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Stability analysis of a stochastic model of absorption of the drugs problem

Asmahan A. Yasir[a,*], Ihsan J. Kadhim[b]

[a]Department of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf, Iraq

[b]Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwania, Iraq

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Abstract

The main objective of this work is to study the stability of the stochastic prototypical issue of drug absorption in the cells of the human body. In addition, we studied the statistical analysis to solve the given model in this paper.

Keywords: Stochastic differential equations (SDEs), Random Dynamical System (RDS), Absorption of Drugs, Stability
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1 Introduction

It is worth noting that stochastic differential equations are used in many applications, including biology, medicine, engineering, environment and economics. Here We will limit Ourselves to studying an important issue in biology and medicine, which is the issue of drug absorption within cells and organs. There are many studies in this field, and we will mention some of them: In 1981, Spiegel [12] presented "A Mathematical Model of Drug Absorption", Mann [8] " Derive the alignment distribution effect of material absorption of the amount of light where he consider this effective quantity to be a random variable. He introduced models for estimating light interception and radiation intensity are modified so as to accommodate a proximate point source of light. For the others researches, see [3, 5, 6]. In order to define (SDE), consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_{tt\geq 0}$ filling the standard conditions, consider an $m-$dimensional Brownian motion $B(t) = (B_1(t), \ldots, B_m(t))^T, t \geq 0$ be defined on the probability space. Let $0 \leq t_0 < T < \infty$. Let $x_0 : x_0 : \Omega \longrightarrow \mathbb{R}^d$ be an $\mathcal{F}_{t_0}$_measurable with $E|x_0|^2 < \infty. Let f : \mathbb{R}^d \times [t_0, T] \longrightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \longrightarrow \mathbb{R}^{d\times m}$ be two Boral measurable functions. Let

$$dx(t) = f(x(t), t)\, dt + g(x(t), t)\, dB(t) \text{ on } t_0 \leq t \leq T \tag{1.1}$$

be the It $\hat{o}$ stochastic differential equation of d- dimensional with initial value $x(t_0) = x_0$. Equation 1.1 is equivalent to the equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\, ds + \int_{t_0}^{t} g(x(s), s)\, dB(s) \text{ on } t_0 \leq t \leq T \tag{1.2}$$

see [9], for more details.

*Asmahan A. Yasir
Email addresses: asmahana.alshemary3@student.uokufa.edu.iq (Asmahan A. Yasir), ihsan.kadhim@qu.edu.iq (Ihsan J. Kadhim)

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Definition 1.1. [7, 9, 10] The stochastic process $x(t)_{t_0\leq t\leq T}$, where $x(t)\in\mathbb{R}^d$ is called a solution of 1.1 if it admits the following axioms:

- $\{x(t)\}is\mathcal{F}_t$- adapted and continuous;
- $\{f(x(t),t)\}\in\mathcal{L}^1([t_0,T];R^d)$ and $g(x(t),t)\ \in\mathcal{L}^2([t_0,T];Rdm)$;
- Equation 1.2 holds for each $t\in[t_0,T]$ with full measure.

If any other solution $\{\breve{x}\,(t)\}$ is undifferentiated from $\{x(t)\}$, then the solution $\{x(t)\}$ is said to be unique , that is

$$\mathbb{P}\left\{x\,(t)=\breve{x}\,(t)\,,\ for\ all\ t_0\leq t\leq T\right\}=1$$

Remark 1.2. [7, 9, 10]

a. Denote the solution of Equation 1.1 by $x\,(t;t_0,x_0)$. Note from Equation 1.2 that for any $s\in[t_0,T]$,

$$(t)=x\,(s)+\int_s^t f\,(x\,(r)\,,r)\,dr+\int_s^t g\,(x\,(r)\,,r)\,dB(r)\text{ons}\leq t\leq T \tag{1.3}$$

But, this is a (SDE) on $[s,T]$ with initial value $x\,(s)=x(s;t_0,\ x_0)$, whose solution is denoted by $x\,(t;s,x\,(s;t_0,x_0))$, therefore the solution of 1.1 fulfills the semigroup axiom

$$x(s;t_0,\ x_0)=x\,(t;s,x\,(s;t_0,x_0))\,,\quad t_0\leq s\leq t\leq T.$$

b. The coefficients f and g can depend on ω in a general manner as long as they are adapted. For further details, one can see Gihman and Skorohod [4].

Definition 1.3. [1, 2] A probability space$(\Omega,\ \boldsymbol{\mathcal{F}},\ \mathbb{P})$ with a collection of functions $\{\theta_t:\ \Omega\longrightarrow\Omega,\ t\in\mathbb{T}\}$ (called transformations) is called metric dynamical system **(MDS)** $\theta\equiv(\Omega,\ \mathbf{F},\ \mathbb{P},\{\theta_t:\ t\ \in\ \mathbb{T}\})$ if

1. $\theta_0=\text{id}$, $\theta_t\circ\theta_s=\theta_{t+s}$ for all $\mathbf{t},\ s\in\mathbb{T}$;
2. $(t,\ \omega)\longmapsto\ \theta_t\omega$ is measurable;
3. $\theta_t\mathbb{P}\ =\mathbb{P}$ for all $t\in\mathbb{T}$, i.e. $\mathbb{P}(\theta_tB)\ =\mathbb{P}(B)$ for all $B\in\mathcal{F}$ and all $t\in\mathbb{T}$.

A set B$\in\mathcal{F}$ is called $\theta-$invariant if θ_tB= B for all t $\in\mathbb{T}$. An MDS θ is called **ergodic** under $\mathbb{P}$ if either $\mathbb{P}$(B)=0 or $\mathbb{P}$(B)=1 for any $\theta-$invariant set B$\in\mathcal{F}$.Throughout, θ will be assumed ergodic.

Definition 1.4. [1, 2] A couple $(\theta,\ \varphi)$ is called random dynamical system (briefly RDS) where t $\theta\equiv(\Omega,\ \boldsymbol{\mathcal{F}},\ \mathbb{P}\{\theta_t:\ t\in\mathbb{T}\})$ is MDS and φ is a measurable function

$$\varphi:\mathbb{T}\times\Omega\times X\longrightarrow X,,(t,\ \omega,\ x)\longmapsto\varphi(t,\ \omega,\ x).$$

With the following axioms

1. the function $x\longmapsto\ \varphi(t,\ \omega,\ x)\equiv\varphi(t,\ \omega)x$ is continuous for every t$\in\mathbb{T}$ and $\omega\in\Omega$,
2. the functions φ(t, ω) := φ(t, ω,) satisfy:
   $\varphi(0,\ \omega)=id,\varphi(t+s,\ \omega)=\varphi(t,\ \theta_s\omega)\circ\phi(s,\ \omega)$ for every t, s $\in\mathbb{T}\ and\omega\in\Omega$

The last axiom called the co-cycle property.

Definition 1.5. (Affine RDS)[1, 2] The affine RDS is a couple $(\theta,\ \varphi)$ where the state space X be a linear Polish space and the co-cycle φ admits the form

$$\varphi\,(t,\ \omega)\,x=\Phi\,(t,\ \omega)\,x+\psi\,(t,\ \omega) \tag{1.4}$$

where $\Phi(t,\ \omega)$ is a co-cycle over θ containing a measurable function $\psi:\mathbb{T}\times\ \Omega\longrightarrow X$ and continuous linear operators of X, and is. The affine RDS is called linear if $\psi(t,\ \omega)\equiv0$ and it is written by LRDS. If $(\theta,\ \Phi)$ is a LRDS, then the co-cycle axiom for ϕ given by 1.4 is equivalent to

$$\psi\,(t+s,\ \omega)=\Phi\,(t,\ \theta_s\omega)\,\psi\,(s,\ \omega)+\ \psi\,(t,\ \theta_s\omega)\,,\ t,\ s\ \geq\ 0 \tag{1.5}$$

Definition 1.6. (Random Set)[1, 2] Let (X, d) be a metric space. The set-valued function $\omega \longmapsto D(\omega) \neq \emptyset$ is called a random set if the function $\omega \longmapsto \text{dist}_X(x,\ D(\omega))$ is measurable for any $x \in X$. The random set D is called a random closed(compact) set if $D(\omega)$ is closed (resp. compact) for each $\omega \in \Omega$. A set $\{D(\omega)\}$ is said to be bounded random set if there exist $x_0 \in X$ and a random variable r(ω)¿0 such that

$$D(\omega \subset \{x \in X : d(x, x_0) \leq r(\omega)\} \text{ for all } \omega \in \Omega.$$

For simplicity, the random set $\omega \longmapsto D(\omega)$ denoted by D or $\{D(\omega)\}$

Definition 1.7. [2] We call that the random set D is tempered w.r.t. MDS θ if for every $\omega \in \Omega$ there is $r : \Omega \longrightarrow \mathbb{R}$ and $y \in X$ with $D(\omega) \subset \{x : \text{dist}_X(x, y) \leq r(\omega)\}$, and $r(\omega)$ satisfy

$$\sup_{\tau \in \mathbb{T}} \left\{ e^{-\alpha|\tau|} \left| r\left(\theta_\tau \omega\right) \right| \right\} < \infty \text{ for every} \alpha > 0 \text{ and } \omega \in \Omega \tag{1.6}$$

A random variable $v : \Omega \longrightarrow X$ is called tempered, if the single random set $\{v\}$ is tempered.

Definition 1.8. (Universe of sets)[2] A collection $\mathcal{D}$ of random closed sets is called universe of sets if it is closed under inclusions.

Definition 1.9. (Absorbing Set)[2] Let $\mathcal{D}$ be the universe. A random closed set B is called **absorbing** for the RDS $(\theta,\ \varphi)$ in $\mathcal{D}$, if for any $D \in \mathcal{D}$ and for any ω there exists $t_0(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) \subset B(\omega) \text{ for all } t \geq t_0(\omega).$$

Definition 1.10. (Dissipative RDS)[2] We call that $(\theta,\ \varphi)$ be **dissipative** in $\mathcal{D}$, if for every $\omega \in \Omega$ there is random variable $r(\omega)$, $x_0 \in X$ an absorbing set B in $\mathcal{D}$,

$$B(\omega) \subset B_{r(\omega)}(x_0) \equiv \{x : \text{dist}_X(x, x_0) \leq r(\omega)\}.$$

Definition 1.11. [2] A random variable $u : \Omega \longmapsto X$is said to be an equilibrium (or fixed point, stationary solution) of the RDS $(\theta,\ \varphi)$ if it is invariant under ϕ, i.e. if

$$\varphi(t,\ \omega) u(\omega) = u(\theta_t \omega) \text{ for all } t \geq 0 \text{ and all } \omega \in \Omega$$

Definition 1.12. (Lyapunov Exponent)[1, 2] Let $(\theta,\ \varphi)$ be a LRDS with X a separable Banach space as a state space. If for some full measure set $\Omega^* \subset \Omega$ the smallest number λsatisfy :

$$\lambda(\omega, x) : = \lim_{t \to +\infty} \frac{1}{t} \log \|\varphi(t,\ \omega) x\|, \omega \in \Omega^*,\ t > 0 \tag{1.7}$$

is called the Lyapunov exponent for $(\theta,\ \varphi)$.

Proposition 1.13. [7, 11]: Consider the SDE

$$dx = (ax + c)\, dt + (bx + d)\, d\beta,\ x(0) = x_0 \tag{1.8}$$

where β is a usual Brownian motion and a, b, c, and d are constants . The solution of 1.8 is given by

$$dx = \psi(t) \left(x_0 + (c - bd) \int_0^1 \frac{1}{\psi(s)} ds + d \int_0^1 \frac{1}{\psi(s)} d\beta(s)\right) \tag{1.9}$$

where

$$\psi(t) = \exp\left(\left(a - \frac{1}{2} b^2\right) t + b\beta(t)\right) \tag{1.10}$$

in financial applications, scalar linear SDEs with multiplicative noise are quite typical, because they can be used to model processes that are strictly positive. An example of such an SDE is the Black-Scholes model.

2 Drugs Absorption in the Cells:

In this section we show that how can the problem of absorption of drugs in the cells and organs is modeled as a stochastic differential equation. Also, we solve this equation by using Ito integral. In this article we have studied the linear pharmacokinetic movement of a single-compartment system.

2.1 Mathematical Stochastic model

A liquid carries a drug into an organ with a volume of $V cm^3$ at a rate of $a\ cm^3/sec$, and out of it at a rate of $b\ cm^3/sec$, the concentration of the drug in the liquid entering is $c\ g/cm^3$. Let X epitomized to the concentration of drug in the organs (g/cm^3). Thus, the mass of the drug is known as

$$\left(V\ cm^3\right)\left(X\frac{g}{cm^3}\right) = VXg. \tag{2.1}$$

Suppose that the concentration depending on the time t and volume V is fixed. Then 2.1 becomes

$$V\frac{dX_t}{dt} = ac - bX_t. \tag{2.2}$$

In our work, the model 2.2 will be analyzed with the feature of white noise according to the effect of environmental fluctuation in reality. To capture this effect it is necessary to establish the stochastic equation according to the deterministic model 2.2. Suppose the rate of fluid exit is affected by external conditions, specifically,

$$b \longmapsto b + \sigma\frac{dB}{dt}$$

where B_t is a usual Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ filling the standard conditions and $\sigma > 0$. Then we achieve the following SDE

$$V\frac{dX_t}{dt} = ac - \left(b + \sigma\frac{dB}{dt}\right)X_t \tag{2.3}$$

or

$$dX_t = \frac{b}{V}\left(\frac{ac}{b} - X_t\right)dt + \frac{\sigma}{V}X_t dB\left(t\right) \tag{2.4}$$

where

$$f\left(X_t, t\right) = \frac{b}{V}\left(\frac{ac}{b} - X_t\right) \text{ and } L\left(X_t, t\right) = \frac{\sigma}{V}X_t. \tag{2.5}$$

The Equation 2.4 is linear stochastic differential equation, and by Proposition 1.13 its solution is given by

$$X_t = \psi\left(t\right)\left(X_0 + \frac{ac}{V}\int_0^t \frac{1}{\psi\left(s\right)}ds\right) \tag{2.6}$$

where

$$\psi\left(t\right) = \exp\left[\left(-\frac{b}{V} - \frac{1}{2}\frac{\sigma^2}{V^2}\right)t - \frac{\sigma}{V}B\left(t\right)\right]. \tag{2.7}$$

Remark 2.1. Especially, this is often used to model interest rate dynamics. According to the model, as X_t increases above some "mean value" $\frac{ac}{b}(> 0)$, the drift term $\frac{b}{V}\left(\frac{ac}{b} - X_t\right)$ will become negative. Then $dX_t < 0$ and X_t will decrease. On the other hand, as X_t drops below the value $\frac{ac}{b}$, then $\frac{b}{V}\left(\frac{ac}{b} - X_t\right) > 0$. Then $dX_t > 0$ and X_t will increase. Hence, we may expect that X_t will eventually move towards the value ac and, indeed, we shall see $E[X_t] \longrightarrow ac$ as $t \longrightarrow \infty$.

By Theorem (2.4.1) from Igor [2], the (RDS) generated by Equation 2.4 is given by the metric dynamical system $\theta : \mathbb{R} \times \Omega \to \Omega$, and the cocycle

$$\varphi\left(t, \omega\right)X_t = X_0 \exp\left[\left(\frac{-b}{V} - \frac{\sigma^2}{2V^2}\right)t - \frac{\sigma}{v}B_t\right] + \frac{ac}{V}\int_0^t \exp\left[\left(\frac{-b}{V} - \frac{\sigma^2}{2V^2}\right)\left(t - s\right) - \frac{\sigma}{V}\left(B_t - B_s\right)\right]ds \tag{2.8}$$

3 Stability Analysis of Absorption of Drugs

The Lyapunov exponent for (θ, φ) is the number $\frac{-b}{v} - \frac{\sigma^2}{2v^2}$ .We can chick that by using theorem (2.1.2) from Igor [2]

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log \left\| x \exp\left[\left(\frac{-b}{v} - \frac{\sigma^2}{2v^2}\right) t - \frac{\sigma}{v} B_t\right]\right\| &= \lim_{t\to\infty} \frac{1}{t} \log\left(\exp\left[. \left(\frac{-b}{v} - \frac{\sigma^2}{2v^2}\right) t - \frac{\sigma}{v} B_t.\right] \|x\|\right) \\ &= \lim_{t\to\infty} \frac{1}{t} \log\left(\exp\left[\left(\frac{-b}{v} - \frac{\sigma^2}{2v^2}\right) t - \frac{\sigma}{v} B_t\right] + \lim_{t\to\infty} \frac{1}{t} \log \|x\|\right) \\ &= \frac{-b}{v} - \frac{\sigma^2}{2v^2}.\end{aligned}$$

Since $\frac{-b}{v} - \frac{\sigma^2}{2v^2} < 0$, we have (θ, φ) is dissipative. The RDS (θ, φ) has a unique equilibrium $u(\omega)$ (see Igor [2]). So

$$u(\omega) = \lim_{t\to+\infty} \psi(t, \theta_{-t}\omega)$$

$$u(\omega) = \lim_{t\to+\infty} \frac{ac}{v} \int_0^t \exp\left[\left(\frac{-b}{v} - \frac{\sigma^2}{2v^2}\right)(t-s) - \frac{\sigma}{v}\left(B_t(\theta_{-t}\omega) - B_s(\theta_{-t}\omega)\right)\right] ds$$

$$u(\omega) = \frac{ac}{v} \int_{-\infty}^0 \exp\left[-\left(\frac{-b}{v} - \frac{\sigma^2}{2v^2}\right)\tau - \sigma B_\tau(\omega)\right] dt, \quad \tau = s - t.$$

Then $u(\omega)$ is measurable w.r.t. σ-algebra $(\mathcal{F}_-)$, and the it is an exponentially stable (because $\frac{-b}{v} - \frac{\sigma^2}{2v^2} < 0$) (see the Definition 1.10.1, proposition (1.9.3) in [2]).

4 Statistical Analysis of Absorption of Drugs

In this section the expected and variance of the solution of the our model are computed. For more details on how to find the expectation and variance see [11]. The equation 2.8 shows obviously that X_t ruins positive providing $X_0 > 0$.

$$E\left[\exp\left\{-\frac{\sigma^2}{2V^2}(t-s) - \frac{\sigma}{V}(B_t - B_s)\right\}\right] = 1 \text{ for } 0 \le s \le t < \infty.$$

To compute the mean

$$\begin{aligned}E[X_t] &= X_0 e^{-\frac{b}{V}t} + \frac{ac}{V} \int_0^t e^{-\frac{b}{V}(t-s)} ds \\ &= X_0 e^{-\frac{b}{V}t} + \frac{ac}{b}\left[1 - e^{-\frac{b}{V}t}\right] = \frac{ac}{b} + (X_0 - \frac{ac}{b}) e^{-\frac{b}{V}t}.\end{aligned}$$

This implies

$$\lim_{t\longrightarrow\infty} E[X_t] = \frac{ac}{b}$$

as expected. To compute the variance, observing that

$$Var[X_t] = EXt2 - (EXt)2,$$

we henceforth compute the second moment. By using the Ito formula, it is easy to see that

$$\begin{aligned}\frac{d(E\left[X_t^2\right])}{dt} &= 2\frac{ac}{V} E[X_t] - \left(2\frac{b}{V} - \frac{\sigma^2}{V^2}\right) E\left[X_t^2\right] \\ &= 2\frac{ac}{V}\left[\frac{ac}{b} + \left(X_0 - \frac{ac}{b}\right) e^{-\frac{b}{V}t}\right] - \left(2\frac{b}{V} - \frac{\sigma^2}{V^2}\right) E\left[X_t^2\right] .\end{aligned}$$

If $2\frac{b}{V} = \frac{\sigma^2}{V^2}$, then

$$E\left[X_t^2\right] = X_0^2 + 2\frac{b}{V}(\frac{ac}{b})^2\ t\ + 2\frac{ac}{b}\left(X_t - \frac{ac}{b}\right)\left(1 - e^{-\frac{b}{V}t}\right) \longrightarrow \infty\ as\ t \longrightarrow \infty.$$

So, $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. Instead, if $2\frac{b}{V} \neq \frac{\sigma^2}{V^2}$, then

$$\begin{aligned} E\left[X_t^2\right] &= e^{-\left(2\frac{b}{V} - \frac{\sigma^2}{V^2}\right)t}\left(X_0^2 + \int_0^t 2\frac{ac}{V}\left[\frac{ac}{b} + \left(X_0 - \frac{ac}{b}\right)e^{-\frac{b}{V}t}\right]e^{\left(2\frac{b}{V} - \frac{\sigma^2}{V^2}\right)u}du\right) \\ &= e^{-(2\frac{b}{V} - \frac{\sigma^2}{V^2})t}[X_0^2 + \frac{2\frac{b}{V}\left(\frac{ac}{b}\right)^2}{2\frac{b}{V} - \frac{\sigma^2}{V^2}}(e^{\left(2\frac{b}{V} - \frac{\sigma^2}{V^2}\right)u} - 1)] + 2\frac{ac}{V}\left(X_0 - \frac{ac}{b}\right)\mathcal{J}(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}(t) = \int_0^t e^{\left(\frac{b}{V} - \frac{\sigma^2}{V^2}\right)u}du &= \begin{cases} \frac{1}{\frac{b}{V} - \frac{\sigma^2}{V^2}}\left(e^{(\frac{b}{V} - \frac{\sigma^2}{V^2})t} - 1\right), & if\ \frac{b}{V} \neq \frac{\sigma^2}{V^2} \\ t, & if\ \frac{b}{V} = \frac{\sigma^2}{V^2} \end{cases} \\ &= \begin{cases} \frac{V^2}{bV - \sigma^2}\left(e^{(\frac{bV - \sigma^2}{V^2})t} - 1\right), & if\ \frac{b}{V} \neq \frac{\sigma^2}{V^2} \\ t, & if\ \frac{b}{V} = \frac{\sigma^2}{V^2} \end{cases} \end{aligned}$$

Consequently, $\lim_{t\longrightarrow\infty} E\left[X_t^2\right] = \begin{cases} \infty, & if\ 2\frac{b}{V} < \frac{\sigma^2}{V^2} \\ \frac{2\frac{b}{V}\left(\frac{ac}{b}\right)^2}{2\frac{b}{V} - \frac{\sigma^2}{V^2}}, & if\ 2\frac{b}{V} > \frac{\sigma^2}{V^2} \end{cases}$ Brief the arguments exceeding provides

$$\begin{aligned} \lim_{t\longrightarrow\infty} Var[X_t] &= \begin{cases} \infty, & if\ 2\frac{b}{V} \leq \frac{\sigma^2}{V^2} \\ \frac{\left(\frac{ac}{b}\right)^2\frac{\sigma^2}{V^2}}{2\frac{b}{V} - \frac{\sigma^2}{V^2}}, & if\ 2\frac{b}{V} > \frac{\sigma^2}{V^2} \end{cases} \\ &= \begin{cases} \infty, & if\ 2\frac{b}{V} \leq \frac{\sigma^2}{V^2} \\ \frac{a^2}{b^2}\frac{c^2\sigma^2}{2bV - a^2}, & if\ 2\frac{b}{V} > \frac{\sigma^2}{V^2} \end{cases} \end{aligned}$$

5 Two Special Cases

Case 1. When the rate of drug entry equals the rate of liquid excretion, this means when $a = b$. Then the mean

$$E\left[X_t\right] = c + (X_0 - c)e^{-\frac{b}{V}t}.$$

This implies

$$\lim_{t\longrightarrow\infty} E\left[X_t\right] = c$$

as expected. The variance,

$$Var[X_t] = E\left[X_t^2\right] - \left(E\left[X_t\right]\right)^2,$$

we henceforth compute the second moment. By using the Ito formula, it is easy to see that

$$\frac{d(E\left[X_t^2\right])}{dt} = 2\frac{ac}{V}\left[c + (X_0 - c)\,e^{-\frac{b}{V}t}\right] - \left(\frac{2bv - \sigma^2}{V^2}\right)E\left[X_t^2\right].$$

If $2bV = \sigma^2$, then

$$E\left[X_t^2\right] = X_0^2 + 2\frac{b}{V}c^2\ t\ + 2c\,(X_t - c)\left(1 - e^{-\frac{b}{V}t}\right) \longrightarrow \infty\ as\ t \longrightarrow \infty$$

so $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. Instead, if $2bV \neq \sigma^2$, then

$$E\left[X_t^2\right] = e^{-(\frac{2bV-\sigma^2}{V^2})t}[X_0^2 + \frac{2bc^2V}{2bV-\sigma^2}(e^{(\frac{bV-\sigma^2}{V^2})t} - 1)] + 2\frac{ac}{V}\left(X_0 - c\right)\mathcal{J}(t),$$

where

$$\mathcal{J}\left(t\right) = \int_0^t e^{(\frac{bV-\sigma^2}{V^2})u}du = \begin{cases} \frac{V^2}{bV-\sigma^2}\left(e^{(\frac{bV-\sigma^2}{V^2})t} - 1\right), & if\ \frac{b}{V} \neq \frac{\sigma^2}{V^2} \\ t, & if\ \frac{b}{V} = \frac{\sigma^2}{V^2} \end{cases}$$

Consequently,

$$\lim_{t\longrightarrow\infty} E\left[X_t^2\right] = \begin{cases} \infty, & if\ 2bV < \sigma^2 \\ \frac{2bc^2V}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

Brief the arguments exceeding provides

$$\lim_{t\longrightarrow\infty} Var[X_t] = \begin{cases} \infty, & if\ 2bV \leq \sigma^2 \\ \frac{c^2\sigma^2}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

Case 2. When the drug entry rate is equal to the liquid exit rate and the original concentration of the drug in the liquid is equal to zero, this means when $a = b$ and $X_0 = 0$. Then the mean

$$E\left[X_t\right] = c(1 - e^{-\frac{b}{V}t}).$$

This implies

$$\lim_{t\longrightarrow\infty} E\left[X_t\right] = c$$

as expected. The variance,

$$Var[X_t] = E\left[X_t^2\right] - \left(E\left[X_t\right]\right)^2,$$

we henceforth compute the second moment. By using the Ito formula, it is easy to see that

$$\frac{d(E\left[X_t^2\right])}{dt} = 2\frac{ac}{V}\left[c + \left(X_0 - c\right)e^{-\frac{b}{V}t}\right] - \left(\frac{2bv - \sigma^2}{V^2}\right)E\left[X_t^2\right].$$

If $2bV = \sigma^2$, then

$$E\left[X_t^2\right] = X_0^2 + 2\frac{b}{V}c^2\ t\ + 2c\left(X_t - c\right)\left(1 - e^{-\frac{b}{V}t}\right) \longrightarrow \infty\ as\ t \longrightarrow \infty$$

so $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. Instead, if $2bV \neq \sigma^2$, then

$$E\left[X_t^2\right] = e^{-(\frac{2bV-\sigma^2}{V^2})t}[X_0^2 + \frac{2bc^2V}{2bV-\sigma^2}(e^{(\frac{bV-\sigma^2}{V^2})t} - 1)] + 2acVX0 - cJ(t),$$

where

$$\mathcal{J}\left(t\right) = \int_0^t e^{(\frac{bV-\sigma^2}{V^2})u}du = \begin{cases} \frac{V^2}{bV-\sigma^2}\left(e^{(\frac{bV-\sigma^2}{V^2})t} - 1\right), & if\ \frac{b}{V} \neq \frac{\sigma^2}{V^2} \\ t, & if\ \frac{b}{V} = \frac{\sigma^2}{V^2} \end{cases}$$

Consequently,

$$\lim_{t\longrightarrow\infty} E\left[X_t^2\right] = \begin{cases} \infty, & if\ 2bV < \sigma^2 \\ \frac{2bc^2V}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

Brief the arguments exceeding provides

$$\lim_{t\longrightarrow\infty} Var[X_t] = \begin{cases} \infty, & if\ 2bV \leq \sigma^2 \\ \frac{c^2\sigma^2}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

Case 3. When the drug entry rate is equal to the liquid exit rate and the original concentration of the drug in the liquid is equal to zero, this means when a=b and $X_0 = 0$. Then the mean

$$E[X_t] = c(1 - e^{-\frac{b}{V}t}).$$

This implies

$$\lim_{t \longrightarrow \infty} E[X_t] = c$$

as expected. The variance,

$$Var[X_t] = E\left[X_t^2\right] - (E[X_t])^2,$$

we henceforth compute the second moment. By using the Ito formula, it is easy to see that

$$\frac{d(E\left[X_t^2\right])}{dt} = 2\frac{ac^2}{V}\left[1 - e^{-\frac{b}{V}t}\right] - \left(\frac{2bv - \sigma^2}{V^2}\right) E\left[X_t^2\right].$$

If $2bV = \sigma^2$, then

$$E\left[X_t^2\right] = 2\frac{b}{V}c^2\ t\ + 2c\left(X_t - c\right)\left(1 - e^{-\frac{b}{V}t}\right) \longrightarrow \infty\ as\ t \longrightarrow \infty$$

so $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. Instead, if $2bV \neq \sigma^2$, then

$$E\left[X_t^2\right] = e^{-(\frac{2bV-\sigma^2}{V^2})t}\left[\frac{2bc^2V}{2bV - \sigma^2}\left(e^{\left(\frac{bV-\sigma^2}{V^2}\right)t} - 1\right)\right] - 2\frac{ac^2}{V}\mathcal{J}(t),$$

where

$$\mathcal{J}(t) = \int_0^t e^{(\frac{bV-\sigma^2}{V^2})u}du = \begin{cases} \frac{V^2}{bV-\sigma^2}\left(e^{(\frac{bV-\sigma^2}{V^2})t} - 1\right), & if\ \frac{b}{V} \neq \frac{\sigma^2}{V^2} \\ t, & if\ \frac{b}{V} = \frac{\sigma^2}{V^2} \end{cases}$$

Consequently,

$$\lim_{t \longrightarrow \infty} E\left[X_t^2\right] = \begin{cases} \infty, & if\ 2bV < \sigma^2 \\ \frac{2bc^2V}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

Brief the arguments exceeding provides

$$\lim_{t \longrightarrow \infty} Var[X_t] = \begin{cases} \infty, & if\ 2bV \leq \sigma^2 \\ \frac{c^2\sigma^2}{2bV-\sigma^2}, & if\ 2bV > \sigma^2 \end{cases}$$

6 Conclusions

In this paper, we analyze the stability and statistic of the SDE that represents the process of drug absorption into the cells of the human body. The Lyapunov exponent for RDS (θ, φ) generated by stochastic model is given by $\frac{-b}{V} - \frac{\sigma^2}{2V^2}$ is less than zero, and hence $(\theta, \varphi)$is dissipative. Also it has an equilibrium which is almost exponentially stable. The expected value of the solution of the given stochastic model is $\frac{ac}{b} + (X_0 - \frac{ac}{b})e^{-\frac{b}{V}t}$ and when $t \longrightarrow \infty$, the expected value becomes $\frac{ac}{b}$. If $2\frac{b}{V} = \frac{\sigma^2}{V^2}$, then $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. If $2\frac{b}{V} \leq \frac{\sigma^2}{V^2}$ then $Var[X_t] \longrightarrow \infty$ as $t \longrightarrow \infty$. If $2\frac{b}{V} > \frac{\sigma^2}{V^2}$, then $Var[X_t] \longrightarrow \frac{a^2}{b^2}\frac{c^2\sigma^2}{2bV-a^2}t \longrightarrow \infty$. Also, we study two special cases. The first case is when the rate of drug entry equals the rate of liquid excretion, this means when a=b. The second case when $a = b$ and $X_0 = 0$.

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