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Non-expansive linear random dynamical systems

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Abstract

The aim of this work is to present and study the concepts of non-expansive, quasi-nonexpansive, α-nonexpansive and asymptotically-nonexpansive of the linear random dynamical systems, and several essential facts be given. Also, the relationships between these concepts are shown.

Keywords: Random Dynamical Systems, non-expansive, $\alpha-$nonexpansive, asymptotically-nonexpansive
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1 Introduction

The main objetive of the theory of dynamical system is the know of the global orbit structure of maps and flows.The new approach of study the dynamical system is so call Random Dynamical Systems(RDS$_s$). The Random Dynamical Systems are an importance in the modeling of several phenomena in biology, physics, etc. In 1945 Ulam and von Neumann the first lesson the RDS . L. Arnold and I.D. Chueshov [3] to introduce the concept of RDS.

The aim of this work is the study of non-expansitivity in linear random dynamical system where the phase space consider as a Banach space. In the following some previous studies. In 2011, Koji Aoyama, Fumiaki Kohsaka [1] present the type of α- nonexpansive functions in Banach spaces. This type satisfies the type of non-expansive functions and addition, they obtain a fixed point theorem for α-nonexpansive functions in uniformly convex Banach spaces. I. J. Kadhim and A. H. Khalil [6] study the expansive random operator modeled with uniform random dynamical systems. Sami Atailia and other [4] were reflected generalized contractions of Suzuk type. In 2021, Pant, R. and others [7] extant certain fixed point consequences for a non-expansive and $\alpha-$ Krasnosel'ski$\hat{1}$ functions. Furthermore, they extant certain convergence results of one parameter non-expansive semi-groups.

Definition 1.1 ([5]). The $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a **metric dynamical system** (Shortly MDS) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the function $\theta : \mathbb{T} \times \Omega \to \Omega$ satisfy

1. θ is measurable,
2. $\theta(0, \omega) = \omega$, for every $\omega \in \Omega$,
3. $\theta_{t+s}(\omega) = (\theta_t \circ \theta_s)(\omega)$ for ever $\omega \in \Omega, t, s \in \mathbb{T}$ and
4. $\mathbb{P}(\theta_t F) = \mathbb{P}(F)$for every $F \in \mathcal{F}$ and every $t \in \mathbb{T}$.

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Definition 1.2 (Random Dynamical System). [2] Let X be a topological space and θ be a MDS. A topological random dynamical system on X over θ is a function $\varphi : \mathbb{T} \times \Omega \times X \to X$, admit the following properties:

1. $\varphi(\cdot, \omega, \cdot) : \mathbb{T} \times X \to X$ is continuous for every $\omega \in \Omega$.
2. The mapping $\varphi(t, \omega)\, \varphi(t, \omega, \cdot) : X \to X$ form a cocycle on $\theta(\cdot)$, that is satisfy
$\varphi(0, \omega)\, x = x$.
$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}, \omega \in \Omega$.

Definition 1.3 (Linear RDS$_\mathrm{S}$). [2] A linear random dynamical system LRDS is an RDS (θ, φ) on the Banach space X such that $\varphi(t, \omega)$ is linear operators of X for each $t \in \mathbb{T}, \omega \in \Omega$.

Definition 1.4. [5] Suppose that (X, d) be a metric space which is a measurable space with Borel σ-field $\mathcal{B}(X)$ and $(\Omega, \mathcal{F})$ be a measurable space. The set-valued function $A : \Omega \to \mathcal{B}(X), \omega \longmapsto A(\omega)$, is a **random set** if the mapping $\omega \longmapsto d(x, A(\omega))$ is measurable for each $x \in X$. The random set $A(\omega)$ is called a **random closed(compact) set** , if it is closed (compact) for all $\omega \in \Omega$.

Definition 1.5. [2] Let $A : \Omega \to \mathcal{B}(X)$ take values in the subspace of closed subsets of a Polish space X. Then:

1. A is a random closed set if and only if the set $\{\omega : A_\omega \cap U \neq \emptyset\}$ is measurable for all open sets $U \subset X$.
2. graph(A) $\in \mathcal{F} \bigotimes \mathcal{B}$,if A is a random closed set.

Definition 1.6 (Invariance Property). [2] Let $(\theta,\ \varphi)$ be a (RDSS). A multifunction $\omega \longmapsto D(\omega)$ is said to be

1. forward invariant with respect to $(\theta,\ \varphi) if \varphi(t,\ \omega) D(\omega) \subseteq D(\theta_t \omega), \forall\ t > 0, \omega \in \Omega$, i.e. if $x \in D(\omega) implies \varphi(t,\ \omega) x \in D(\theta_t \omega)$ for all $t \geq 0$ and $\omega \in \Omega$;
2. backward invariant with respect to $(\theta,\ \varphi)$ if $\varphi(t,\ \omega)\, D(\omega) \supseteq D(\theta_t \omega), \forall\ \ t > 0, \omega \in \Omega$, i.e. for every $t > 0, \omega \in \Omega$ and $y \in D(\theta_t \omega)$ there exists $x \in D(\omega)$ such that $\varphi(t,\ \omega)\, x = y; \varphi(t,\ \omega)\ D(\omega) \subseteq D(\theta_t \omega), t < 0$.

Definition 1.7. [5] Let (θ, φ) be a (RDS). A a random variable $x : \Omega \longrightarrow X$ is called a random fixed point if there exists a full measure set $\widetilde{\Omega}$ in Ω such that $\varphi(t,\ \omega)\, x(\omega) = x(\theta_t \omega)$, for all $t \in \mathbb{R}, \omega \in \widetilde{\Omega}$. The set of all random fixed points of $(\theta,\ \varphi)$ is symbolized FX^{Ω}.

Definition 1.8. [8] Let $\mathfrak{D}$ be a family of random closed sets is called a universe of sets if it is closed with respect to inclusions (i.e. if $D_1 \in \mathfrak{D}$ and a random closed set D_2 possesses the property $D_2(\omega)\ \subset D_1(\omega)$ for all $\omega \in \Omega$, then $D_2 \in \mathfrak{D}$).

2 Main Results

Here some types of the nonexpansitivity are introduces and studied in LRDS' s , where the phase space X considered as a Banach space, also it is considers as a measurable space with Borel σ-algebra.

Definition 2.1. A LRDS (θ, φ) is said to nonexpansive if for every random variables $x, y : \Omega \to X$ and

$$\mathbb{P}\{\omega : x(\omega) \neq y(\omega)\} = 1$$

we have

$$\left\| \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\ \right\| \leq \|x(\omega) - y(\omega)\|.$$

Remark 2.2. If x and y are any two points (not necessarily random) in X, then the above definition can be formulated as follows:

A LRDS (θ, φ) is said to be a **nonexpansive** if for every $x, y \in X$ and $x \neq y$ we have

$$\|\varphi(t, \theta_{-t}\omega)\, x - \varphi(t, \theta_{-t}\omega)\, y\| \leq \|x - y\|.$$

Definition 2.3. A LRDS (θ, φ) is said to be a **quasi-nonexpansive** if for all $x \in X^{\Omega}$ and $z \in FX^{\Omega}$

$$\left\| \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - z\ (\theta_{-t}\omega) \right\| \leq \|x(\omega) - z(\omega)\|.$$

Definition 2.4. Let (θ,φ) be a LRDS and Y be a nonempty subset of X is said to be satisfies **condition** (RE_μ) on Y if there exists $\mu \geq 1$ such that

$$\|x(\omega) - \varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega)\ \| \leq \mu\|x(\omega) - \varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega)\| + \|x(\omega) - y(\omega)\|$$

for all random variables $x, y : \Omega :\longrightarrow Y$. (θ,φ) satisfies condition (RE_μ) on Y when satisfies (RE_μ) for some $\mu \geq 1$.

Proposition 2.5. If a LRDS (θ,φ) is satisfies condition (RE) with $FX^\Omega \neq \emptyset$ then (θ,φ) is quasi-nonexpansive

Proof . Let $x \in X^\Omega$ and $z \in FX^\Omega$. Since (θ,φ) satisfies condition (RE),

$$\begin{aligned}\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - z\ (\omega)\right\| &\leq \left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - z\ (\omega)\right\| + \|z(\omega) - z\ (\omega)\| \\ &\leq \mu\left\|z\ \ (\omega) - \varphi(t,\theta_{-t}\omega)\, z(\theta_{-t}\omega)\right\| + \|z(\omega) - x(\omega)\| \\ &\leq \mu\|z\ \ (\omega) - z(\omega)\| + \|z(\omega) - x(\omega)\| \\ &\leq \mu\|z(\omega) - x(\omega)\|.\end{aligned}$$

This means that (θ,φ) is quasi-nonexpansive. □

Definition 2.6. A LRDS (θ,φ) is called be a generalized α-nonexpansive if there exists $\alpha \in [0,1)$ and a full measure $\widetilde{\Omega}$ of Ω such that

$$\frac{1}{2}\left\|x(\omega) - \varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| \leq \|x(\omega) - y(\omega)\|$$

implies

$$\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| \leq \max\{P(x,y), Q(x,y)\} \tag{2.1}$$

for every $x, y \in X$, where

$$P(x,y) := \alpha\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - x(\omega)\right\| + \alpha\left\|\varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega) - y(\omega)\right\| + (1-2\alpha)\|x(\omega) - y(\omega)\|$$
$$Q(x,y) := \alpha\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - y(\omega)\right\| + \alpha\left\|\varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega) - x(\omega)\right\| + (1-2\alpha)\|x(\omega) - y(\omega)\|.$$

Definition 2.7. Let Y is a nonempty convex subset of X and (θ,φ) LRDS, φ_α is said to be a α –convex if there exists $\alpha \in (0,1)$ such that,

$$\varphi_\alpha(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) = (1-\alpha\ )x(\omega) +\ \alpha\ \varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega)$$

for all random variable $x : \Omega \longrightarrow Y$.

Definition 2.8. Let Y is a nonempty convex subset of X. The LRDS (θ,φ) is called asymptotically regular if for every sequences $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to +\infty$ there exists a full measure $\widetilde{\Omega} \subset \Omega$ such that

$$\lim_{n,m\to\infty}\left\|\varphi(t_n,\theta_{-t_n}\omega)\, x(\theta_{-t_n}\omega) - \varphi(t_m,\theta_{-t_m}\omega)\, x(\theta_{-t_m}\omega)\right\| = 0,$$

for all $n, m \in \mathbb{N}$ and $m > n$ and $\omega \in \widetilde{\Omega}$.

Definition 2.9. The LRDS (θ,φ) is called asymptotically-nonexpansive if for all $x, y\ \in X$, with $x \neq y$ and sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to +\infty$, there exists a full measure $\widetilde{\Omega}$ of Ω such that

$$\lim_{n\to\infty}\left\|\varphi(t_n,\theta_{-t_n}\omega)\, x\theta_{-t_n}\omega) - \varphi(t_n,\theta_{-t_n}\omega)\, y(\theta_{-t}\omega)\right\| = 0,\ \omega \in\ \widetilde{\Omega}.$$

Definition 2.10. The LRDS (θ,φ) is called a symptotically-expansive if for all $x, y \in X, x \neq y$, and sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to +\infty$, there exists a full measure $\widetilde{\Omega}$ of Ω

$$\lim_{n\to\infty}\left\|\varphi(t_n,\theta_{-t_n}\omega)\, x(\theta_{-t_n}\omega) - \varphi(t_n,\theta_{-t_n}\omega)\, y(\theta_{-t_n}\omega)\right\| = \infty\ ,\ \omega \in\ \widetilde{\Omega}.$$

Definition 2.11. A LRDS (θ,φ) is call α-nonexpansive if there is an $0 < \alpha < 1$ such that for all $x, y \in X$

$$\begin{aligned}\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| \leq\ & \alpha\left\|\varphi(t,\theta_{-t}\omega)\, x(\theta_{-t}\omega) - y(\omega)\right\| \\ &+ \alpha\left\|\varphi(t,\theta_{-t}\omega)\, y(\theta_{-t}\omega) - x(\omega)\right\| +\ (1-2\alpha)\|x(\omega) - y(\omega)\|.\end{aligned}$$

Definition 2.12. A LRDS (θ, φ) is a α-nonexpansive is called **uniformly a symptotically regular** (u.a.r.) if for any $\mathcal{S} \in \mathbb{R}^+$ and any closed random set D in the universal
$\mathcal{D} \equiv$ universal

$$\lim_{t\to\infty} \sup_{x\in D(\theta_{-t}\omega)} \left\|\varphi(s+t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| = 0$$

Definition 2.13. A LRDS (θ, φ) is called a generalized contraction type-$\boldsymbol{\beta}$ if there exists $\beta \in (0,1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ where $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 1$ a full measure $\widetilde{\Omega}$ of Ω such that for every $x, y \in X$,

$$\beta\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| \le \|x(\omega) - y(\omega)\|$$

implies

$$\begin{aligned} \left\|\varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| &\le \alpha_1\|x(\omega) - y(\omega)\| + \alpha_2\Big(\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| \\ &\quad + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\|\Big) \\ &\quad + \le \alpha_3\Big(\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\|\Big). \end{aligned} \tag{2.2}$$

Theorem 2.14. If LRDS (θ, φ) be a generalized contraction type-β (with $\beta = 1/2$) then (θ, φ) is a generalized α-nonexpansive.

Proof . Suppose that every $x, y \in X$,

$$\frac{1}{2}\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| \le \|x(\omega) - y(\omega)\|.$$

By hypothesis (θ, φ) is a generalized contraction type- β (with $\beta = 1/2$), then by definition we have

$$\begin{aligned} \left\|\varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| &\le \alpha_1\|x(\omega) - y(\omega)\| + \alpha_2\Big(\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| \\ &\quad + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\|\Big) \\ &\quad \le \alpha_3\Big(\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\|\Big). \end{aligned} \tag{2.3}$$

Case 1.

$$\begin{aligned} &\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| \\ &\qquad \ge \left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\|. \end{aligned}$$

Then 2.3 becomes

$$\begin{aligned} &\left\|\varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| \le \alpha_1\|x(\omega) - y(\omega)\| \\ &\qquad + (\alpha_2 + \alpha_3)\Big(\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\|\Big). \end{aligned}$$

Take $\alpha = \alpha_2 + \alpha_3$, since $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 1, \alpha_1 = 1 - 2\alpha$, the above condition becomes

$$\begin{aligned} \left\|\varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| &\le \alpha\left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| + \alpha\left\|y(\omega) - \varphi(n, \theta_{-n}\omega)\, y(\theta_{-t}\omega)\right\| \\ &\quad + (1 - 2\alpha)\|x(\omega) - y(\omega)\|. \end{aligned} \tag{2.4}$$

Case 2.

$$\begin{aligned} \left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\| + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| &< \left\|x(\omega) - \varphi(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega)\right\| \\ &\quad + \left\|y(\omega) - \varphi(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega)\right\|. \end{aligned}$$

Then (3) reduces to

$$\left\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq\alpha_1\|x(\omega)-y(\omega)\|+(\alpha_2+\alpha_3)\Big(\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|$$
$$+\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|\Big).$$

Again, take $\alpha=\alpha_2+\alpha_3$, since $\alpha_1+2\alpha_2+3\alpha_3=1$, $\alpha_1=1-2\alpha$, thus the above condition becomes

$$\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\|\leq\alpha\|x\left(\omega\right)-\varphi\left(t,\theta_{-t}\omega\right)y\left(\theta_{-t}\omega\right)\|+\alpha\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|$$
$$+(1-2\alpha)\|x(\omega)-y(\omega)\|. \tag{2.5}$$

Let

$$P\left(x,y\right):=\alpha\left\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-x(\theta_{-t}\omega)\right\|+\alpha\left\|(t,\theta_{-t}\omega)\,y(\theta_{-t}\omega)-y(\omega)\right\|+(1-2\alpha)\|x(\omega)-y(\omega)\|$$

$$Q\left(x,y\right):=\alpha\left\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-y(\omega)\right\|+\alpha\left\|\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)-x(\omega)\right\|+(1-2\alpha)\|x(\omega)-y(\omega)\|.$$

Then, by 2.4 and 2.5 we have

$$\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\leq\max\left\{P\left(x,y\right),Q(x,y)\right\}.$$

This proves the theorem. □

Lemma 2.15. Suppose (θ,φ) a LRDS and a generalized contraction of type-β and $\beta\in\left[\frac{1}{2}\ ,1\right)$. Then

$$\left\|x(\omega)-\varphi\left(t,\theta_t\omega\right)y(\theta_{-t}\omega)\right\|\leq\frac{2+\alpha_{\ 1}+\alpha_2\ +3\alpha_3}{1-\alpha_2-\alpha_3}\Big(\left\|x(\omega)-\varphi\left(t,\theta_t\omega\right)x(\theta_{-t}\omega)\ \right\|+\|\ x(\omega)-y(\omega)\ \|\right\|\Big).$$

Proof . By the triangular inequality, to admit

$$\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|+\left\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|.$$

Now, if

$$\left\|\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq\alpha_1\|(x\omega)-y(\omega)\|$$
$$+\alpha_2\Big(\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|\Big)+\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|$$
$$+\alpha_3\Big(\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|+\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|\Big)$$

then

$$\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq\alpha_1\|(x\omega)-y(\omega)\|+(1+\alpha_2)\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|$$
$$+\alpha_2\Big(\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\Big)$$
$$+\alpha_3\Big(\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|+\left\|y(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|\Big).$$

Similarly

$$\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq\alpha_1\|x\left(\omega\right)-y\left(\omega\right)\|+(1+\alpha_2)\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|$$
$$+\alpha_2\Big(\|x\left(\omega\right)-y(\omega)\|\Big)+\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|$$
$$+\alpha_3\Big(\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|+\|x\left(\omega\right)-y(\omega)\|+\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|\Big).$$

It follows that

$$(1-\alpha_2-\alpha_3)\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)y(\theta_{-t}\omega)\right\|\leq(\alpha_1+\alpha_2+\alpha_3)\left\|x(\omega)-\varphi\left(t,\theta_{-t}\omega\right)x(\theta_{-t}\omega)\right\|.$$

Thus

$$\left\|x(\omega)-\varphi\left(t, \theta_{-t} \omega\right) y(\theta_{-t} \omega)\right\| \leq (\frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3})\|x\left(\omega\right)-y(\omega)\|+(\frac{1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3})\left\|x(\omega)-\varphi\left(t, \theta_{-t} \omega\right) x(\theta_{-t} \omega)\right\|$$
$$=\|x\left(\omega\right)-y(\omega)\|+(\frac{1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3})\left\|x(\omega)-\varphi\left(t, \theta_{-t} \omega\right) x(\theta_{-t} \omega)\right\|.$$

□

Proposition 2.16. Suppose that (θ, φ) a LRDS and a generalized contraction of type- β. Then (θ, φ) satisfies condition (RE)

Proof . Take $\mu=\frac{2+\alpha_1+\alpha_2\ +3\propto_3}{1-\propto_2-\propto_3} \geq 1$ in Lemma 2.15, then (θ, φ) satisfies the condition (RE)

$$\begin{aligned}
\frac{2+\alpha_1+\alpha_2\ +3\propto_3}{1-\propto_2-\propto_3} \geq 1 \quad & \text{if and only if} \quad 2+\alpha_1+\alpha_2+3\alpha_3 \geq 1-\alpha_2\ -\alpha_3 \\
& \text{if and only if} \quad 1+\alpha_1+2\alpha_2\ +4\alpha_3 \geq 0 \\
& \text{if and only if} \quad 1+\alpha_1+2\alpha_2\ +2\alpha_3+2\alpha_3 \geq 0 \qquad (\text{since } \alpha_1+2\alpha_2\ +2\alpha_3 \geq 1) \\
& \text{if and only if} \quad 1+1+2\alpha_3 \geq 0 \\
& \text{if and only if} \quad 2+2\alpha_3 \geq 0 \\
& \text{if and only if} \quad \alpha_3 \geq -1.
\end{aligned}$$

□

Definition 2.17. A set of random variable X^{Ω} is called **almost surly closed** if for every sequence $\{x_n\} \in X^{\Omega}$, we have $x_n \longrightarrow x \in X^{\Omega}$ almost surly .

Definition 2.18. Let X a normed linear space, and let $x_n,\ x \in X^{\Omega}$. We say that x_n, converges strongly almost surly to x, and write $x_n \longrightarrow x$, if there exists a full measure Ω^* subset of Ω

$$\lim_{n \longrightarrow \infty}\|x_n(\omega)-x(\omega)\|=0, \qquad \text{for all } \omega \in \Omega^*.$$

Theorem 2.19. Let LRDS (θ, φ) satisfy the condition (RE) with $FX^{\Omega} \neq \emptyset$, then

1. FX^{Ω} is almost surly closed.
2. If the subset Y is convex, then FX^{Ω} is convex.
3. If the subset Y is convex compact and X strictly(convex)compact, $\varphi\left(t, \theta_{-t} \omega\right): X \longrightarrow X$ is continuous mapping, then for any $y_0 \in Y, \alpha \in (0,1)$, the $\alpha-$ convex process $\{\ \varphi_\alpha\left(n, \theta_{-n} \omega\right) y_0\}$ converges to some $y^* \in FX^{\Omega}$.

Proof .

1. Define $FX^{\Omega}=\{x: \Omega \longrightarrow X: \varphi\left(t, \theta_{-t} \omega\right) x(\theta_{-t} \omega)=x(\omega),\ \ \forall\ t \in \mathbb{R}\ \}$. Let $\{x_n\} \subset FX^{\Omega}$ such that

$$x_n \longrightarrow x \text{ almost surly} \tag{2.6}$$

This means that $\mathbb{P}\left\{\omega: x_n(\omega) \longrightarrow x\ (\omega)\right\}=1$. Since $\varphi\ \left(t,\ \theta_{-t} \omega\ \right): X \longrightarrow X$ is continuous, we have

$$\varphi\ \left(t\ ,\ \theta_{-t} \omega\ \right) x_n(\theta_{-t} \omega) \longrightarrow \varphi\left(t\ ,\ \theta_{-t} \omega\right) x(\theta_{-t} \omega).$$

But $\varphi\left(t\ ,\ \theta_{-t} \omega\right) x_n(\theta_{-t} \omega)=x_n(\omega)$, so

$$x_n(\omega) \longrightarrow \varphi\left(t,\ \theta_{-t} \omega\right) x(\theta_{-t} \omega). \tag{2.7}$$

Thus by the uniqueness of the limit of the sequence we have

$$\varphi\left(t,\ \theta_{-t} \omega\right) x(\theta_{-t} \omega)=x(\omega).$$

This means that $x \in FX^{\Omega}$. Consequently FX^{Ω} is almost surly closed.

2. Since X is strictly convex, Y is convex. Let $\beta \epsilon\ (0,1\ )$ and $x, y \in FX^{\Omega}$ with $x \neq y$, put $z(\omega) = \beta x(\omega) + (1-\beta)\, y(\omega) \in Y$, for every $\omega \in \Omega$. Then $z(\omega) \in Y$, for every $\omega \in \Omega$. Since (φ, θ) is satisfies condition (RE),

$$\|x(\omega) - \varphi\,(t, \theta_{-t}\omega)\, z(\theta_{-t}\omega)\| \leq \|x(\omega) - \varphi(t, \theta_{-t}\omega) x(\theta_{-t}\omega)\| + \|x(\omega) - z(\omega)\| = \|x(\omega) - z(\omega)\|.$$

From strict convexity of X, there is a $\mu \in [0,1]$ such that

$$\begin{aligned} \varphi\,(t, \theta_{-t}\omega)\, z(\theta_{-t}\omega) = &\ \mu \varphi\,(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) + (1-\mu)\, \varphi\,(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega) \\ = &\ \mu x(\omega) + (1-\mu)\, y(\omega) \end{aligned}$$

implies

$$\begin{aligned} (1-\mu)\, \|x(\omega) - y(\omega)\| &= \|\varphi\,(t, \theta_{-t}\omega)\, x(\theta_{-t}\omega) - \varphi\,(t, \theta_{-t}\omega)\, z(\theta_{-t}\omega)\| \\ &\leq \|x(\omega) - z(\omega)\| = (1-\beta)\|x(\omega) - y(\omega)\| \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \|x(\omega) - y(\omega)\| &= \|\varphi\,(t, \theta_{-t}\omega)\, y(\theta_{-t}\omega) - \varphi\,(t, \theta_{-t}\omega)\, z(\theta_{-t}\omega)\| \\ &\leq \|y\,(\omega) - z\,(\omega)\| = \beta \|x\,(\omega) - y\,(\omega)\|. \end{aligned} \tag{2.9}$$

From 2.8 and 2.9

$$1 - \mu \leq 1 - \beta \text{ and } \mu \leq \beta \text{ then } \mu = \beta.$$

Hence $\varphi\,(t, \theta_{-t}\omega)\, z(\theta_{-t}\omega) = z(\omega)$, and $z \in FX^{\Omega}$

3. Let us define (y_n) by $y_n(\omega) = \varphi_\alpha\,(t, \theta_{-t}\omega)\, y_0(\theta_{-t}\omega),\ \ y_0 \in Y$, where

$$\varphi_\alpha\,(t, \theta_{-t}\omega)\, y_0(\theta_{-t}\omega) = (1-\alpha)\, y_0(\omega) + \alpha \varphi\,(t, \theta_{-t}\omega)\, y_0(\theta_{-t}\omega), \quad \alpha \in (0,1).$$

Since Y is compact, there is a subsequence $(y_{nk}(\omega))$ of $(y_n(\omega))$ converges (almost surly) to some $y^*(\omega) \in Y$ .Since $\varphi\,(t, \theta_{-t}\omega)$ is continuous, from given we have $FX^{\Omega} \neq \emptyset$, We prove that $y^* \in FX^{\Omega}$ let $x_0 \in FX^{\Omega}$

$$\begin{aligned} \|y_n(\omega) - x_0(\omega)\| &= \|\varphi_\alpha\,(t, \theta_{-t}\omega)\, y_0(\theta_{-t}\omega) - x_0(\omega)\| \\ &\leq \|\varphi_\alpha\,(t, \theta_{-t}\omega)\, y_0(\theta_{-t}\omega) - x_0(\omega)\| = \|y_{t-1}(\omega) - x_0(\omega)\|. \end{aligned}$$

Therefore, $\{\|y_n(\omega) - x_0(\omega)\|\}$ is a decreasing sequence for every $\omega \in \Omega$ which bounded below by 0. So, it converges. Furthermore, since φ_α is a continuous,

$$\begin{aligned} \|y^*(\omega) - x_0(\omega)\| &= \lim_{k \longrightarrow \infty} \left\|y_{n_{k+1}}(\omega) - x_0(\omega)\right\| \\ &\leq \lim_{k \longrightarrow \infty} \left\|y_{n_{k+1}}(\omega) - x_0(\omega)\right\| \\ &= \lim_{k \longrightarrow \infty} \|\varphi_\alpha\,(t, \theta_{-t}\omega)\, y_{n_k}(\theta_{-t}\omega) - x_0(\omega)\| \\ &= \|\varphi_\alpha\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\| \\ &= (1-\alpha)\, \|y^*(\omega) + \alpha \varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - y_0(\omega)\| \\ &= (1-\alpha)\, \|y^*(\omega) - x_0(\omega)\| + \alpha \|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\|. \end{aligned} \tag{2.10}$$

Since $\alpha \neq 0$, it implies that

$$\|y^* - x_0\| \leq \|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\|.$$

Since φ is a quasi-nonexpansive

$$\|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\| \leq \|y^*(\omega) - x_0(\omega)\|.$$

From the above ,we get

$$\|y^*(\omega) - x_0(\omega)\| = \|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\|. \tag{2.11}$$

From 2.10, we have

$$\begin{aligned} \|y^*(\omega) - x_0(\omega)\| &\leq \|(1-\alpha)\, y^*(\omega) + \alpha \varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\| \\ &\leq (1-\alpha)\, \|y^*(\omega) - x_0(\omega)\| + \alpha \|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\| \\ &= \|y^*(\omega) - x_0(\omega)\|. \end{aligned}$$

Therefore,

$$\|(1-\alpha)\, y^*(\omega) + \alpha\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\| = (1-\alpha)\,\|y^*(\omega) - x_0(\omega)\| \\ + \alpha\|\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega)\|.$$

Since X is strictly convex, either $\varphi\,(t, \theta_{-t}\omega)\, y^*(\theta_{-t}\omega) - x_0(\omega) = c(y^*(\omega) - x_0(\omega))$, for some $c > 0$ or $y^*(\omega) = x_0(\omega)$ from 2.11, we have $c = 1$,then,

$$\varphi\,(t, \theta_{-t}\omega)\, y^*\,(\theta_{-t}\omega) = y^*\,(\omega) \text{ therefore } y^* \in FX^{\Omega}.$$

Since $\lim_{n\to\infty} \|y_n(\omega) - x_0(\omega)\|$ exists the sequence $\{y_{n_k}(\omega)\}$ converges strongly almost surly to $y^*(\omega)$, $\{y_n\}$ converges strongly almost surly to $y^* \in FX^{\Omega}$.

□

3 Conclusions

In this paper we obtained, if (θ, φ) be a generalized contraction type-β then satisfies condition (RE), and if $(\beta = 1/2)$, then (θ, φ) is a generalized α-nonexpansive . we also showed if (θ, φ) condition (RE) and $FX^{\Omega} \neq \emptyset$ and Ysubset convex compact and $\varphi\,(t, \theta_{-t}\omega)$ is continuous then for any $y_0 \in Y$, $\alpha \in (0, 1)$, the α-convex process $\{\varphi_\alpha\,(n, \theta_{-n}\omega)\, y_0\}$ converges to some $y^* \in FX^{\Omega}$.

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