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Linear maps which are local (g, h)-ternary derivations from $*$-module extension Banach algebras into their periodical duals

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Abstract

We introduce a $*$-module extension Banach algebras to generalized the results of Niazi and Miri. Precisely, every local (g, h)-ternary derivation from a $*$-module extension Banach algebra into one of its periodical duals is (g, h)-ternary derivation.

Keywords: $*$-module extension Banach algebras, (g, h)-derivations, (g, h)-generalized derivations, ternary (triple) derivations.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra with identity, and let X be a Banach $\mathcal{A}$-bimodule. The l^1- direct sum Banach algebra related to $\mathcal{A}$ and X, denoted by $\mathcal{A} \oplus X$, is the module extension with the algebraic operations which are defined as follows;

$$(s, n) + (r, m) = (s + r, n + m), r(s, n) = (rs, rn), (s, n) r = (sr, nr),$$

$$(s, n)(r, m) = (sr, sm + nr), \text{for all } s, r \in \mathcal{A}, n, m \in X.$$

And it is obvious that $\mathcal{A} \oplus X$ is a Banach algebra with the following norm;

$$\| (s, n) \| = \| s \| + \| n \|, \text{ for all} s \in \mathcal{A}, n \in X.$$

There are many researchers studied this type of Banach algebras from different sides; see for example [9, 11]. A $*$-module extension Banach algebra is module extension Banach algebra $\mathcal{A} \oplus X$ with involution mapping $* : \mathcal{A} \oplus X \to \mathcal{A} \oplus X$, denoted by $* - \mathcal{A} \oplus X$, such that the mapping $*$: satisfying the properties:

$$((s, n) + (r, m))^* = (s, n)^* + (r, m)^*, (1, 0)^* = (1, 0),$$

$$((s, n)\ (r, m))^* = (r, m)^* (s, n)^*, \left((s, n)^*\right)^* = (s, n),$$

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for all $(s,n),(r,m)$ in $*-\mathcal{A}\oplus X$, and $(1,0)$ is the unit element of $*-\mathcal{A}\oplus X$. A linear mapping D from $*-\mathcal{A}\oplus X$ into Banach $(*-\mathcal{A}\oplus X)$-bimodule U is called a (g,h)-derivation if it satisfies: for all $(s,n),(r,m)$ in $*-\mathcal{A}\oplus X$, $D((s,n)(r,m)) = D(s,n)\,h(r,m) + g(s,n)\;D(r,m)$, where $g,h : *-\mathcal{A}\oplus X \to U$ are linear maps [1]. According to [8], a local (g,h)-ternary derivation on a Jordan ternary J is a linear mapping D satisfies: for every $s\in J$ there exists (g,h)-ternary derivation d_s on J, depending on s with $D(s) = d_s(s)$. Mackey show that all bounded local ternary derivation on a JBW^*-ternary is a ternary derivation [8, Theorem 5.11]. For C^*-algebras, M. Burgos et al. generalized the Mackey's result. By assuming that C^*-algebra $\mathcal{A}$ is a Jordan ternary with the ternary product: $\{s,r,c\} = \frac{1}{2}\left(sr^*c + cr^*s\right)$, for all $s,r,c\in\mathcal{A}$ [3]. And in [4, Theorem 2.4], M. Burgos et al. show that all continuous local ternary derivation on a JB^*-ternary is a ternary derivation. In [10, Theorem 3.9], Niazi and Miri show that all continuous local ternary derivation defined on a C^*-algebra $\mathcal{A}$ into one of its periodical duals is ternary derivation. We recall that for a C^*-algebra $\mathcal{A}$, let D be a bounded linear map defined on a unital $*-\mathcal{A}\oplus X$ into any of its periodical duals. The self-adjoint set of a $*-\mathcal{A}\oplus X$ will denoted by $(*-\mathcal{A}\oplus X)_{sa}$. In this paper, we improved the Niazi's and Miri's result in [10], for $*$-module extension Banach algebras $*-\mathcal{A}\oplus X$ with the following triple product: for all $(s,n),(r,m),(c,z)\in *-\mathcal{A}\oplus X, \{(s,n),(r,m),(c,z)\} = \frac{1}{2}\left((s,n)(r,\ m)^*(c,z) + (c,z)(r,m)^*(s,n)\right)$. By proving that: every continuous local (g,h)-ternary derivation $D : *-\mathcal{A}\oplus X \to (*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ being a (g,h)-(ternary) triple derivation (Theorem 3.8).

2 Ternary modules

In this section we present some definitions and proposition which are useful for our results. Recall that from [10], a (g,h)-triple derivation from a Jordan ternary J into a ternary J-module U is a conjugate linear (linear) mapping $D : J \to U$, defined by $D\{s,r,c\} = \{D(s),\ h(r),\ h(c)\} + \{g(s),\ D(r), h(c)\} + \{g(s),\ g(r), D(c)\}$, for all s,r,c in J, where $g,h :\ J \to U$ are linear maps. Suppose J is a Jordan ternary and U is a ternary J-module, then for every $u\in U$ and $y\in J$, the mapping $\delta(u,\ y) :\ J \to U$, defined by

$$\delta(u,y)(x) = \{u,\ g(y), h(x)\} - \{g(y), u,\ h(x)\}, (x\ \in\ J) \tag{2.1}$$

is a (g,h)-triple derivation. An inner (g,h)-triple derivation is the finite sum of the previous derivations (2.1). Also, a (g,h)-derivation defined on a duple (associative) algebra E into J-bimodule U is a linear mapping $D :\ E \to U$ fulfilling: $D(s\ r) = D(s)\ h(r) + g(s)\ D(r)$, for all $s,\ r\in E$, where $g,h : E \to U$ are linear maps. And, D is said to be a Jordan (g,h)-derivation if it satisfies: for all $s\in E, D\left(s^2\right) = D(s)\ h(s) + g(s)\ D(s)$ or equivalently for all $s,\ r\in E, D(s\circ r) = D(s)\ \bullet (g,h)(r) + (g,h)(s)\bullet D(r)$, where $s\circ r = (s\ r + r\ s)\ /2$ and $D(s)\ \bullet (g, h)(r) = (D(s)\ h(r) + g(r)\ D(s))/2$. A linear mapping D from a unital algebra E into an J-bimodule U is called a (g,h)-generalized derivation if it satisfies: for all $s,\ r\in E, D(s\ r) = D(s)\ h(r) + g(s)\ D(r) - g(s)\ D(1)\,h(r)$. Note that if $D(1) = 0, (g,h)$-generalized derivation is (g,h)-derivation.

Proposition 2.1. [10] Suppose $\mathcal{A}$ is a Banach $*$-algebra and $\mathrm{n}\in N$. For all $s, r\in\mathcal{A}$ and $f\in\mathcal{A}^{\mathrm{n}}$, we have $\{f,s,r\} = \{r,s,f\} = \frac{1}{2}\left(fs\,r^* + r^*\,s\,f\right)$, $\{s,f,r\} = \frac{1}{2}\left(s^*f^*\,r^* + r^*f^*\,s^*\right)$, whenever n is odd, and $\{f,s,r\} = \{r,s,f\} = \frac{1}{2}\left(f\ s^*\ r + r\ s^*f\right)$, $\{s,f,r\} = \frac{1}{2}\left(s\ f^*\ r + r\ f^*\ s\right)$, whenever n is even.

3 Local (g,h)-ternary derivations on $*$-module extension Banach algebra

Throughout this section, the symbol $*-N\oplus M$ will denote closed $*$-submodule extension algebra of unital $*$-module extension Banach algebra $*-\mathcal{A}\oplus X$, we assume that $*-N\oplus M$ have the unit element of $*-\mathcal{A}\oplus X, g,h : *-N\oplus M \to (*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ are continuous homomorphisms, also $D\circ *(s,\ n) = D\left((s,\ n)^*\right)$, for all $(s,\ n)\, in *-\mathcal{A}\oplus X$. We begin by the following lemma:

Lemma 3.1. Let $D :\ *-N\oplus M \to (*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ be a local (g,h)-triple derivation with $*-N\oplus M$ being commutative. If $(s,\ n)^*\ (r,\ m) = (r,\ m)^*\ (c,z) = (0,0)$, for all $(s,\ n), (r,\ m),\ (c,z)$ in $*-N\oplus M$, then $\{g(s,\ n), D(r,\ m),\ h(c,z)\} = (0,0)$.

Proof . The proof is comparable to that of [10, Lemma 3.1]. □

Lemma 3.2. Suppose $*-N\oplus M$ is a commutative unital $*$-module extension Banach algebra. Let U be a Banach space and a bounded map $\Phi : *-N\oplus M\times\ *-N\oplus M \to U$ be conjugate linear in the second variable and linear in the first variable. If $\Phi((s,n),(r,m)) = (0,0)$, for every $(s,\ n), (r,\ m)$ in $*$-$N\oplus M$ with $(s,\ n)^*\ (r,\ m) = (0,0)$, then $\Phi((s,n),(r,m)) = \Phi((1,0),(s,\ n)^*(r,m))$, for all $(s,\ n), (r,\ m)\in *-N\oplus M$.

Proof . Let us consider the following mapping $S : N \times N \to$ U defined by $\Phi((s,n),(r,m)) = S(s,r)$, for all $s,\ r \in N$. Since $\Phi((s,n),(r,m)) = S(s,r) = 0$, for every $(s,\ n),(r,\ m) \in * - N \oplus M$ with $(s,\ n)^*\ (r,\ m) = (0,0)$. For all $\varphi \in U^*$, from [5, Theorem 1.10], there exists ϕ, ψ in N^*, such that

$$\varphi \circ \Phi((s,n),(r,m)) = \varphi \circ S(s,r) = \phi\,(r^* s) + \psi\,(s\ r^*)\,.$$

Since $* - N \oplus M$ is commutative, we have that

$$\varphi \circ \Phi((s,n),(r,m)) = \varphi \circ S(s,r) = (\phi + \psi)\,(r^* s)\,. \tag{3.1}$$

Also, we can obtain

$$\varphi \circ \Phi\,((1,0),(s,n)^*(r,m)) = \varphi \circ S\,(1, s^* r) = (\phi + \psi)\,(r^* s)\,. \tag{3.2}$$

By combining (3.1) and (3.2), we have that

$$\varphi \circ \Phi((s,n),(r,m)) = \varphi \circ \Phi\,((1,0),(s,n)^*(r,m))\,.$$

Now, by applying Hahn-Banach theorem, we get the desired result. □

Lemma 3.3. Let $D :\ * - N \oplus M \to\ * - \mathcal{A} \oplus X$ be a continuous linear operator. The statements are equivalent:

1. $g\,(s,\ n)\ \ D\,(r,\ m)\ \ h(c,z) = (0,0)$, whenever $(s,\ n)\ \ (r,\ m) = (r,\ m)\ \ (c,z) = (0,0)$ in $* - N \oplus M$;
2. $g\,(s,\ n)\ \ D\,(r,\ m)\ \ h(c,z) = (0,0)$, whenever $(s,\ n)\ \ (r,\ m) = (r,\ m)\ \ (c,z) = (0,0)$ in $(* - N \oplus M)_{sa}$;
3. D is a (g,h)-generalized derivation.

Proof . The proof is like to that of [2, Proposition 2.8], [7, Proposition 1.1]. □

Proposition 3.4. Let $D :\ * - N \oplus M \to (* - \mathcal{A} \oplus X\ )^{(\mathrm{n})}$ be a bounded local (g,h)-triple derivation with $* - N \oplus M$ being commutative. The following statements hold:

1. For all $(s,\ n)\,,\ (r,\ m)\,,\ (c,z),(f,k) \in * - N \oplus M$, we have the identity
$$\begin{aligned}\{g(s,n), D((r,m)(c,z)), h(f,k)\} = &\{g(s,n), D(r,m), h\,((c,z)^*(f,k))\} + \{g\,((r,m)^*(s,n))\,, D(c,z), h(f,k)\} \\ &- \{g\,((r,m)^*(s,n))\,, D(1,0), h\,((c,z)^*(f,k))\}\end{aligned} \tag{3.3}$$
2. For all $(r,\ m) \in * - N \oplus M and\ \ (s,\ n)\,,\ (c,z)\,,(f,k)$ in $(* - N \oplus M)^{**}$, we have the identity
$$\begin{aligned}\{g^{**}(s,n), D^{**}((r,m)(c,z)), h^{**}(f,k)\} = &\{g^{**}(s,n), D(r,m), h^{**}\,((c,z)^*(f,k))\} \\ &+ \{g^{**}\,((s,n)(r,m)^*)\,, D^{**}(c,z), h^{**}(f,k)\} \\ &- \{g^{**}\,((s,n)(r,m)^*)\,, D(1,0), h^{**}\,((c,z)^*(f,k))\}\end{aligned} \tag{3.4}$$
3. If $(s,\ n)^*\ \ (r,\ m) = (r,\ m)^*\ \ (c,z) = (0,0)$, for all $(s,\ n)\,,\ (r,\ m)\,,(c,z)$ in $* - N \oplus M$. Then $g\,(s,\ n)\ \ D\,(r,\ m)^*\ \ h(c,z) = (0,0)$ and $g\,(s,\ n)\ \ D\left((r,\ m)^*\right)^*\ \ h(c,z) = (0,0)$.
4. If $D\,(1,0) = (0,0)$. Then the following statements hold:
   (a) D is (h,g)-derivation, whenever n is even.
   (b) $D \circ *$ is (h,g)-derivation, whenever n is odd.
5. $D(1,0)^* = -\ D(1,0)$.

Proof .

1. Suppose n is an odd integer, let us take $(s,\ n)\,,\ (r,\ m)$ in $* - N \oplus M$, and consider the following map $U_{(s,\ n),(r,\ m)} :$ $* - N \oplus M \times * - N \oplus M \to (* - \mathcal{A} \oplus X)^{(\mathrm{n})}$ defined by
$$U_{(s,\ n),(r,\ m)}\,((c,z),(f,k)) = \{g\,(s,\ n)\,,\ D\,((r,\ m)\,(c,z))\,,\ h\,(f,k)\}.$$
From Proposition 2.1, we have that
$$U_{(s,\ n),(r,\ m)}\,((c,z),(f,k)) = \frac{1}{2}\left(g\,(s,\ n)^*\,D\,((r,m)\,(c,z))^*\,h\,(f,k)^* + h\,(f,k)^*\,D\,((r,\ m)\,(c,z))^*\,g\,(s,\ n)^*\right),$$

for every $(c,z),(f,k)$ in $*-N\oplus M$. In the odd cases of n, D is a conjugate linear mapping, we deduce that $U_{(s,\ n),(r,\ m)}\left((c,z),(f,k)\right)$ is linear in (c,z) and conjugative linear in (f,k). Therefore, if $(s,\ n)^{*}\ (r,\ m)=(0,0)$, then applying Lemma 3.1, we obtain $U_{(s,\ n),(r,\ m)}\left((c,z),(f,k)\right)=(0,0)$, for all (c,z), $(f,k)\, in*-N\oplus M$ with $(c,z)^{*}\ (f,k)=(0,0)$. Lemma 3.2 assures that

$$\begin{aligned}\{g(s,n),D((r,m)(c,z)),h(f,k)\} =&U_{(s,n),(r,m)}((c,z),(f,k))=U_{(s,n),(r,m)}\left((1,0),(c,z)^{*}(f,k)\right)\\ =&\{g(s,n),D(r,m),h\left((c,z)^{*}(f,k)\right)\}\end{aligned}\tag{3.5}$$

for all $(s,n)\,,(r,m)$, $(c,z),(f,k)$ in $*-N\oplus M$. Let $(c,z),(f,k)$ in $*-N\oplus M$, we consider the following map $F_{(c,z),(f,k)}:*-N\oplus M\times *-N\oplus M\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ define by

$$F_{(c,z),(f,k)}\left((r,m)\,,(s,n)\right)=\{g\left(s,n\right),D\left((r,m)\left(c,z\right)\right),h\left(f,k\right)\}-\left\{g\left(s,n\right),D\left(r,m\right),h((c,z)^{*}\left(f,k\right))\right\}.$$

By Proposition 2.1, and D is a conjugate linear mapping, we have that $F_{(c,z),(f,k)}\left((r,m)\,,(s,n)\right)$ is linear in (r,m) and conjugative linear in (s,n). From (3.5), we get $F_{(c,z),(f,k)}\left((r,m)\,,(s,n)\right)=(0,0)$, for all $(r,m)\,,(s,n)\in *-N\oplus M$ with $(r,m)^{*}(s,n)=(0,0)$. Hence Lemma 3.2 assures that $F_{(c,z),(f,k)}\left((r,m)\,,(s,n)\right)=F_{(c,z),(f,k)}((1,0),(r,m)^{*}(s,n))$, for all $(r,m)\,,(s,n)\in *-N\oplus M$, which completes the desired identity. With the exception of minor differences in the conjugacy of the variables and involutions, the same argument holds true for even integers.

2. The proof is like to that of [10, Proposition 3.4].
3. The proof is comparable to that of [10, Proposition 3.5].
4. (a) Suppose n is even and let us consider the mapping $G:*-N\oplus M\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ defined by $G(s,n)=D\left((s,n)^{*}\right)^{*}$. Applying part (3), we see that $g(s,n)\ G(r,m)\ h(c,z)=(0,0)$, for all $(s,n)^{*}(r,m)=(r,m)^{*}(c,z)=(0,0)$ in $*$ - $N\oplus M$. Lemma 3.3 assures that G is a (g,h)-generalized derivation. Therefore,

$$\begin{aligned}D((s,n)(r,m)) =&G\left((r,m)^{*}(s,n)^{*}\right)^{*}\\ =&\left(G\left((r,m)^{*}\right)h(s,n)^{*}g(r,m)^{*}G\left((s,n)^{*}\right)-g(r,m)^{*}G(1,0)h(s,n)^{*}\right)^{*}\\ =&h(s,n)G\left((r,m)^{*}\right)^{*}+G\left((s,n)^{*}\right)^{*}g(r,m)-h(s,n)G(1,0)^{*}g(r,m)\\ =&h(s,n)D(r,m)+D(s,n)g(r,m)-h(s,n)D(1,0)g(r,m)\\ =&h(s,n)D(r,m)+D(s,n)g(r,m).\end{aligned}$$

   Hence, D is a (h,g)-derivation.

   (b) Suppose n is odd and let us consider the following mapping $G:*-N\oplus M\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ defined by $G(s,n)=D(s,n)^{*}$. Part (3) assures that $g(s,n)G(r,m)h(c,z)=(0,0)$, for every $(s,n)^{*}(r,m)=(r,m)^{*}(c,z)=(0,0)$ in $*-N\oplus M$. Lemma 3.3 implies that G is a (g,h)-generalized derivation. Thus,

$$\begin{aligned}D\circ*((s,n)(r,m)) =&D\left((r,m)^{*}(s,n)^{*}\right)=G\left((r,m)^{*}(s,n)^{*}\right)^{*}\\ =&\left(G\left((r,m)^{*}\right)h(s,n)^{*}+g(r,m)^{*}G\left((s,n)^{*}\right)-g(r,m)^{*}G(1,0)h(s,n)^{*}\right)^{*}\\ =&h(s,n)G\left((r,m)^{*}\right)^{*}+G\left((s,n)^{*}\right)^{*}g(r,m)-h(s,n)G(1,0)^{*}g(r,m)\\ =&h(s,n)D\left((r,m)^{*}\right)+D\left((s,n)^{*}\right)g(r,m)-h(s,n)D(1,0)g(r,m)\\ =&h(s,n)(D\circ*)(r,m)+(D\circ*)(s,n)g(r,m).\end{aligned}$$

   Therefore, $(D\circ*)$ is a (h,g)-derivation.
5. The proof is similar to that of [10, Lemma 3.8].

□

Lemma 3.5. Let $D:\ *-\mathcal{A}\oplus X\to\ *-\mathcal{A}\oplus X$ be a continuous local (g,h)-ternary derivation such that $D\left(1,0\right)=(0,0)$, then D is symmetric map, for all $(s,\ n)\in\ *-\mathcal{A}\oplus X$.

Proof . The proof is comparable to that of ([[3], Theorem 9]). □

Proposition 3.6. Let $D:\ *-\mathcal{A}\oplus X\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ be a continuous local (g,h)-ternary derivation such that $D\left(1,0\right)=(0,0)$, then $D\left((s,\ n)^{*}\right)=D\left(s,\ n\right)^{*}$, for all $(s,\ n)\in\ *-\mathcal{A}\oplus X$.

Proof . Suppose n is even, the general form of the proof of Lemma 3.5 is to prove the statement. The same argument still holds true if n is odd, with some involutions changing. However, in order to be complete, we provide the case of odd integers. Suppose n is odd integer. Assume that $(s,\ n)$ is a self-adjoint in $*-\mathcal{A}\oplus X$ and $*-N\oplus M$ is the commutative closed $*$-submodule extension algebra of $*-\mathcal{A}\oplus X$ generated by the unit of $*-\mathcal{A}\oplus X$ and $(s,\ n)$. Since $D|_{*-N\oplus M}:\ *-N\oplus M\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ is a continuous local (g,h)-ternary derivation such that $D\,(1,0)=(0,0)$. Applying Proposition 2.4 (4), we have that $(D\circ\ *)\,|_{*-N\oplus\ M}\ =\ D|_{*-N\oplus M}\circ\ *$ is a (h,g)-derivation. Thus, for a unitary element (e_1,e_2) in $*-N\oplus M$ with $g\,(e_1,e_2)\ \ h\,(e_1,e_2)^*=(1,0)$, we get

$$\begin{aligned}(D\circ *)\left((e_1,e_2)^*(e_1,e_2)\right)=&(D\circ *)\left((e_1,e_2)^*\right)g\,(e_1,e_2)+h\,(e_1,e_2)^*(D\circ *)\,(e_1,e_2)\\=&D\,(e_1,e_2)\,g\,(e_1,e_2)+h\,(e_1,e_2)^*D\left((e_1,e_2)^*\right).\end{aligned}$$

Since $(D\circ\ *)\left((e_1,e_2)^*(e_1,e_2)\right)=D\left((e_1,e_2)^*(e_1,e_2)\right)=D(1,0)=(0,0)$, we have

$$D(e_1,e_2)=-\ h(e_1,e_2)^*\ D((e_1,e_2)^*)\ \ g(e_1,e_2)^*. \tag{3.6}$$

Furthermore, since D is a local (g,h)-ternary derivation, there is a (g,h)-ternary derivation $d_{(e_1,e_2)}$ with $D(e_1,e_2)=d_{(e_1,e_2)}(e_1,e_2)$. So

$$\begin{aligned}D\,(e_1,e_2)=&d_{(e_1,e_2)}(e_1,e_2)=d_{(e_1,e_2)}\left((e_1,e_2)\,(e_1,e_2)^*\,(e_1,e_2)\right)\\=&d_{(e_1,e_2)}\left\{(e_1,e_2)\,,\ (e_1,e_2)\,,(e_1,e_2)\right\}\\=&\left\{d_{(e_1,e_2)}\,(e_1,e_2)\,,h\,(e_1,e_2)\,,h\,(e_1,e_2)\right\}+\left\{g(e_1,e_2),d_{(e_1,e_2)}(e_1,e_2),h(e_1,e_2)\right\}\\&+\left\{g(e_1,e_2),g(e_1,e_2),d_{(e_1,e_2)}(e_1,e_2)\right\}\\=&\{D(e_1,e_2),h(e_1,e_2),h(e_1,e_2)\}+\{g(e_1,e_2),D(e_1,e_2),h(e_1,e_2)\}+\{g(e_1,e_2),g(e_1,e_2),D(e_1,e_2)\}\,.\end{aligned}$$

And Proposition 2.1 assures that

$$D(e_1,e_2)=D(e_1,e_2)+\frac{1}{2}\ (\ g(e_1,e_2)^*\ D(e_1,e_2)^*\ h(e_1,e_2)^*+h(e_1,e_2)^*\ D(e_1,e_2)^*\ g(e_1,e_2)^*)+D(e_1,e_2),$$

this leads to

$$D\,(e_1,e_2)=-\frac{1}{2}(g\,(e_1,e_2)^*\quad D\,(e_1,e_2)^*\,h\,(e_1,e_2)^*+h\,(e_1,e_2)^*\,D\,(e_1,e_2)^*\,g(e_1,e_2)^*). \tag{3.7}$$

We obtain by combining equations (3.6) and (3.7) that

$$h(e_1,e_2)^*D((e_1,e_2)^*)g(e_1,e_2)^*=\frac{1}{2}(g(e_1,e_2)^*D(e_1,e_2)^*h(e_1,e_2)^*+h(e_1,e_2)^*D(e_1,e_2)^*g(e_1,e_2)^*).$$

which gives

$$D((e_1,e_2)^*)\ =\ D(e_1,e_2)^*.$$

Since $*-N\oplus M$ is the unitary elements' linear span, this leads to $D((r,\ m)^*)\ =\ D\,(r,\ m)^*$, for all $(r,\ m)$ in $*-N\oplus M$. The self-adjoint element $(s,\ n)$ is arbitrary, therefore $D\,(r,\ m)^*\ =\ D\,(r,\ m)$, for every $(r,\ m)\in(*-\mathcal{A}\oplus X)_{sa}$, by the linearity of D, which completes the desired result. □

Proposition 3.7. Every local (g,h)-ternary derivation $D:*-\mathcal{A}\oplus X\to\ *-\mathcal{A}\oplus\ X$ is a (g,h)-ternary derivation.

Proof . The proof is like to that of [3, Theorem 10]. □

Theorem 3.8. Every bounded local (g,h)-ternary derivation $D:\ *-\mathcal{A}\oplus X\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ is a (g,h)-ternary derivation.

Proof . Suppose $D:\ *-\mathcal{A}\oplus X\to(*-\mathcal{A}\oplus X)^{(\mathrm{n})}$ is a continuous local (g,h)-ternary derivation and put $\widetilde{D}=D-\delta(\frac{1}{2}D\,(1,0)\,,(1,0))$. Since $\delta(\frac{1}{2}D\,(1,0)\,,(1,0))$ is a continuous (g,h)-ternary derivation, we have that $\widetilde{D}$ is also a continuous local (g,h)-ternary derivation. Now, n is either odd or even, from Proposition 2.1 and Proposition 3.4 (5), we get

$$\begin{aligned}\widetilde{D}(1,0) &= D(1,0) - \delta\left(\frac{1}{2}D(1,0),(1,0)\right)(1,0),\\ &= D(1,0) - \left(\left\{\frac{1}{2}D(1,0), g(1,0), h(1,0)\right\} - \left\{g(1,0), \frac{1}{2}D(1,0), h(1,0)\right\}\right),\\ &= D(1,0) - \frac{1}{2}D(1,0) + \frac{1}{2}D(1,0)^*,\\ &= D(1,0) - \frac{1}{2}D(1,0) - \frac{1}{2}D(1,0),\\ &= (0,0).\end{aligned}$$

Assume that (s,n) is a self-adjoint in $*-\mathcal{A}\oplus X$ and $*-N\oplus M$ denotes $*-$ submodule extension algebra of $*-\mathcal{A}\oplus X$ generated by (s,n) and the unit of $*-\ \mathcal{A}\oplus X$, which is commutative. Since $\widetilde{D}\Big|_{*-N\oplus M}$ is a continuous local (g,h) ternary derivation with $\widetilde{D}(1,0)=(0,0)$, Proposition 3.4 (4) assures that $\widetilde{D}\Big|_{*-N\oplus M}$ is a (h,g)-derivation, when n is even and $(\widetilde{D}\circ *)\Big|_{*-N\oplus M} = \widetilde{D}\Big|_{*-N\oplus M}\circ *$ is a (h,g)-derivation, when n is odd. Thus, we have that

$$\widetilde{D}\left((s,\ n)^2\right) = \widetilde{D}\,(s,\ n)\ \ g\,(s,\ n) + h\,(s,\ n)\ \ \widetilde{D}\,(s,\ n)\,. \tag{3.8}$$

For each self-adjoint elements $(s,\ n)\,,\ (r,\ m)$ in $*-\mathcal{A}\oplus X$, we conclude from (3.8) that

$$\widetilde{D}\left(((s,n)+(r,m))^2\right) = \widetilde{D}((s,n)+(r,m))\ g((s,n)+(r,m)) + h((s,n)+(r,m))\widetilde{D}((s,n)+(r,m)). \tag{3.9}$$

We obtain by combining equations (3.8) and (3.9) that

$$\widetilde{D}((s,n)\circ(r,m)) = \widetilde{D}(s,n)\bullet(h,g)\ (r,m) + (h,g)\ (s,n)\bullet\widetilde{D}(r,m) \tag{3.10}$$

for every $(s,\ n)\,,\ (r,\ m)\in(*-\mathcal{A}\oplus X)_{sa}$.

Let us now explore the integer n the two distinct cases of odd and even, one at a time. The same argument for Proposition 3.7 could be used to show that $\widetilde{D}$ is a (g,h)-ternary derivation, whenever n is even. Let n be odd. We see from linearity of $\widetilde{D}\circ *$ with equation (3.10) that $(\widetilde{D}\circ *)((s,n)\circ(r,m)) = (\widetilde{D}\circ *)(s,n)\bullet(h,g)(r,m) + (h,g)(s,n)\bullet(\widetilde{D}*)(r,m)$, for every $(s,n),(r,m)\in *-\ \mathcal{A}\oplus X$, this implies that $\widetilde{D}\circ *$ is a Jorden (h,g)-derivation. We have from [[6], Theorem 6.2] that $\widetilde{D}\circ *$ is an associative (h,g)-derivation and by applying Proposition 3.6, we obtain

$$\begin{aligned}
&\widetilde{D}\left\{(s,\ n)\,,\ (r,\ m)\,,(c,z)\right\} = \tfrac{1}{2}\left(\widetilde{D}\circ\ *\right)\left[(c,z)^*(r,\ m)\,(s,\ n)^* + \left((c,z)\,(r,\ m)^*\,(s,\ n)\right)^*\right],\\
&= \tfrac{1}{2}\Big[\ \left(\widetilde{D}\circ\ *\right)\left((c,z)^*\ (r,\ m)\ (s,\ n)^*\right) + \left(\widetilde{D}\circ\ *\right)\left((c,z)\ (r,\ m)^*\ (s,\ n)\right)^*\Big],\\
&= \tfrac{1}{2}\Big[\ \left(\widetilde{D}\circ\ *\right)\left((c,z)^*\ (r,\ m)\right)\ g\,(s,\ n)^* + h\left((c,z)^*(r,\ m)\right)\ \left(\widetilde{D}\circ\ *\right)\left((s,\ n)^*\right)\\
&+ \left(\left(\widetilde{D}\circ\ *\right)\left((c,z)\ (r,\ m)^*\right)\ g\,(s,\ n) + h\left((c,z)\,(r,\ m)^*\right)\ \left(\widetilde{D}\circ\ *\right)(s,\ n)\right)^*\Big],\\
&= \tfrac{1}{2}\Big[(\widetilde{D}\circ *)\left((c,z)^*\right)\ g(r,m)\ g(s,n)^* + h(c,z)^*(\widetilde{D}\circ *)(r,m)\ g(s,n)^*\\
&+h(c,z)^*\ h(r,m)\ (\widetilde{D}\circ *)\left((s,n)^*\right) + \Big((\widetilde{D}\circ *)(c,z)\ g(r,m)^*\ g(s,n)+\\
&h(c,z)\ (\widetilde{D}\circ *)\left((r,m)^*\right)\ g(s,n) + h(c,z)\ h(r,m)^*\ (\widetilde{D}\circ *)(s,n)\Big)^*\Big],\\
&= \tfrac{1}{2}\Big[(\widetilde{D}\circ *)\left((c,z)^*\right)\ g(r,m)\ g(s,n)^* + h(c,z)^*\ (\widetilde{D}\circ *)(r,m)\ g(s,n)^*\\
&+h(c,z)^*\ h(r,m)\ (\widetilde{D}\circ *)\left((s,n)^*\right) + g(s,n)^*\ g(r,m)(\widetilde{D}\circ *)\left((c,z)^*\right)\\
&+g(s,n)^*\ (\widetilde{D}\circ *)(r,m)\ h(c,z)^* + (\widetilde{D}\circ *)\left((s,n)^*\right)\ h(r,m)\ h(c,z)^*\Big],\\
&= \tfrac{1}{2}\Big[\widetilde{D}(c,z)\ g(r,m)\ g(s,n)^* + h(c,z)^*\ \widetilde{D}(r,m)^*\ g(s,n)^* + h(c,z)^*\ h(r,m)\ \widetilde{D}(s,n) + g(s,n)^*\ g(r,m)\ \widetilde{D}(c,z)\\
&+g(s,n)^*\ \widetilde{D}(r,m)^*\ h(c,z)^* + \widetilde{D}(s,n)\ h(r,m)\ h(c,z)^*\Big],\\
&= \{\widetilde{D}(s,n),\ h(r,m),\ h(c,z)\} + \{g(s,n),\ \widetilde{D}(r,m),\ h(c,z)\} + \{g(s,n),\ g(r,m),\ \widetilde{D}(c,z)\}.
\end{aligned}$$

Therefore, $\widetilde{D}$ is (g, h)-ternary derivation. Since $D = \widetilde{D} + \delta(\frac{1}{2}D\,(1,0)\,,(1,0))$ is the sum of two (g, h)-ternary derivations. So, we have that D is (g, h)-ternary derivation. □

The following result is a generalization of Theorem 3.9 in [10].

Corollary 3.9. Let $\mathcal{A}$ be a unital C^*-algebra. Each continuous local ternary derivation $D:\ \mathcal{A} \to \mathcal{A}^{(\mathrm{n})}$ is a ternary derivation.

Proof . By theorem 3.8, taking g and h to be the identity maps and X=0. □

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