in\mathbb{R}. \tag{3.5}$$

According to Theorem 2.1, there exists a unique solution $F:\mathbb{R}\to Y$ of Eq. (1.4) such that satisfies the following inequality

$$\left\|f(x)-F(x)\right\|\leq\frac{2\mathcal{K}^2(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon}{2^\beta-\mathcal{K}},\ \ \mathcal{K}<2^\beta$$

for all $x\in\mathbb{R}$. Thus, from the last inequality and in view of (3.2), (3.3) and (3.4), we conclude that

$$\left\|g(x)-F(x)\right\|\leq\frac{2\mathcal{K}^3(2\mathcal{K}^2+\mathcal{K}+1)\varepsilon}{2^\beta-\mathcal{K}}+2\mathcal{K}^2(\mathcal{K}+1)\,\varepsilon,\quad \mathcal{K}<2^\beta,$$

for all $x\in\mathbb{R}$. □

Theorem 3.2. Let $d > 0$ and $\varepsilon \geq 0$ be fixed. If the functions $f, g : \mathbb{R} \to Y$ such that $f(0) = 0$ satisfy the functional inequality

$$\left\| f\left(\sqrt{x^2 + y^2}\right) - g(x) - f(y) \right\| \leq \varepsilon \tag{3.6}$$

for all $(x, y) \in \mathbb{R}^2$ with $|x| + |y| \geq d$. Then there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.5) and satisfies the following inequalities

$$\left\| f(x) - F(x) \right\| \leq \frac{4\mathcal{K}^3(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta$$

and

$$\left\| g(x) - F(x) \right\| \leq \frac{4\mathcal{K}^4(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon}{2^\beta - \mathcal{K}} + 4\mathcal{K}3(\mathcal{K}+1)2\,\varepsilon, \quad \mathcal{K} < 2^\beta$$

for all $x \in \mathbb{R}$.

Proof . Let us consider the difference operator $C : \mathbb{R}^2 \to Y$ defined as:

$$C(x, y) = f\left(\sqrt{x^2 + y^2}\right) - g(x) - f(y),$$

for all $x, y \in \mathbb{R}$. Notice that

$$\begin{aligned} C(x, y) =& f\left(\sqrt{x^2 + y^2}\right) + g(t) - f\left(\sqrt{x^2 + y^2 + t^2}\right) + f\left(\sqrt{x^2 + y^2 + t^2}\right) - f\left(\sqrt{y^2 + t^2}\right) - g(x) \\ &+ f\left(\sqrt{y^2 + t^2}\right) - g(t) - f(y) \\ =& - C\left(\sqrt{x^2 + y^2}, t\right) + C\left(\sqrt{y^2 + t^2}, x\right) + C(y, t), \end{aligned}$$

for all $x, y \in \mathbb{R}$. Assume that $|x| + |y| < d$ and let $t \in \mathbb{R}$ such that $|t| = d$. So,

$$\sqrt{x^2 + y^2} + |t| \geq d,$$

$$\sqrt{y^2 + t^2} + |x| \geq d$$

and

$$|y| + |t| \geq d$$

for all $x, y, t \in \mathbb{R}$. This implies that

$$\left\| C\left(\sqrt{x^2 + y^2}, t\right) \right\| \leq \varepsilon, \;\; \left\| C\left(\sqrt{y^2 + t^2}, x\right) \right\| \leq \varepsilon, \;\; \left\| C(y, t) \right\| \leq \varepsilon.$$

for all $x, y, t \in \mathbb{R}$. Using the triangle inequality, we get

$$\left\| C(x, y) \right\| \leq 2\mathcal{K}(\mathcal{K}+1)\varepsilon \tag{3.7}$$

for all $x, y \in \mathbb{R}$. Now, according to Theorem 3.1, there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.5) such that satisfies the following inequalities

$$\left\| f(x) - F(x) \right\| \leq \frac{4\mathcal{K}^3(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\,\varepsilon}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta$$

and

$$\left\| g(x) - F(x) \right\| \leq \frac{4\mathcal{K}^4(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\varepsilon}{2^\beta - \mathcal{K}} + 4\mathcal{K}^3(\mathcal{K}+1)^2\varepsilon, \;\; \mathcal{K} < 2^\beta$$

for all $x \in \mathbb{R}$. □

Corollary 3.3. Suppose that $f, g : \mathbb{R} \to Y$ be two functions, with $f(0) = 0$, satisfy the equation

$$f\left(\sqrt{x^2 + y^2}\right) - g(x) - f(y) = 0 \tag{3.8}$$

for all $(x, y) \in \mathbb{R}^2$ with $|x| + |y| \geq d$. Then, the functional equation (3.8) holds for all $x, y \in \mathbb{R}$.

Let us define the set B as

$$B := \{(x,y) \in \mathbb{R}^2 : |x| < d \text{ and } |y| < d\}$$

for some $d > 0$. Indeed, we have

$$\{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq 2d\} \subset \mathbb{R}^2 - B.$$

Then, we present the following corollary as a direct consequence of Theorem 3.2.

Corollary 3.4. Assume that a mapping $f : \mathbb{R} \to Y$ such that $f(0) = 0$ and satisfies the inequality (3.6) for all $(x,y) \in \mathbb{R}^2 - B$ and some $\varepsilon \geq 0$. Then there exists a unique solution $F : \mathbb{R} \to Y$ of Eq. (1.5) that satisfies the inequality (2.2).

By similar method of the proof of Corollary 2.5, we can prove the following corollary.

Corollary 3.5. Suppose that $f, g : \mathbb{R} \to Y$ be two functions, with $f(0) = 0$, satisfy the condition

$$\left\| f\left(\sqrt{x^2+y^2}\right) - g(x) - f(y)\right\| \to 0, \ \ as \ |x| + |y| \to \infty. \tag{3.9}$$

Then f, g satisfy the functional equation (1.5).

Proof . From the condition (3.9), we get that there exists a strictly positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ monotonically decreasing to 0 such that

$$\left\| f\left(\sqrt{x^2+y^2}\right) - g(x) - f(y)\right\| \leq \varepsilon_n \tag{3.10}$$

for all $x, y \in \mathbb{R}$ with $|x| + |y| > n$. Hence, it follows from (3.10) and Theorem 3.2 that there exists a unique solution $F_n : \mathbb{R} \to Y$ of Eq. (1.5) such that

$$\left\| f(x) - F_n(x)\right\| \leq \frac{4\mathcal{K}^3(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_n}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta \tag{3.11}$$

and

$$\left\| g(x) - F_n(x)\right\| \leq \frac{4\mathcal{K}^4(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_n}{2^\beta - \mathcal{K}} + 4\mathcal{K}^3(\mathcal{K}+1)^2\ \varepsilon_n, \quad \mathcal{K} < 2^\beta \tag{3.12}$$

for all $x \in \mathbb{R}$. Let $l, m \in \mathbb{N}$ such that $m \geq l$. Since $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a monotonically decreasing to 0 and in view of (3.11) and (3.12), we get

$$\begin{aligned}\left\| f(x) - F_m(x)\right\| &\leq \frac{4\mathcal{K}^3(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_m}{2^\beta - \mathcal{K}} \\ &\leq \frac{4\mathcal{K}^3(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_l}{2^\beta - \mathcal{K}}, \quad \mathcal{K} < 2^\beta\end{aligned}$$

and

$$\begin{aligned}\left\| g(x) - F_m(x)\right\| &\leq \frac{4\mathcal{K}^4(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_m}{2^\beta - \mathcal{K}} + 4\mathcal{K}^3(\mathcal{K}+1)^2\ \varepsilon_m \\ &\leq \frac{4\mathcal{K}^4(\mathcal{K}+1)(2\mathcal{K}^2+\mathcal{K}+1)\ \varepsilon_l}{2^\beta - \mathcal{K}} + 4\mathcal{K}^3(\mathcal{K}+1)^2\ \varepsilon_l, \quad \mathcal{K} < 2^\beta.\end{aligned}$$

Then the uniqueness of F_n implies that $F_m = F_l$. Hence, letting $n \to \infty$ in (3.11) and (3.12), we deduce that $f = g = F_m$ which satisfies Eq. (1.5). □

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[3] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.

[4] J. Chung, D. Kim and J.M. Rassias, *Stability of Jensen-type functional equations on restricted domains in a group and their asymptotic behaviors*, J. App. Math. **2012** (2012), Article ID 691981.

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[6] M.E. Gordji and H. Khodaei, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, Nonlinear Anal. **71** (2009), 5629–5643.

[7] M.E. Gordji and M. Parviz, *On the Hyers–Ulam stability of the functional equation* $f\left(\sqrt[2]{x^2+y^2}\right) = f(x) + f(y)$, Nonlinear Funct. Anal. Appl. **14** (2009), 413–420.

[8] M.E. Gordji, H. Khodaei, A. Ebadian and G.H. Kim, *Nearly radical quadratic functional equations in p-2-normed spaces*, Abstr. Appl. Anal. **2012** (2012), Article ID 896032.

[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[10] K. Jun and H. Kim, *On the stability of an n-dimensional quadratic and additive functional equation*, Math. Inequal. Appl. **9** (2006), 153–165.

[11] S.M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), 126–137.

[12] S.S. Kim, Y.J. Cho and M.E. Gordji, *On the generalized Hyers-Ulam-Rassias stability problem of radical functional equations*, J. Inequal. Appl. **2012** (2012), Article ID 186.

[13] J.M. Rassias, *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl. **276** (2002), 747–762.

[14] J.M. Rassias and M.J. Rassias, *On the Ulam stability of Jensen and Jensen type mappings on restricted domains*, J. Math. Anal. Appl. **281** (2003), 516–524.

[15] J.M. Rassias and M.J. Rassias, *Asymptotic behavior of alternative Jensen and Jensen type functional equations*, Bull. Sci. Math. **129** (2005), 545–558.

[16] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[17] Th.M. Rassias, *Problem 16; 2. Report of the 27th international symposium on functional equations*, Aequationes Math. **39** (1990), 292–293.

[18] Th.M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl. **158** (1991), 106–113.

[19] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai Math. **XLIII** (1998), 89–124.

[20] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta. Appl. Math. **62** (2000), no. 1, 23–130.

[21] F. Skof, *Proprietà locali e approssimazione di operatori Rend*, Semin. Mat. Fis. Milano **53** (1983), 113–129.

[22] J. Tabor, *Stability of Cauchy functional equation in quasi-Banach spaces*, Ann. Pol. Math. **83** (2004), 243–255.

[23] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, John-Wiley & Sons Inc., New York, 1964.