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Solving fractional two-dimensional nonlinear weakly singular partial integro-differential equation by using Fibonacci polynomials

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Abstract

The main aim of the present study is to expand the operational matrix method for solving the fractional two-dimensional nonlinear weakly singular partial integro-differential equations. To do this, firstly, we use and present the operational matrix of fractional integration of two-dimensional Fibonacci polynomials. Then, by using the obtained operational matrices to approximate the fractional derivative of the solution of the considered equation, we convert the original problem to a nonlinear system of equations. Also, we present the error analysis of the proposed method by a theorem. Finally, we present and solve some numerical examples to illustrate the proposed method.

Keywords: Singular integral equation, Two-dimensional Fibonacci polynomials, Operational matrix, Fractional calculus
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1 Introduction

Differential equations as well as fractional integro-differential equations (FIDEs) has many applications in various sciences such as engineering, chemistry, physics and biology [2, 26, 34]. Due to the complexity of determining the analytical solution of these equations, researchers had to develop the numerical methods to solve them. So, recently, many researchers presented some numerical methods for solving fractional integral equations (FIEs) and FIDEs by the help of wavelets and orthogonal polynomials [1, 14, 20, 27, 37, 38].

Numerical methods based on operational matrices (OMs) have recently been considered by researchers as one of the efficient methods for solving various linear or nonlinear problems such as FIEs and FIDEs. In these methods, the principal problem becomes a system of linear or nonlinear equations. To see some proposed operational matrices methods for solving FIEs and FIDEs, one can refer to [11, 16, 39, 40, 41].

Solving partial differential equations has long been of interest to authors. Due to the widespread use of weakly singular fractional order partial integro-differential equations (WSFPIDEs), many researchers have proposed different numerical methods to solve these equations. Among the proposed methods, we can mention to [12, 13, 18, 22, 32, 33].

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Recently, the use of polynomials as well as fractional-order functions such as Euler and Fibonachi polynomials to solve FIDEs and WSFPIDEs has been considered by researchers. In [28], the authors applied fractional-order euler polynomials for solving weakly singular fractional-order delay integro-differential equations. The authors of [31] used Fibonachi wavelets to solve two classes of time-varying delay problems. Solving time-fractional Telegraph equations with Dirichlet boundary conditions by Fibonachi wavelet is done in [35]. In [6], the authors proposed a numerical method based on finite-difference Fibonachi collocation method to solve two-dimensional fractional-order reaction advection sub-diffusion equation. To study the other methods, one can refer to [8, 9, 30, 36].

The operational matrix (OM) methodology was also applied by Maleknejad to solve the previous equation [17]. Moreover, Babolain et al. recently utilized TFs for the same purpose [3]. Nema et al. utilized Legendre polynomials to solve the Volterra equation within a two-dimensional space [21]. Chebyshev wavelets were constructed by Baghani et al. to provide an equation solving approach for finite-time quadratic fractional control equations [4].

The method of Fibonacci wavelet has been proposed by [6]. J. Sing to solve the equation of fractional advection sub-diffusion. In addition, To solve the pennies bioheat transfer equation, the Fibonacci wavelets (FWs) operational matrix has been employed [8, 9]. The method of FWs to solve telegraph equations [31] and [35] was used by S. Shiralashetti, respectively. Furthermore, the Fibonacci wavelet in [36] Rahimkhani and [30] was presented for fractional-order and solving optimal control problems, respectively. In order to define the integrating the OM for which defined properly by some kind of orthogonal function such as Laguerre series [10], Triangular functions [12], and Block-pulse Functions [7], Bernoulli wavelets [16], Chebyshev polynomials [37], and Legendre polynomials [15]. Fibonacci operational was utlized for solving the fractional initial value problems by researchers [24]. Boubaker function was applied to solve the problems of optimal control and delay optimal control by [24] and [25], respectively. Moreover, [5] Davaeifar and Ordokhani [23] have been applied the Boubaker function to solve Volterra-Fredholm integral equations systems and pantograph delay differential equation, respectively. M. Yi and J. Huang used the OM of wavelet in [39] to approximate the FDEs solution. In addition, F. Mirzaee al., applied such an approach to solving Volterra-Fredholme [19]. Many works exist. There are various researches regarding the fractional Volterra integral equation, such as Bernstein polynomials [41], Wavelets method [38], Bernstein polynomials method [18], Block-pulse function [7], and Collocation method [22]. In some advanced applications, the two-dimensional fractional weakly singular partial integro-differential equation (2DFWSPIDE) is common. For instance, in physics, and especially in the plasma field, traces of these equations can be seen. In particular, mathematicians have down some of this research [32]. The following 2DFWSPIDE system is considered in this article.

$$D_{\iota}^{\theta}u(\epsilon,\iota) = u(\epsilon,\iota) + g(\epsilon,\iota) + \int_0^{\epsilon}\int_0^{\iota}\frac{H(u(s,y))}{(\epsilon-s)^{\alpha}}dyds, \tag{1.1}$$

where $u(\epsilon,\iota)$ is an unknown function. The known functions $H(u(s,y))$, $g(\epsilon,\iota)$ and $u_0(\epsilon)$ are defined on interval $\Omega=[0,1]\times[0,1]$. Also, D_{ι}^{θ} denotes the Caputo fractional derivate. $(\epsilon,\iota)\in\Omega=[0,1]\times[0,1]$, $0<\alpha<1, 0<\theta\leq 1$ and with the initial conditionals $u(\epsilon,0)=u_0(\epsilon)$, of $u(\epsilon,\iota)$ respect to variable ι of order θ.

This paper is organized as follow: Fractional derivatives basis concepts are provided in section 2. Some necessary Fibonacci polynomials properties, fractional-order derivatives, the integration of operational matrix, and other operational matrices are presented in section 3, and section 4 describes the method. Finally, the accuracy and efficiency of the proposed scheme using numerical solutions for some examples are demonstrated in Section 5. The tables show the results obtained from the performance of the technique to solve the related examples. These conclusions confirm the validity of the proposed solution.

2 Preliminaries

In this section, we recall the fundamental characteristics and definitions of the fractional integral and derivative.

Definition 2.1. [40]. The Riemann-Liouville fractional integral operator I^{γ_1} of order $\gamma_1\geq 0$, of a function $u\in C_{\mu}, \mu\geq 1$, is defined as

$$(I^{\gamma_1})u(\epsilon) = \begin{cases} \frac{1}{\Gamma(\gamma_1)}\int_0^{\epsilon}\frac{f(s)}{(\epsilon-s)^{1-\gamma_1}}ds, & s>0, \\ u(\epsilon), & \gamma_1=0. \end{cases}$$

For $\gamma_2\geq -1$, we have

$$I^{\gamma_1}\epsilon^{\gamma_2} = \frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+\gamma_1+1)}\epsilon^{\gamma_1+\gamma_2}.$$

Definition 2.2. [20]. The Caputo partial fractional derivative of $u(\epsilon,\iota)$ with respect to ϵ of order $\gamma_1>0$ is define as

$$({}^cD_\epsilon^{\gamma_1}u)(\epsilon,\iota)=\frac{\partial^{\gamma_1}u(\epsilon,\iota)}{\partial\epsilon^{\gamma_1}}=\begin{cases}\frac{1}{\Gamma(n-\gamma_1)}\int_0^\epsilon\frac{\partial^n u(s,t)}{\partial s^n}\frac{ds}{(\epsilon-s)^{\gamma_1+1-n}}, & n-1<\gamma_1<n, \quad n\in N,\\ \frac{\partial^n u(\epsilon,\iota)}{d\epsilon^n}, & \gamma_1=n.\end{cases}$$

Definition 2.3. [40] The mixed Caputo partial fractional derivative of order (γ_1,γ_2), $\gamma_1,\gamma_2>0$ is defined as

$$({}^cD_{\epsilon,\iota}^{\gamma_1,\gamma_2}u)(\epsilon,\iota)=\frac{\partial^{\gamma_1+\gamma_2}u(\epsilon,\iota)}{\partial\epsilon^{\gamma_1}\partial\iota^{\gamma_2}}=\frac{1}{\Gamma(m-\gamma_1)\Gamma(n-\gamma_2)}\int_0^\epsilon\int_0^\iota\frac{\partial^{n+m}u(s,y)}{\partial s^n\partial y^m}\frac{dyds}{(\epsilon-s)^{\gamma_1+1-m}(\iota-y)^{\gamma_2+1-n}},$$

for $m-1<\gamma_2<m$, $n-1<\gamma_1<n$.

Lemma 2.4. [20]. If $n-1<\gamma_1\le n$, $n\in\mathbb{N}$, then $D_\epsilon^{\gamma_1}I^{\gamma_1}u(\epsilon,\iota)=u(\epsilon,\iota)$, and:

$$I^{\gamma_1}D_\epsilon^{\gamma_1}u(\epsilon,\iota)=u(\epsilon,\iota)-\sum_{r=0}^{n-1}\frac{\partial^r u(0^+,t)}{\partial\epsilon^r}\frac{\epsilon^r}{r!},\quad \epsilon>0.$$

3 Fibonacci polynomials (FPs)

The FPs can be written as [31]

$$\tilde{E}_n(\epsilon)=\begin{cases}1, & n=0,\\ \epsilon, & n=1,\\ \epsilon\tilde{E}_{n-1}(\epsilon)+\tilde{E}_{n-2}(\epsilon), & n>1.\end{cases}$$

The closed form of FPs are as follows:

$$\tilde{E}_n(\epsilon)=\frac{\beta^n-\gamma^n}{\beta-\gamma}, \tag{3.1}$$

where β and γ are the roots of the companion polynomial $\alpha^2-\epsilon\alpha-1$ of the recursion. The FPs can be represented in the power form as [31]

$$\tilde{E}_n(\epsilon)=\sum_{p=0}^{\lfloor\frac{n}{2}\rfloor}\begin{pmatrix}n-i\\ i\end{pmatrix}\epsilon^{n-2i},\quad n\ge 0.$$

Suppose that $\tilde{E}(\epsilon)=[\tilde{E}_0(\epsilon),\tilde{E}_1(\epsilon),\tilde{E}_2(\epsilon),...,\tilde{E}_n(\epsilon)]^T$. For the matrix form of $\tilde{E}(\epsilon)$, we have:

$$\tilde{E}(\epsilon)=RT_n(\epsilon), \tag{3.2}$$

where

$$T(\epsilon)=[1,\epsilon,\epsilon^2,\epsilon^3,...,\epsilon^n]^T, \tag{3.3}$$

and

$$R_n=\begin{pmatrix}1&0&0&0&0&0&0&\cdots\\0&1&0&0&0&0&0&\cdots\\1&0&1&0&0&0&0&\cdots\\0&2&0&1&0&0&0&\cdots\\1&0&3&0&1&0&0&\cdots\\0&3&0&4&0&1&0&\cdots\\1&0&6&0&5&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

For the FPs, in [31], the following result is proved

$$\int_0^1\tilde{E}_n(\epsilon)\tilde{E}_m(\epsilon)d\epsilon=\sum_{p=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{q=0}^{\lfloor\frac{m}{2}\rfloor}\begin{pmatrix}n-i\\ i\end{pmatrix}\begin{pmatrix}m-i\\ i\end{pmatrix}\frac{1}{n+m-2i-2j+1}.$$

3.1 Function approximation

A function $u(\epsilon) \in L^2([0,1])$ can be expanded in terms of FPs as follows:

$$u(\epsilon) \approx u_n(\epsilon) = \sum_{i=0}^{n} b_i \tilde{E}_i(\epsilon) = b^T \tilde{E}(\epsilon) = \tilde{E}^T(\epsilon) b, \tag{3.4}$$

where

$$b = [b_0, b_1, ..., b_n,]^T. \tag{3.5}$$

To expand $u(\epsilon, \iota)$ in terms of two-dimensional fibonacci polynomials (2DFPs), first, we define 2DFPs as follows:

$$\tilde{E}_{i,j}(\epsilon, \iota) = \tilde{E}_i(\epsilon)\tilde{E}_j(\iota). \tag{3.6}$$

Now, we define

$$\Phi(\epsilon, \iota) = [\tilde{E}_{0,0}(\epsilon, \iota), \tilde{E}_{0,1}(\epsilon, \iota), ..., \tilde{E}_{0,m}(\epsilon, \iota), \tilde{E}_{1,0}(\epsilon, \iota), ..., \tilde{E}_{n,m}(\epsilon, \iota)]^T. \tag{3.7}$$

Obviously, we can write $\Phi(\epsilon, \iota)$ as the following form:

$$\Phi(\epsilon, \iota) = E(\epsilon) \otimes E(\iota), \tag{3.8}$$

where $\otimes$ is the Kronecker product and

$$\tilde{E}(\epsilon) = [\tilde{E}_0(\epsilon), \tilde{E}_1(\epsilon), ..., \tilde{E}_n(\epsilon)]^T, \ \tilde{E}(\iota) = [\tilde{E}_0(\iota), \tilde{E}_1(\iota), ..., \tilde{E}_m(\iota)]^T. \tag{3.9}$$

Suppose that $u(\epsilon, \iota) \in L^2([0,1) \times [0,1))$. Clearly, we can expand $u(\epsilon, \iota)$ in terms of 2DFPs as

$$u(\epsilon, \iota) \backsimeq \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} \tilde{E}_{ij}(\epsilon, \iota) = C^T \Phi(\epsilon, \iota) = \Phi^T(\epsilon, \iota) C, \tag{3.10}$$

where

$$C = [c_{0,0}, c_{0,1}, ..., c_{0,n}, c_{1,0}, ..., c_{1,n}, ..., c_{n,0}, ..., c_{n,m}]^T.$$

3.2 Operational matrix of integration

Here, we obtain the operational matrix of integration (OMI) of FPs . To do this, we have:

$$\int_0^\epsilon \tilde{E}(t)dt = R \int_0^\epsilon T(t)dt \simeq RPT(\epsilon) = RPR^{-1}\tilde{E}(\epsilon) = \Upsilon \tilde{E}(\epsilon), \tag{3.11}$$

where $\Upsilon = RPR^{-1}$, and

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By using Eq. (3.11), the OMI based on 2DFPs with respect to variable ϵ is obtained as follows:

$$\begin{aligned} \int_0^\epsilon \Phi(t, y)dt &= \int_0^\epsilon (\tilde{E}(t) \otimes \tilde{E}(y))dt = (\int_0^\epsilon \tilde{E}(t)dt) \otimes \tilde{E}(y) \\ &= (\Upsilon \tilde{E}(\epsilon)) \otimes (I\tilde{E}(\iota)) = (\Upsilon \otimes I)(\tilde{E}(\epsilon) \otimes \tilde{E}(y)) \\ &= \hat{\Upsilon}_\epsilon \Phi(\epsilon, y), \end{aligned} \tag{3.12}$$

where I is a identify matrix. In a similar way, we get

$$\begin{aligned} \int_0^\iota \Phi(\epsilon, s)ds &= \int_0^\iota (\tilde{E}(\epsilon) \otimes \tilde{E}(s))ds = \tilde{E}(\epsilon) \otimes (\int_0^\iota \tilde{E}(s)ds) \\ &= I\tilde{E}(\epsilon) \otimes (\Upsilon \tilde{E}(\iota)) = (I \otimes \Upsilon)(\tilde{E}(\epsilon) \otimes \tilde{E}(\iota)) \\ &= \hat{\Upsilon}_\iota \Phi(\epsilon, \iota). \end{aligned} \tag{3.13}$$

Now, for mixed variable, we conclude that

$$\begin{aligned}\int_0^{\epsilon}\int_0^{\iota}\Phi(x,y)dxdy &= \int_0^{\epsilon}\int_0^{\iota}(\tilde{E}(x)\otimes\tilde{E}(y))dxdy = (\int_0^{\epsilon}\tilde{E}(x)dx)\otimes(\int_0^{\iota}\tilde{E}(y)dy)\\ &= (\Upsilon_{\epsilon}\tilde{E}(\epsilon))\otimes(\Upsilon_{\iota}\tilde{E}(\iota)) = (\Upsilon_{\epsilon}\otimes\Upsilon_{\iota})(\tilde{E}(\epsilon)\otimes\tilde{E}(\iota))\\ &= \hat{\Upsilon}_{\epsilon\iota}\Phi(\epsilon,\iota), \end{aligned} \tag{3.14}$$

where Υ_{ϵ} and $\hat{\Upsilon}_{\epsilon\iota}$ are matrixes of order $(n+1)^2$ and $(n+1)^2\times(n+1)^2$, respectively. Also, we have:

$$\hat{\Upsilon}_{\epsilon\iota} = \begin{pmatrix} \Upsilon & O & \dots & O\\ O & \Upsilon & \dots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \dots & \Upsilon \end{pmatrix}.$$

3.3 Operational matrix of fractional integration (OMFI)

By using Definition (2.1), we have

$$\int_0^{\epsilon}\frac{r^i}{(\epsilon-r)^{1-\theta}}dr = \frac{\Gamma(\theta)\Gamma(1+i)}{\Gamma(1+i+\theta)}\epsilon^{i+\theta}. \tag{3.15}$$

So, we get

$$\int_0^{\epsilon}\frac{\tilde{E}(r)}{(\epsilon-r)^{1-\theta}}dr = R\int_0^{\epsilon}\frac{T_n(r)}{(\epsilon-r)^{1-\theta}}dr = R\Theta_n(\epsilon), \tag{3.16}$$

where

$$\Theta_n(\epsilon) = \left[\frac{\Gamma(\theta)}{\Gamma(1+\theta)}\epsilon^{\theta}, \frac{\Gamma(\theta)}{\Gamma(2+\theta)}\epsilon^{1+\theta}, \cdots, \frac{\Gamma(\theta)\Gamma(1+n)}{\Gamma(1+n+\theta)}\epsilon^{n+\theta}\right]^T. \tag{3.17}$$

The vector $\Theta_n(\epsilon)$ can be written in the following matrix from

$$\Theta_n(\epsilon) = \Theta T_n^{\theta}(\epsilon), \tag{3.18}$$

where

$$\begin{aligned}\Theta &= [\rho_{ij}]_{(n+1)\times(n+1)}, \quad \rho_{i,j} = \begin{cases} \frac{\Gamma(\theta)\Gamma(i+1)}{\Gamma(i+1+\theta)}, & i=j,\\ & i,j=0,1,\cdots,n.\\ 0, & i\neq j, \end{cases}\\ T_N^{\theta}(\epsilon) &= [\epsilon^{\theta},\epsilon^{\theta+1},...,\epsilon^{\theta+N}]. \end{aligned} \tag{3.19}$$

Now, we approximate $\epsilon^{\theta+i}$ in terms of FPs for $i=0,1,...,N$. So, we have

$$\epsilon^{\theta+i} \simeq \sum_{k=0}^{n}\delta_k^i E_k(\epsilon) = \Delta_i^T\tilde{E}(\epsilon) = \tilde{E}^T(\epsilon)\Delta_i, \tag{3.20}$$

where

$$\Delta_i = [\delta_0^i,\delta_1^i,...,\delta_n^i]^T.$$

By defining $(N+1)\times(N+1)$ matrix $\Delta = [\Delta_0,\Delta_1,...,\Delta_n]$ and by using Eqs.(3.19) and (3.20), we get the following result:

$$T_n^{\theta}(\epsilon) = \Delta^T\tilde{E}(\epsilon). \tag{3.21}$$

Hence, from Eqs. (3.16),(3.18) and (3.21), we have

$$\int_0^{\epsilon}\frac{\tilde{E}(r)}{(\epsilon-r)^{1-\theta}}ds = R\Delta^T\Theta\tilde{E}(\epsilon) = V\tilde{E}(\epsilon), \tag{3.22}$$

where $V = R\Delta^T\Theta$. Clearly, we have:

$$\begin{aligned}\int_0^\epsilon \frac{\Phi(r,\iota)}{(\epsilon-r)^{1-\theta}}dr &= \int_0^\epsilon \frac{\tilde{E}(r)\otimes\tilde{E}(\iota)}{(\epsilon-r)^{1-\theta}}dr = (\int_0^\epsilon \frac{\tilde{E}(r)}{(\epsilon-r)^{1-\theta}}dr)\otimes\tilde{E}(\iota)\\ &= (V\tilde{E}(\epsilon))\otimes(I\tilde{E}(\iota)) = (V\otimes I)(\tilde{E}(\epsilon)\otimes\tilde{E}(\iota) = V_\epsilon\Phi(\epsilon,\iota)\end{aligned} \tag{3.23}$$

where V is defined in Eq. (3.22). In a similar manner, we conclude that

$$\begin{aligned}\int_0^\iota \frac{\Phi(\epsilon,r)}{(\iota-r)^{1-\theta}}dr &= \int_0^\iota \frac{\tilde{E}(\epsilon)\otimes\tilde{E}(r)}{(\iota-r)^{1-\theta}}dr = \tilde{E}(\epsilon)\otimes(\int_0^\iota \frac{\tilde{E}(r)}{(\iota-r)^{1-\theta}}dr)\\ &= (I\tilde{E}(\epsilon))\otimes(V\tilde{E}(\iota)) = (I\otimes V)(\tilde{E}(\epsilon)\otimes\tilde{E}(\iota)) = V_\iota\Phi(\epsilon,\iota)\end{aligned} \tag{3.24}$$

where

$$V_\iota = I\otimes V = \begin{pmatrix} V & O & O & \dots & O\\ O & V & O & \dots & O\\ O & O & \ddots & \ddots & O\\ \vdots & \vdots & \ddots & \ddots & O\\ O & O & \cdots & O & V\end{pmatrix}_{(n+1)^2\times(n+1)^2}.$$

Now, we can obtain OMFI of $\Phi(\epsilon,\iota)$ as follows:

$$\begin{aligned}\int_0^\epsilon\int_0^\iota \frac{\Phi(s,y)}{(\epsilon-s)^\alpha}dyds &= \int_0^\epsilon\int_0^\iota \frac{\tilde{E}(s)\otimes\tilde{E}(y)}{(\epsilon-s)^\alpha}dyds\\ &= (\int_0^\epsilon \frac{\tilde{E}(s)}{(\epsilon-s)^\alpha}ds)\otimes(\int_0^\iota \tilde{E}(y)dy)\\ &= (\hat{V}\tilde{E}(\epsilon)\otimes(\Upsilon\tilde{E}(\iota)) = (\hat{V}\otimes\Upsilon)(\tilde{E}(\epsilon)\otimes\tilde{E}(\iota)) = L\Phi(\epsilon,\iota),\end{aligned} \tag{3.25}$$

where $L = \hat{V}\otimes\Upsilon$ is called OMFI of $\Phi(\epsilon,\iota)$. The product of two matrices of 2DFPs satisfies the following proposition

$$\Phi(\epsilon,\iota)\Phi^T(\epsilon,\iota)C \simeq \tilde{C}\Phi(\epsilon,\iota), \tag{3.26}$$

where $\tilde{C}$ is a matrix of order $(m+1).(n+1)\times(m+1).(n+1)$, and C is arbitrary vector [8]. To approximate $[u(\epsilon,\iota)]^p$ in terms of 2DFPS, we have

$$\begin{aligned}[u(\epsilon,\iota)]^2 &\simeq (U^T\Phi(\epsilon,\iota))(\Phi^T(\epsilon,\iota)U)\\ &= U^T\hat{U}\Phi(\epsilon,\iota) = \Phi^T(\epsilon,\iota)C_2,\end{aligned}$$

$$\begin{aligned}[u(\epsilon,\iota)]^3 &\simeq (U^T\Phi(\epsilon,\iota))(\Phi^T(\epsilon,\iota)C_2)\\ &= U^T\hat{C}_2\Phi(\epsilon,\iota) = \Phi^T(\epsilon,\iota)C_3,\\ &\vdots\end{aligned}$$

$$\begin{aligned}[u(\epsilon,\iota)]^P &\simeq (U^T\Phi(\epsilon,\iota))(\Phi^T(\epsilon,\iota)C_{p-1})\\ &= U^T\hat{C}_{p-1}\Phi(\epsilon,\iota) = \Phi^T(\epsilon,\iota)C_p,\end{aligned}$$

where $C_2 = (U^T\hat{U})^T$, $C_3 = (U^T\hat{C}_2)^T$ and $C_p = (U^T\hat{C}_{p-1})^T$.

4 The method of solution

Here, we use OMFI of fractional 2DFPs to find the approximate solution of Eq. (1.1). The functions $u(\epsilon,\iota)$, $[H(u(\epsilon,\iota))$, $u(\epsilon,\iota)$ and $G(\epsilon,\iota)$ may be approximated as the following

$$\begin{aligned}
&u(\epsilon,\iota) = U^T\Phi(\epsilon,\iota) = \Phi^T(\epsilon,\iota)U,\\
&g(\epsilon,\iota) = G^T\Phi(\epsilon,\iota) = \Phi^T(\epsilon,\iota)G,\\
&H(u(\epsilon,\iota)) = [u(\epsilon,\iota)]^p = \Phi^T(\epsilon,\iota)C_p,\\
&H(u(\epsilon,\iota)) = \lambda_{ij}u(\epsilon,\iota) = P\Phi(\epsilon,\iota)\\
&\Phi(\epsilon,\iota)\Phi^T(\epsilon,\iota)C = \tilde{C}^T\Phi(\epsilon,\iota),\\
&u(\epsilon,0) = u_0(\epsilon) = D^T\Phi(\epsilon,\iota),
\end{aligned} \tag{4.1}$$

where $\lambda_{ij}u(\epsilon,\iota) = \frac{\partial^{i+j}u(\epsilon,\iota)}{\partial\epsilon^i\partial\iota^j}$. By implementation of I^θ with respect to variable ι on the both side of Eq. (1.1), we have:

$$u(\epsilon,\iota) = u_0(\epsilon) + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{u(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{g(\epsilon,r)}{(\iota-r)^{1-\theta}}dr\frac{1}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{H(u(s,y))}{(\epsilon-s)^\alpha}dydsdr.$$

By supposing $H(u(s,y)) = u(s,y)^p$, and using Eq. (4.1), we get

$$U^T\Phi(\epsilon,\iota) = D^T\Phi(\epsilon,\iota) + \frac{U^T}{\Gamma(\theta)}\int_0^\iota \frac{\Phi(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{G^T}{\Gamma(\theta)}\int_0^\iota \frac{\Phi(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{C_p}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{\Phi(s,y)}{(\epsilon-s)^\alpha}dydsdr.$$

Using Eqs.(3.24) and (3.25) in the above equation, we have

$$U^T\Phi(\epsilon,\iota) = D^T\Phi(\epsilon,\iota) + \frac{U^T}{\Gamma(\theta)}V_\iota\Phi(\epsilon,\iota) + \frac{G^T}{\Gamma(\theta)}V_\iota\Phi(\epsilon,\iota) + \frac{C_p}{\Gamma(\theta)}LV_\iota\Phi(\epsilon,\iota),$$

by using the properties of FPs, the above equation can be rewritten as

$$U^T = D^T + \frac{U^T}{\Gamma(\theta)}V_\iota + \frac{G^T}{\Gamma(\theta)}V_\iota + \frac{C_p}{\Gamma(\theta)}LV_\iota. \tag{4.2}$$

Eq. (4.2) indicates a system of nonlinear equations which may be solved by using known methods. Therefore, by solving this system and substituting U^T in Eq. (4.2), $u(\epsilon,\iota)$ is obtained as a numerical solution of Eq. (1.1). Now, we consider $H(u(\epsilon,\iota)) = \lambda_{ij}u(\epsilon,\iota)$ in Eq. (1.1). We have:

$$u(\epsilon,\iota) = u_0(\epsilon) + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{u(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{g(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{\lambda_{ij}u(s,y)}{(\epsilon-s)^\alpha}dydsdr.$$

Clearly, we get the following result

$$\begin{aligned}
\lambda_{ij}u(\epsilon,\iota) = \frac{\partial^{i+j}u(\epsilon,\iota)}{\partial\epsilon^i\partial\iota^j} &= U^T\frac{\partial^{i+j}\Phi(\epsilon,\iota)}{\partial\epsilon^i\partial\iota^j} = U^T\frac{\partial^i W^j\Phi(\epsilon,\iota)}{\partial\epsilon^i} = U^T W^j\frac{\partial^i\Phi(\epsilon,\iota)}{\partial\epsilon^i}\\
&= U^T W^j\Xi^i\Phi(\epsilon,\iota).
\end{aligned}$$

Hence

$$\lambda_{ij}u(\epsilon,\iota) = P\Phi(\epsilon,\iota), \tag{4.3}$$

where $P = U^T W^j\Xi^i$, Ξ and W are the operational matrices of differentiation of $\Phi(\epsilon,\iota)$ with respect to variables ϵ and ι, respectively. Therefore, by applying (4.3) in (4.3), we have the following equation:

$$\begin{aligned}
U^T\Phi(\epsilon,\iota) =& D^T\Phi(\epsilon,\iota) + \frac{U^T}{\Gamma(\theta)}\int_0^\iota \frac{\Phi(s,y)}{(\iota-r)^{1-\theta}}dr + \frac{G^T}{\Gamma(\theta)}\int_0^\iota \frac{\Phi(\epsilon,\iota)}{(\iota-r)^{1-\theta}}dr + \frac{P}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{\Phi(s,y)}{(\epsilon-s)^\alpha}dydsdr\\
=& D^T\Phi(\epsilon,\iota) + \frac{U^T}{\Gamma(\theta)}V_\iota\Phi(\epsilon,\iota) + \frac{G^T}{\Gamma(\theta)}V_\iota\Phi(\epsilon,\iota) + \frac{P}{\Gamma(\theta)}LV_\iota\Phi(\epsilon,\iota).
\end{aligned} \tag{4.4}$$

Hence

$$U^T = D^T + \frac{U^T}{\Gamma(\theta)}V_\iota + \frac{G^T}{\Gamma(\theta)}V_\iota + \frac{P}{\Gamma(\theta)}LV_\iota. \tag{4.5}$$

Eq. (4.5) indicates a system of nonlinear equations which may be solved using a suitable numerical method such as Newton's numerical method. Then, by solving this system, we can obtain U^T. By substituting U^T in Eq. (4.1), $u(\epsilon,\iota)$ is obtained as a numerical solution of Eq. (1.1).

5 Error analysis

In this section, we apply the following 2-norm for $u \in L^2(\Omega)$

$$\|u(x,y)\|_2 = \left(\int_0^1\int_0^1 (u(x,y))^2 dxdy\right)^{\frac{1}{2}}.$$

Theorem 5.1. [29] If h be an integrable function and $l : [a,b] \to R$ be a continuous function that its sign dose not change on interval $[a,b]$, then there exists a constant $\rho \in (a,b)$ such that

$$\int_a^b l(\epsilon)h(\epsilon)d\epsilon = l(\rho)\int_a^b h(\epsilon)d\epsilon. \tag{5.1}$$

Theorem 5.2. Suppose that $u(\epsilon,\iota)$ and $u_N(\epsilon,\iota)$ be the exact and approximate solution of Eq. (4.2), respectively. Also, assume that

(1). $1 - \frac{\eta}{\Gamma(\theta)} > 0$,
(2). $\|H(u_N(\epsilon,\iota) - \hat{H}(u_N(\epsilon,\iota)))\|\iota_2 \leq \varrho$,
(3). $\theta > \frac{1}{2}$ *and* $\alpha < \frac{1}{2}$,

where η is a constant number and $\hat{H}(u_N(\epsilon,\iota))$ is the approximation of $H(u_N(\epsilon,\iota))$ by using FPs. Then, the error bound would be obtained as follows

$$\|u(\epsilon,\iota) - u_N(\epsilon,\iota)\|_2 \leq \frac{\sqrt{B_0} + \frac{\sqrt{B_1}+\sqrt{B_2}}{\sqrt{2\theta+1}\Gamma(\theta)} + \frac{\eta_2}{\Gamma(\theta)}}{1 - \frac{\eta_1}{\Gamma(\theta)}},$$

where η_2 is a constant number and

$$\|u(\epsilon,0) - u_N(\epsilon,0)\|_2 \leq \sqrt{B_0},$$
$$\|u(\epsilon,\rho_1) - u_N(\epsilon,\rho_1)\|_2 = \left(\int_0^1 (u(\epsilon,\rho_1) - u_N(\epsilon,\rho_1))^2)d\epsilon\right)^{\frac{1}{2}} \leq \sqrt{B_1},\ for\ fix\ \rho_1 \in (0,\iota),$$
$$\|g(\epsilon,\rho_2) - g_N(\epsilon,\rho_2)\|_2 = \left(\int_0^1 (g(\epsilon,\rho_2) - g_N(\epsilon,\rho_2))^2)d\epsilon\right)^{\frac{1}{2}} \leq \sqrt{B_2},\ for\ fix\ \rho_2 \in (0,\iota).$$

Proof: By using the Lipschitz condition, we have

$$\begin{aligned}\|H(u(\epsilon,\iota) - \hat{H}(u_N(\epsilon,\iota))\|_2 &\leq \|H(u(\epsilon,\iota) - H(u_N(\epsilon,\iota))\|_2 + \|H(u(\epsilon,\iota) - \hat{H}(u_N(\epsilon,\iota))\|_2 \\ &\leq L\|u(\epsilon,\iota) - u_N(\epsilon,\iota)\|_2 + \varrho.\end{aligned} \tag{5.2}$$

Approximation of Eq (4.2) 2DFPs is as follows

$$u_N(\epsilon,\iota) = u_N(\epsilon,0) + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{u_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{g_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr + \frac{1}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{\hat{H}(u_N(s,y))^k}{(\epsilon-s)^\alpha}dydsdr.$$

where $\hat{H}(\epsilon,\iota)$, $g_N(\epsilon,\iota)$, $u_N(\epsilon,\iota)$, and $u_N(\epsilon,\iota)$ are approximation of $H(\epsilon,\iota)$, $g(\epsilon,\iota)$, $u(\epsilon,\iota)$, and $u(\epsilon,\iota)$ by using 2DFPs, respectively. Thus,

$$\begin{aligned} u(\epsilon,\iota)-u_N(\epsilon,\iota) &\le u(\epsilon,0)-u_N(\epsilon,0)+\frac{1}{\Gamma(\theta)}\int_0^\iota \frac{u(\epsilon,r)-u_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr+\frac{1}{\Gamma(\theta)}\int_0^\iota \frac{g(\epsilon,r)-g_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr \\ &+\frac{1}{\Gamma(\theta)}\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{H(u(s,y))-\hat{H}(u_N(s,y))}{(\epsilon-s)^\alpha}dydsdr. \end{aligned}$$

Hence

$$\begin{aligned} \|u(\epsilon,\iota)-u_N(\epsilon,\iota)\|_2 \le &\|u(\epsilon,0)-u_N(\epsilon,0)\|_2+\frac{1}{\Gamma(\theta)}\|\int_0^\iota \frac{u(\epsilon,r)-u_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr\|_2+\frac{1}{\Gamma(\theta)}\|\int_0^\iota \frac{g(\epsilon,r)-g_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr\|_2 \\ &+\frac{1}{\Gamma(\theta)}\|\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{H(u(s,y))-\hat{H}(u_N(s,y))}{(\epsilon-s)^\alpha}dydsdr\|_2. \end{aligned}$$

By using Theorem (5.1), there are $\rho_1,\rho_2\in(0,\iota)$, such that

$$\int_0^\iota \frac{u(\epsilon,r)-u_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr=(u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1))\int_0^\iota \frac{dr}{(\iota-r)^{1-\theta}}=\frac{1}{\theta}\iota^\theta(u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1)), \tag{5.3}$$

$$\int_0^\iota \frac{g(\epsilon,r)-g_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr=(g(\epsilon,\rho_2)-g_N(\epsilon,\rho_2))\int_0^\iota \frac{dr}{(\iota-r)^{1-\theta}}=\frac{1}{\theta}\iota^\theta(g(\epsilon,\rho_1)-g_N(\epsilon,\rho_1)). \tag{5.4}$$

So,

$$\begin{aligned} \left\|\int_0^\iota \frac{u(\epsilon,r)-u_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr\right\|_2^2 &= \left\|\frac{1}{\theta}\iota^\theta(u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1))\right\|_2^2 \\ &= \frac{1}{\theta^2}\int_0^1\int_0^1 \iota^{2\theta}(u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1))^2 d\epsilon d\iota \\ &= \frac{1}{\theta^2}\int_0^1 \iota^{2\theta}d\iota\int_0^1 (u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1))^2 d\epsilon \\ &= \frac{1}{\theta^2}\|\iota^\theta\|_2^2\|(u(\epsilon,\rho_1)-u_N(\epsilon,\rho_1))\|_2^2 \\ &\le \frac{1}{\theta^2(2\theta+1)}B_1. \end{aligned} \tag{5.5}$$

By using similar way, we have

$$\|\int_0^\iota \frac{g(\epsilon,r)-g_N(\epsilon,r)}{(\iota-r)^{1-\theta}}dr\|_2^2\le\frac{1}{\theta^2(2\theta+1)}B_2. \tag{5.6}$$

By using (5.2), we have

$$\begin{aligned} \|\frac{H(u(\epsilon,\iota))-\hat{H}(u_N(\epsilon,\iota))}{(\iota-r)^{1-\theta}(\epsilon-s)^\alpha}\|_2 &\le \|H(u(\epsilon,\iota)-\hat{H}(u_N(\epsilon,\iota)\|_2\|\frac{1}{(\iota-r)^{1-\theta}}\|_2\|\frac{1}{(\epsilon-s)^\alpha}\|_2 \\ &\le (L\|u(\epsilon,\iota)-u_N(\epsilon,\iota)\|_2+\varrho)\|\frac{1}{(\iota-r)^{1-\theta}}\|_2\|\frac{1}{(\epsilon-s)^\alpha}\|_2 \\ &= \frac{L}{\sqrt{(2\theta-1)}\sqrt{1-2\alpha}}\|u(\epsilon,\iota)-u_N(\epsilon,\iota)\|_2+\frac{\varrho}{\sqrt{(2\theta-1)}\sqrt{1-2\alpha}} \\ &= \eta_1\|u(\epsilon,\iota)-u_N(\epsilon,\iota)\|_2+\eta_2. \end{aligned} \tag{5.7}$$

Now, by the help of Eq. (5.7), we have

$$\left\|\int_0^\iota \frac{1}{(\iota-r)^{1-\theta}}\int_0^\epsilon\int_0^r \frac{H(u(s,y))-\hat{H}(u_N(s,y))}{(\epsilon-s)^\alpha}dydsdr\right\|_2\le$$

$$\int_0^\iota \int_0^\epsilon \int_0^r \| \frac{H(u(s,y)) - \hat{H}(u_N(s,y))}{(\iota - r)^{1-\theta}(\epsilon - s)^\alpha} \|_2 dydsdr \leq \eta_1 \|u(s,y) - u_N(s,y)\|_2 + \eta_2. \tag{5.8}$$

By substituting Eqs.(5.5)-(5.8) in Eq. (5.2), we conclude that

$$\|u(\epsilon,\iota) - u_N(\epsilon,\iota)\|_2 \leq \frac{\sqrt{B_0} + \frac{\sqrt{B_1}+\sqrt{B_2}}{\theta\sqrt{2\theta+1}\Gamma(\theta)} + \frac{\eta_2}{\Gamma(\theta)}}{1 - \frac{\eta_1}{\Gamma(\theta)}}.$$

6 Numerical examples

This section considers some numerical examples to demonstrate the efficiency of the proposed method.

Example 6.1. Consider the following FWS2DPVIE [33]

$$D_\iota^\theta u(\epsilon,\iota) = u(\epsilon,\iota) + g(\epsilon,\iota) + \int_0^\epsilon \int_0^\iota \frac{(u(s,y))^2}{(\epsilon - s)^{\frac{1}{2}}} dyds, \tag{6.1}$$

where

$$g(\epsilon,\iota) = \frac{2}{\Gamma(3-\theta)}\iota^{2-\iota} - 2\epsilon - x^2 - \iota^2 - \frac{2}{315}\epsilon^{\frac{1}{2}}\iota(128\epsilon^4 + 112\epsilon^2\epsilon^2 + 63\epsilon^4).$$

The exact solution of this example is $u(\epsilon,\iota) = \epsilon^2 + \iota^2$. The numerical results calculated by the proposed method are shown in Table 1. The absolute errors (AEs) for $n = 2$ and $\theta = 0.9, \theta = 0.95$ have been plotted in Fig. 1.

Example 6.2. In the next example, the following FWS2DPVIE is considered [33]

$$D_\iota^\theta u(\epsilon,\iota) = u(\epsilon,\iota) + g(\epsilon,\iota) + \int_0^\epsilon \int_0^\iota \frac{k(\epsilon,\iota,s,y)}{(\epsilon - s)^{\frac{1}{2}}} u_{st}(s,y) dyds, \tag{6.2}$$

with

$$g(\epsilon,\iota) = \frac{1}{\Gamma(2-\theta)}\iota^{1-\theta} - \epsilon - \iota.$$

The exact solution is $u(\epsilon,\iota) = \epsilon + \iota$. The numerical results calculated by the proposed method are shown in Table 2. The absolute error function for $n = 2$ and $\theta = 0.9, \theta = 0.95$ have been plotted in Fig. 2.

Example 6.3. In the final example, the below FWS2DPVIE is considered [33]

$$D_\iota^\theta u(\epsilon,\iota) = u(\epsilon,\iota) + g(\epsilon,\iota) + \int_0^\epsilon \int_0^\iota \frac{u_{st}(s,y)}{(\epsilon - s)^{\frac{1}{2}}} dyds, \tag{6.3}$$

with

$$g(\epsilon,\iota) = \frac{2}{\Gamma(3-\theta)}\epsilon\iota^{2-\theta} - \epsilon\iota^2 - \frac{8}{3}\epsilon^{\frac{3}{2}}\iota.$$

The exact solution of this example is $u(\epsilon,\iota) = \epsilon\iota^2$. Table 3 shows the numerical results obtained by the presented method. The absolute error function for $n = 2$ and $\theta = 0.9, \theta = 0.95$ has been plotted in Figure 3.

7 Conclusion

The numerical solution of 2DFWSPIDE is regarded as one of the most exceedingly difficult problems to solve. As a result, we developed a technique for solving 2DFWSPIDE depending on Fibonacci polynomials and their characteristics. as a result we sought to solve twin sets of 2DFWSPIDE. Therefore, it is possible to reduce 2DFWSPIDE to algebraic equations by the operational matrices utilization and the properties of two-dimensional Fibonacci polynomials. For further investigation, a couple of theorems about the error analysis and method accuracy are presented as well. The high degree of accuracy of the method confirms the obtained numerical finding. The reliability and simplicity of the method are demonstrated using linear and nonlinear examples of 2DFWSPIDE.

Table 1: The AEs for Example 6.1.

$\epsilon = \iota$	$n = 2,\, \theta = 0.9$	$n = 2,\, \theta = 0.9$	$n = 2,\, \theta = 0.95$	$n = 2,\, \theta = 0.95$
	Bernoulli method	Present method	Bernoulli method	Present method
0.1	4.636916×10^{-10}	7.874328×10^{-10}	6.697149×10^{-10}	8.540981×10^{-10}
0.2	2.359472×10^{-9}	6.068483×10^{-10}	2.635334×10^{-9}	9.084345×10^{-10}
0.3	2.306380×10^{-9}	5.943746×10^{-9}	3.095219×10^{-9}	5.212702×10^{-10}
0.4	7.125630×10^{-9}	7.300353×10^{-9}	3.416147×10^{-9}	7.436020×10^{-10}
0.5	3.880506×10^{-8}	2.760653×10^{-9}	2.574417×10^{-8}	2.689306×10^{-10}
0.6	1.110388×10^{-7}	1.984143×10^{-8}	7.611416×10^{-7}	7.909325×10^{-8}
0.7	2.475725×10^{-7}	3.530115×10^{-7}	1.701313×10^{-7}	7.578905×10^{-8}
0.8	4.775898×10^{-7}	8.322934×10^{-7}	3.267807×10^{-7}	8.320080×10^{-8}
0.9	8.357132×10^{-7}	1.895932×10^{-7}	5.684273×10^{-7}	9.983065×10^{-7}

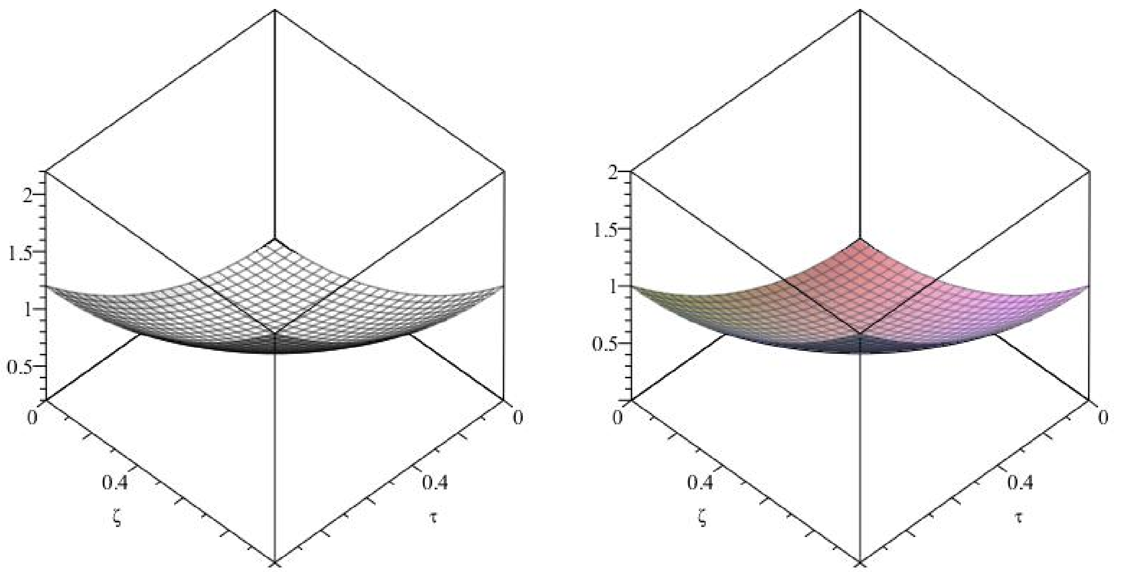

Figure 1: Comparing numerical (Right) and exact (Left) solutions, $u(\epsilon, \iota)$, with $n = 2$ for Example 6.1.

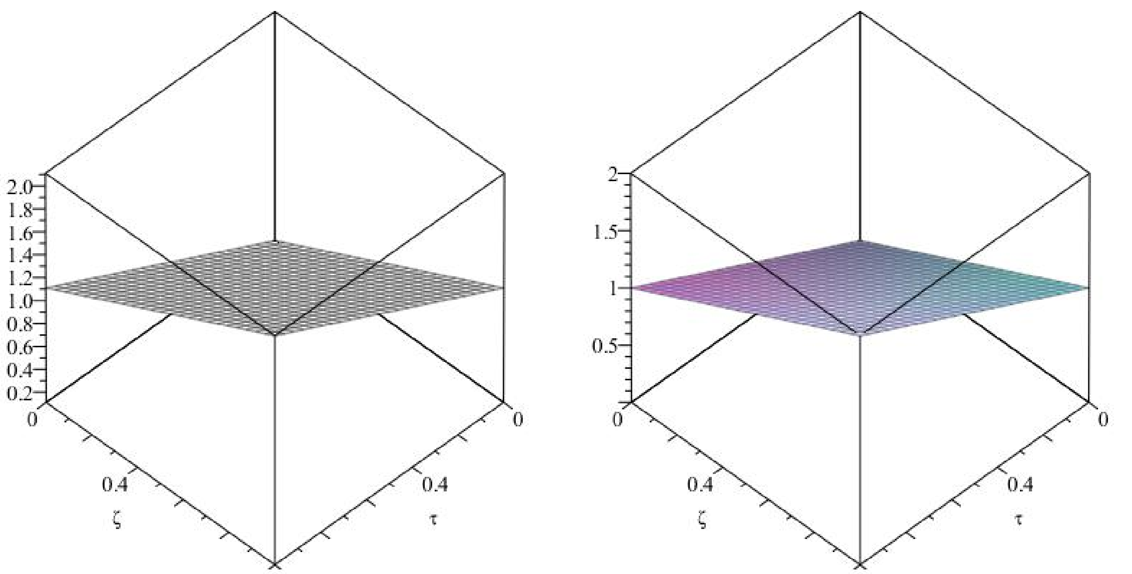

Figure 2: Comparing numerical (Right) and exact (Left) solutions, $u(\epsilon, \iota)$, with $n = 2$ for Example 6.2.

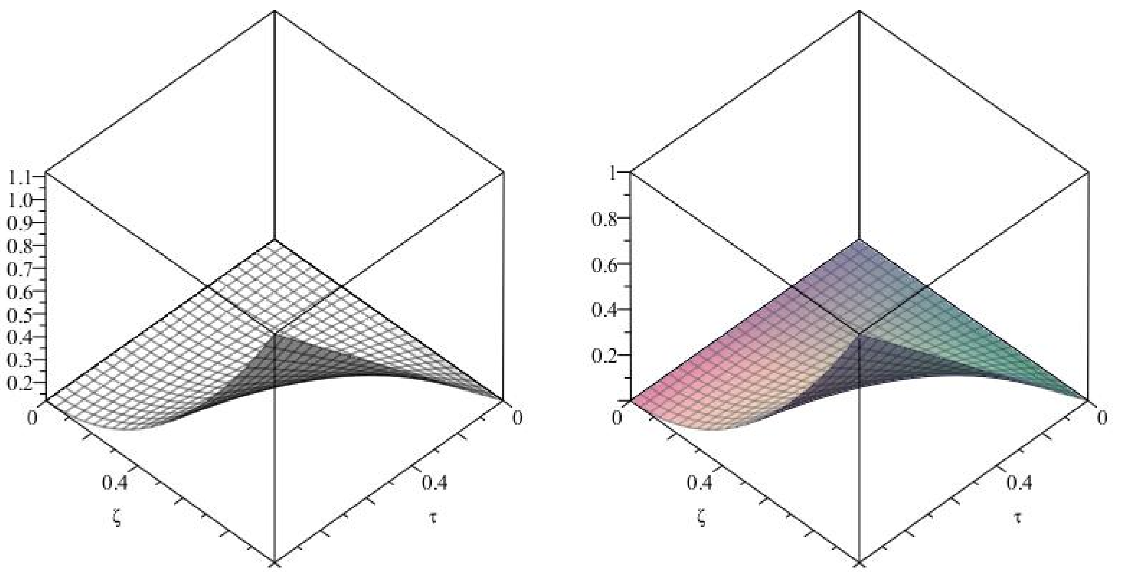

Figure 3: Comparing numerical (Right) and exact (Left) solutions, $u(\epsilon, \iota)$, with $n = 2$ for Example 6.3.

Table 2: The AEs for Example 6.2.

$\epsilon = \iota$	$n = 2$, $\theta = 0.9$	$n = 2$, $\theta = 0.9$	$n = 2$, $\theta = 0.95$	$n = 2$, $\theta = 0.95$
	Bernoulli method	Proposed method	Bernoulli method	Proposed method
0.1	1.934907×10^{-12}	2.365101×10^{-13}	1.054650×10^{-12}	5.369840×10^{-13}
0.2	3.761703×10^{-13}	3.651080×10^{-13}	2.157539×10^{-13}	9.320150×10^{-13}
0.3	7.484526×10^{-13}	5.320501×10^{-14}	3.911223×10^{-13}	6.598040×10^{-14}
0.4	1.438961×10^{-12}	7.321500×10^{-13}	7.659788×10^{-13}	8.996001×10^{-14}
0.5	1.695357×10^{-12}	8.970501×10^{-12}	9.088153×10^{-13}	1.195101×10^{-13}
0.6	1.517639×10^{-13}	3.856017×10^{-13}	8.196318×10^{-13}	5.169503×10^{-14}
0.7	9.058072×10^{-13}	5.020277×10^{-12}	4.984284×10^{-13}	6.962574×10^{-14}
0.8	1.401384×10^{-12}	6.905407×10^{-12}	5.479496×10^{-14}	8.963514×10^{-13}
0.9	1.620197×10^{-12}	6.950150×10^{-14}	8.400383×10^{-13}	4.502177×10^{-14}

Table 3: The AEs for Example 6.3.

	$n = 2$, $\theta = 0.9$	$n = 2$, $\theta = 0.9$	$n = 2$, $\theta = 0.95$	$n = 2$, $\theta = 0.95$
$\epsilon = \iota$	Bernoulli method	Proposed method	Bernoulli method	Proposed method
0.1	2.177053×10^{-12}	6.381405×10^{-13}	1.046189×10^{-12}	1.320850×10^{-12}
0.2	4.190303×10^{-12}	9.301045×10^{-13}	1.903131×10^{-12}	2.365048×10^{-13}
0.3	2.109463×10^{-11}	6.440284×10^{-12}	9.939731×10^{-12}	6.118087×10^{-13}
0.4	4.911192×10^{-11}	6.105018×10^{-10}	2.337588×10^{-11}	9.171058×10^{-10}
0.5	9.097921×10^{-11}	5.101279×10^{-11}	4.420788×10^{-11}	8.621184×10^{-12}
0.6	1.515947×10^{-10}	9.365014×10^{-11}	7.611600×10^{-11}	9.902254×10^{-12}
0.7	2.380176×10^{-10}	9.204840×10^{-11}	1.244645×10^{-10}	1.950476×10^{-11}
0.8	3.594689×10^{-10}	8.320511×10^{-10}	1.963017×10^{-10}	9.854087×10^{-10}
0.9	5.273295×10^{-10}	8.298056×10^{-10}	3.003598×10^{-10}	2.369874×10^{-11}

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