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Fixed point theorems in non-Archimedean G-fuzzy metric spaces with new type contractive mappings

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Abstract

In this article, we extend some recently fixed point theorems in the setting of G-fuzzy metric spaces. We introduce some new concepts of contractions called γ-contractions and γ-weak contractions. We prove some fixed point theorems for mappings providing γ-contractions and γ-weak contractions. On the other hand, we consider a more general class of auxiliary functions in the contractivity condition.

Keywords: Fixed point, Contractive mapping, Weak Contractive mapping, Fuzzy metric space
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1 Introduction

Fixed point theory is a very important concept in mathematics. In 1922, Banach created a famous result called Banach contraction principle in the concept of the fixed point theory which states sufficient conditions for the existence and uniqueness of a fixed point[1].

There are two well-known extensions of the notion of metric space in which imprecise models are considered: fuzzy metric spaces (see [11]) and probabilistic metric spaces [3, 14, 15]. The two concepts are very similar, but they are different in nature. The concept of a fuzzy metric space was introduced in different ways by some authors (see [2, 4]). Gregori and Sapena [4] introduced the notion of fuzzy contractive mappings and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Mihet [8] developed the class of fuzzy contractive mappings of Gregori and Sapena, considered these mappings in non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michalek, and obtained a fixed point theorem for fuzzy contractive mappings. Lots of different types of fixed point theorems has been presented by many authors by expanding the Banach's result, simultaneously (see [16, 17]).

In recent times, many fixed point theorems have been presented in the setting of probabilistic metric space $(X, F, *)$ in which F is a distance distribution function. Many of them have been inspired by their corresponding results on metric spaces. One of the most attractive and effective ways to introduce contractivity conditions in the probabilistic framework is based on considering some terms like in the following expression:

$$\frac{1}{F(x,y,t)} - 1, \quad \text{where} \quad x, y \in X \text{ and } t > 0,$$

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(see [5, 13]). For instance, in [7], Kutbi et al. stated the following result (where Φ and Ψ are appropriate collections of auxiliary functions that we will describe in Section 3).

Theorem 1.1. (Kutbi et al. [7], Theorem 2.1) Let $(X, F, *)$ be a G-complete Menger space and let $f : X \to X$ be a mapping. Assume that there exist a constant $c \in (0, 1)$ and two functions $\phi \in \Phi$ and $\psi \in \Psi$ satisfying the inequality

$$\frac{1}{F(fx, fy, \phi(ct))} - 1 \leq \psi(\frac{1}{F(x, y, \phi(t))} - 1),$$

for all $x, y \in X$ and all $t > 0$ such that $F(x, y, \phi(t)) > 0$. Then f has a unique fixed point.

In 2005, Z. Mustafa and B. Sims introduced a new class of generalized metric spaces (see [9, 10]), which are called G-metric spaces as generalization of metric space (X, d), to develop and to introduce a new fixed point theory for a variety of mappings in this new setting, also to extend known metric space theorems to a more general setting.

In this work, using a mapping $\gamma : [0, 1) \to \mathbb{R}$ we introduce some new types of contractions called γ-contractions and γ-weak contractions. Later, we prove some fixed point theorems for mappings providing γ-contractions and γ-weak contractions in non-Archimedean G-fuzzy metric spaces. Also, some examples are supplied in order to support the usability of our results. On the other hand, we consider a more general class of auxiliary functions which generate some contractive conditions, and we show that the function $t \to 1/t - 1$ (which appears in many fixed point theorems in the fuzzy context) can be replaced by more appropriate and general functions.

Before proving our main results, we recall some basic definitions and facts which will be used later in this paper.

Definition 1.2. [14] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative,

(TN-2) $*$ is continuous,

(TN-3) $*(a, 1) = a$ for every $a \in [0, 1]$,

(TN-4) $*(a, b) \leq *(c, d)$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 1.3. [17] A G-fuzzy metric space is an ordered triple $(X, G, *)$ such that X is a nonempty set, $*$ is a continuous t-norm, and G is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions, for all $s, t > 0$:

(GF-1) $G(x, x, y, t) < 1$ for all $x, y \in X$ with $x \neq y$,

(GF-2) $G(x, x, y, t) \leq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,

(GF-3) $G(x, y, z, t) = 1$, then $x = y = z$,

(GF-4) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,

(GF-5) $G(x, y, z, t + s) \geq G(x, a, a, s) * G(a, y, z, t)$ for all $x, y, z, a \in X$,

(GF-6) $G(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

If, in the above definition, the triangular inequality (GF-5) is replaced by

$$G(x, y, z, \max\{s, t\}) \geq G(x, a, a, s) * G(a, y, z, t)$$

for all $x, y, z, a \in X$ and $s, t > 0$, or equivalently,

$$G(x, y, z, t) \geq G(x, a, a, t) * G(a, y, z, t) \tag{1.1}$$

the triple $(X, G, *)$ is called a non-Archimedean G-fuzzy metric space [6].

Example 1.4. Let X be a nonempty set and let G be a G-metric on X. Denote $*(a, b) = ab$ for all $a, b \in [0, 1]$. For each $t > 0$, $G(x, y, z, t) = t/(t + G(x, y, z))$ is a G-fuzzy metric on X.

Definition 1.5. Let $\{x_n\}$ be a sequence in a G-fuzzy metric space $(X, G, *)$. We will say that:

- $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} G(x_n, x_n, x, t) = 1$; i.e., for all $t > 0$ and all $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_n, x, t) > 1-\lambda$ for all $n \geq n_0$ (in such a case, we will write $\{x_n\} \to x$);
- $\{x_n\}$ is a Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_n, x_m, t) > 1-\lambda$ for all $n, m \geq n_0$. $\{x_n\}$ is a G-Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_n, x_{n+p}, t) > 1-\lambda$ for all $n \geq n_0$ and $p > 0$; in other words, $\lim_{n\to\infty} G(x_n, x_n, x_{n+p}, t) = 1$.
- The G-fuzzy metric space $(X, G, *)$ is called complete (G-complete) if every Cauchy (G-Cauchy) sequence is convergent.

Lemma 1.6. (see [17]) Let $(X, G, *)$ be a G-fuzzy metric space. Then, $G(x, y, z, t)$ is nondecreasing with respect to t for all $x, y, z \in X$.

Lemma 1.7. (see [17]) Let $(X, G, *)$ be a G-fuzzy metric space. Then, G is a continuous function on $X^3 \times (0, \infty)$.

It is easy to prove that a $G(x, y, z, t)$ in a non-Archimedean G-fuzzy metric space $(X, G, *)$ is also nondecreasing with respect to t and a continuous function for all $x, y, z \in X$.

2 New types of contractive mappings

Definition 2.1. Let $\gamma : [0,1) \to \mathbb{R}$ be a strictly increasing continuous mapping and for each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} a_n = 1$ if and only if $\lim_{n\to\infty} \gamma(a_n) = \infty$. Let Γ be the family of all γ functions.

Let $(X, G, *)$ be a non-Archimedean G-fuzzy metric space. A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta > 0$ such that

$$G(Tx, Ty, Tz, t) < 1 \Rightarrow \gamma(G(Tx, Ty, Tz, t)) \geq \gamma(G(x, y, z, t)) + \delta \tag{2.1}$$

for all $x, y, z \in X$, $t > 0$ and $\gamma \in \Gamma$.

When we consider in (2.1) the different types of the mapping γ, then we obtain a variety of contractions, some of them are of a type known in the literature. See the following example:

Example 2.2. The different types of the mapping $\gamma \in \Gamma$ are as follows:

$$\gamma_1 = \frac{1}{(1-x)}, \quad \gamma_2 = \ln\frac{1}{(1-x)}, \quad \gamma_3 = \frac{1}{(1-x)} + x, \quad \gamma_4 = \frac{1}{(1-x^2)}, \quad \gamma_5 = \frac{1}{\sqrt{1-x}}.$$

If $\gamma = \ln \frac{1}{(1-x)}$. Then each mapping $T : X \to X$ satisfying (2.1) is a γ-contraction such that

$$G(Tx, Ty, Tz, t) \geq k(\delta) G(x, y, z, t),$$

for all $x, y, z \in X$, $t > 0$ and $G(Tx, Ty, Tz, t) < 1$, in which $k(\delta) = \frac{G(x,y,z,t)-1+e^{\delta}}{e^{\delta} G(x,y,z,t)} \geq 1$.

Note that from γ and (2.1) it is easy to conclude that every γ-contraction T is a contractive mapping, that is,

$$G(Tx, Ty, Tz, t) > G(x, y, z, t) \tag{2.2}$$

for all $x, y, z \in X$, such that $Tx \neq Ty \neq Tz$. Thus every γ-contraction is a continuous mapping. Now we state one of the main results of the present manuscript.

Theorem 2.3. Let $G(X, G, *)$ be a complete non-Archimedean G-fuzzy metric space and let $T : X \to X$ be a γ-contraction. Then T has a unique fixed point in X.

Proof . Let $x_0 \in X$ be arbitrary and fixed. Define sequence $\{x_n\}$ by

$$Tx_n = x_{n+1}, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

If $x_n = x_{n+1}$, then x_{n+1} is the fixed point of T; then the proof is finished. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Therefore by (2.1), we get

$$\gamma(G(Tx_{n-1}, Tx_{n-1}, Tx_n, t)) \geq \gamma(G(x_{n-1}, x_{n-1}, x_n, t)) + \delta. \tag{2.4}$$

Repeating this process, we have

$$\begin{aligned} \gamma(G(Tx_{n-1}, Tx_{n-1}, Tx_n, t)) &\geq \gamma(G(x_{n-1}, x_{n-1}, x_n, t)) + \delta \\ &= \gamma(G(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}, t)) + \delta \\ &\geq \gamma(G(x_{n-2}, x_{n-2}, x_{n-1}, t)) + 2\delta ... \\ &\geq \gamma(G(x_0, x_0, x_1, t)) + n\delta. \end{aligned} \tag{2.5}$$

Letting $n \to \infty$, from (2.5) we get

$$\lim_{n\to\infty} \gamma(G(Tx_{n-1}, Tx_{n-1}, Tx_n, t)) = +\infty. \tag{2.6}$$

Then, we have

$$\lim_{n\to\infty} G(Tx_{n-1}, Tx_{n-1}, Tx_n, t) = 1. \tag{2.7}$$

With the same process, we have $\lim_{n\to\infty} G(Tx_{n-1}, Tx_n, Tx_n, t) = 1$. Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, we assume that $\{x_n\}$ is not a Cauchy sequence. Then there are $\lambda \in (0,1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $n(k) > m(k) > k$ and

$$G(x_{n(k)}, x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \lambda. \tag{2.8}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying inequality (2.8). Then, we have

$$G(x_{n(k)}, x_{n(k)}, x_{m(k)-1}, t_0) > 1 - \lambda, \tag{2.9}$$

and so, for all $k \in \mathbb{N}$ and from (1.1), we get

$$\begin{aligned} 1 - \lambda &\geq G(x_{n(k)}, x_{n(k)}, x_{m(k)}, t_0) \\ &= G(x_{m(k)}, x_{n(k)}, x_{n(k)}, t_0) \\ &\geq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}, t_0) * G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}, t_0) \\ &\geq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}, t_0) * (1 - \lambda). \end{aligned} \tag{2.10}$$

Letting $k \to \infty$ in (2.10) and using (2.7), we obtain

$$\lim_{k\to\infty} G(x_{n(k)}, x_{n(k)}, x_{m(k)}, t_0) = 1 - \lambda. \tag{2.11}$$

From (1.1), we get

$$G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t_0) \geq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}, t_0) * G(x_{m(k)}, x_{n(k)}, x_{n(k)}, t_0) * G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}, t_0),$$

so, letting $k \to \infty$ and using (2.7), we have

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t_0) \geq 1 - \lambda. \tag{2.12}$$

From (2.8), we obtain

$$\begin{aligned} 1 - \lambda &\geq G(x_{m(k)}, x_{n(k)}, x_{n(k)}, t_0) \\ &\geq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}, t_0) * G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t_0) * G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t_0), \end{aligned} \tag{2.13}$$

and so by taking the limit as $k \to \infty$ in (2.13) and from (2.7) and (2.12), we have

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t_0) = 1 - \lambda. \tag{2.14}$$

By applying inequality (2.1) with $x = y = x_{n(k)}$ and $z = x_{m(k)}$

$$\gamma(G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}, t_0)) \geq \gamma(G(x_{n(k)}, x_{n(k)}, x_{m(k)}, t_0)) + \delta. \tag{2.15}$$

Taking the limit $k \to \infty$ in (2.15), applying (2.1), from (2.11), (2.14), and the continuity of γ, we obtain

$$\gamma(1-\lambda) \geq \gamma(1-\lambda) + \delta,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X. From the completeness of $(X, G, *)$ there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Finally, the continuity of T and G yields

$$G(Tx, Tx, x, t) = \lim_{n\to\infty} G(Tx_n, Tx_n, x_n, t) = \lim_{n\to\infty} G(x_{n+1}, x_{n+1}, x_n, t) = 1.$$

Now, we show that T has a unique fixed point. Suppose that x and y are two fixed points of T. Indeed, if for $x, y \in X$, $Tx = x \neq y = Ty$, then we get $\gamma(G(x, x, y, t)) \geq \gamma(G(x, x, y, t)) + \delta$, which is a contradiction. Thus, T has a unique fixed point. Hence, the proof is completed. □

Example 2.4. Let $X = [0, 1)$, $*(a, b) = \min\{a, b\}$, and

$$G(x,y,z,t) = \begin{cases} 1, & \text{if } x = y = z, \\ \frac{1}{1+\max\{x,y,z\}}, & \text{otherwise.} \end{cases} \tag{2.16}$$

for all $t > 0$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma(x) = 1/1 - x$ for all $x \in [0, 1)$ and define $T : X \to X$ by $T(x) = 2x^2/5$ for all $x \in X$. Clearly, $(X, G, *)$ is a complete non-Archimedean G-fuzzy metric space.

Case 1. We assume that $x, y, z \in (0, 1)$. Since $x^2 < x$, $y^2 < y$ and $z^2 < z$, $\max\{x, y, z\} > \max\{Tx, Ty, Tz\}$. So, there exists a $\delta > 0$ such that

$$\frac{1}{\max\{Tx, Ty, Tz\}} + 1 \geq \frac{1}{\max\{x, y, z\}} + 1 + \delta.$$

It is easy to see that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(G(x, y, z)) + \delta.$$

Case 2. Let $x = 0$ and $y, z \in (0, 1)$. Since $x^2 = 0$, $y^2 < y$ and $z^2 < z$, then $\max\{x, y, z\} = \max\{y, z\} > \max\{Tx, Ty, Tz\} = \max\{Ty, Tz\}$. Hence, we have

$$G(Tx, Ty, Tz, t) = \frac{1}{1 + \max\{Tx, Ty, Tz\}} > \frac{1}{1 + \max\{x, y, z\}} = G(x, y, z, t).$$

So, there exists a $\delta > 0$ such that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(G(x, y, z, t)) + \delta.$$

Case 3. Let $x = y = 0$ and $z \in (0, 1)$, it is easy to see that,

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(G(x, y, z, t)) + \delta.$$

Therefore, T is a γ-contraction. Then all the conditions of Theorem (2.3) hold and T has the unique fixed point $x = 0$.

Definition 2.5. Let $(X, G, *)$ be a non-Archimedean G-fuzzy metric space. A mapping $T : X \to X$ is said to be a γ-weak contraction if there exists a $\delta > 0$ such that $G(Tx, Ty, Tz, t) < 1$ implies that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(\min\{G(x, y, z, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}) + \delta, \tag{2.17}$$

for all $x, y, z \in X$ and $\gamma \in \Gamma$. Note that every γ-contraction is a γ-weak contraction. But the converse is not true.

Example 2.6. Let $X = A \cup B$, where $A = \{1/10, 1/2, 1, 2, 3\}$, $B = [4, 5]$. $*(a, b) = \min\{a, b\}$ and $G(x, y, x, t) = \min\{x, y, z\}/\max\{x, y, z\}$ for all $t > 0$. Clearly, $(X, G, *)$ is a complete non-Archimedean G-fuzzy metric space. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma(x) = 1/\sqrt{1-x}$ for all $x \in [0, 1)$ and define $T : X \to X$ by

$$\begin{cases} \frac{1}{10}, & \text{if } x \in A, \\ \frac{1}{2}, & \text{if } x \in B. \end{cases}$$

Since T is not continuous, T is not γ-contraction by (2.2). Now, we show that T is a γ-weak contraction for all $x \in X$.

Case 1. Let $x = 1$ and $y, z \in B$,

$$\begin{aligned} G(Tx, Ty, Tx, t) =& \frac{1}{5} > \frac{1}{10} = \min\left\{\frac{1}{\max\{y, z\}}, \frac{1}{10}, \frac{1}{2y}, \frac{1}{2z}\right\} \\ =& \min\{G(x, y, x, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}. \end{aligned}$$

So, there exists a $\delta > 0$ such that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(\min\{G(x, y, z, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}) + \delta.$$

Case 2. Let $x \in \{2, 3\}$ and $y, z \in B$,

$$\begin{aligned} G(Tx, Ty, Tx, t) =& \frac{1}{5} > \frac{1}{10x} = \min\left\{\frac{x}{\max\{y, z\}}, \frac{1}{10x}, \frac{1}{2y}, \frac{1}{2z}\right\} \\ =& \min\{G(x, y, x, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}. \end{aligned}$$

So, there exists a $\delta > 0$ such that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(\min\{G(x, y, z, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}) + \delta.$$

Case 3. Let $x \in \{1/10, 1/2\}$ and $y, z \in B$,

$$\begin{aligned} G(Tx, Ty, Tx, t) =& \frac{1}{5} > \frac{x}{\max\{y, z\}} = \min\left\{\frac{x}{\max\{y, z\}}, \frac{1}{10}, \frac{1}{2y}, \frac{1}{2z}\right\} \\ =& \min\{G(x, y, x, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}. \end{aligned}$$

So, there exists a $\delta > 0$ such that

$$\gamma(G(Tx, Ty, Tz, t)) \geq \gamma(\min\{G(x, y, z, t), G(x, x, Tx, t), G(y, y, Ty, t), G(z, z, Tz, t)\}) + \delta.$$

By proving the rest of cases, we get T is a γ-weak contraction.

Theorem 2.7. Let $(X, G, *)$ be a complete non-Archimedean G-fuzzy metric space and let $T : X \to X$ be a γ-weak contraction. Then T has a unique fixed point in X.

Proof . Let $x_0 \in X$ be arbitrary and fixed. Define sequence $\{x_n\}$ by

$$Tx_n = x_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

If $x_n = x_{n+1}$, then x_{n+1} is the fixed point of T; then the proof is finished. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Therefore by (2.17), we have

$$\begin{aligned} \gamma(G(Tx_{n-1}, Tx_{n-1}, Tx_n, t)) \geq& \gamma(\min\{G(x_{n-1}, x_{n-1}, x_n, t), G(x_{n-1}, x_{n-1}, Tx_{n-1}, t), G(x_n, x_n, Tx_n, t)\}) + \delta \\ =& \gamma(\min\{G(x_{n-1}, x_{n-1}, x_n, t), G(_{n-1}, x_{n-1}, x_n, t), G(x_n, x_n, x_{n+1}, t)\}) + \delta \\ =& \gamma(\min\{G(x_{n-1}, x_{n-1}, x_n, t), G(x_n, x_n, x_{n+1}, t)\}) + \delta. \end{aligned} \tag{2.18}$$

If there exists $n \in \mathbb{N}$ such that
$$\min\{G(x_{n-1}, x_{n-1}, x_n, t), G(x_n, x_n, x_{n+1}, t)\} = G(x_n, x_n, x_{n+1}, t),$$
it follows from (2.18) that
$$\begin{aligned}\gamma(G(Tx_{n-1}, Tx_{n-1}, Tx_n, t)) &= \gamma(G(x_n, x_n, x_{n+1}, t))\\ &\geq \gamma(G(x_n, x_n, x_{n+1}, t)) + \delta\\ &> \gamma(G(x_n, x_n, x_{n+1}, t)),\end{aligned}$$
which is a contradiction, therefore,
$$\min\{G(x_{n-1}, x_{n-1}, x_n, t), G(x_n, x_n, x_{n+1}, t)\} = G(x_{n-1}, x_{n-1}, x_n, t), \tag{2.19}$$
for all $n \in \mathbb{N}$. That is, from (2.18) and (2.19) and the property of γ, we obtain
$$\gamma(G(x_n, x_n, x_{n+1}, t)) \geq \gamma(G(x_{n-1}, x_{n-1}, x_n, t)) + \delta,$$
for all $n \in \mathbb{N}$. It implies that
$$\gamma(G(x_n, x_n, x_{n+1}, t)) \geq \gamma(G(x_0, x_0, x_1, t)) + n\delta.$$

By taking $n \to \infty$ we get,
$$\lim_{n\to\infty} \gamma(G(x_n, x_n, x_{n+1}, t)) = +\infty.$$

Then, we have
$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}, t) = 1.$$

So, by the same argument as in the proof of Theorem (2.3), we get $\{x_n\}$ is a Cauchy sequence. From the completeness of $(X, G, *)$ there exists x such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is the fixed point of T. Since γ is continuous, there are two cases.

Case 1. For each $n \in \mathbb{N}$, there exists $i_n \geq n$ such that $x_{i_n+1} = Tx$ and $i_n > i_{n-1}$, where $i_0 = 1$. Then, we get
$$x = \lim_{n\to\infty} x_{i_n+1} = \lim_{n\to\infty} Tx = Tx.$$

This proves that x is the fixed point of T.

Case 2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx$ for all $n \geq n_0$. That is, $Tx_n = x_{n+1} \neq Tx$ and so, $G(Tx_n, Tx_n, Tx, t) < 1$ for all $n \geq n_0$. It follows from (2.17),
$$\begin{aligned}\gamma(G(x_{n+1}, x_{n+1}, Tx, t)) = \gamma(G(Tx_n, Tx_n, Tx, t)) &\geq \gamma(\min\{G(x_n, x_n, x, t), G(x_n, x_n, Tx_n, t), G(x, x, Tx, t)\}) + \delta\\ &= \gamma(\min\{G(x_n, x_n, x, t), G(x_n, x_n, x_{n+1}, t), G(x, x, Tx, t)\}) + \delta.\end{aligned} \tag{2.20}$$

Since $\lim_{n\to\infty} G(x_n, x_n, x, t) = 1$ and $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}, t) = 1$, if $G(x, x, Tx, t) < 1$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we get
$$\min\{G(x_n, x_n, x, t), G(x_n, x_n, x_{n+1}, t), G(x, x, Tx, t)\} = G(x, x, Tx, t).$$

From (2.20), we have
$$\gamma(G(x_{n+1}, x_{n+1}, Tx, t)) \geq \gamma(G(x, x, Tx, t)) + \delta,$$
for all $n \geq \max\{n_0, n_1\}$. Since γ is continuous, taking the limit as $n \to \infty$, we obtain
$$\gamma(G(x, x, Tx, t)) \geq \gamma(G(x, x, Tx, t)) + \delta,$$
which is a contradiction. Therefore, $G(x, x, Tx, t) = 1$; that is, x is the fixed point of T. Now, we prove that the fixed point of T is unique. Let x_1 and x_2 be two fixed points of T. Suppose that $x_1 \neq x_2$; then we have $Tx_1 \neq Tx_2$. From (2.17) we obtain
$$\begin{aligned}\gamma(G(x_1, x_1, x_2, t)) = \gamma(G(Tx_1, Tx_1, Tx_2, t)) &\geq \gamma(\min\{G(x_1, x_1, x_2, t), G(x_1, x_1, Tx_1, t), G(x_2, x_2, Tx_2, t)\}) + \delta\\ &= \gamma(G(x_1, x_1, x_2, t)) + \delta > \gamma(G(x_1, x_1, x_2, t),\end{aligned}$$
which is a contradiction. Then, $G(x_1, x_1, x_2, t) = 1$, that is, $x_1 = x_2$. Therefore, the fixed point of T is unique. $\square$

Example 2.8. Let $(X, G, *)$ be the non-Archimedean G-fuzzy metric space and let T be considered in Example (2.6). Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma(x) = 1/(1 - x^2)$ for all $x \in [0, 1)$. So, T is a γ-weak contraction. Therefore, Theorem (2.7) can be applicable to T and the unique fixed point of T is $1/10$.

3 General contractivity conditions

In this section, we present an extension of Theorem (1.1) in several ways: the metric space is more general, the contractivity condition is better and the involved auxiliary functions form a wider class. The following families of auxiliary functions were considered in [7].

Definition 3.1. Let Φ be the family of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:

(1) $\phi(t) = 0$ if and only if $t = 0$,

(2) $\lim_{t\to\infty} \phi(t) = \infty$,

(3) ϕ is continuous at $t = 0$.

Definition 3.2. Let Ψ be the class of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(1) ψ is nondecreasing

(2) $\psi(0) = 0$,

(3) if $\{a_n\} \subset [0, \infty)$ is a sequence such that $\{a_n\} \to 0$, then $\{\psi^n(a_n)\} \to 0$ (where ψ^n denotes the nth-iterate of ψ).

We shall remind that ψ is continuous at $t = 0$ for functions in Ψ.(Proposition 7 [12])

Definition 3.3. We shall denote by $\mathcal{H}$ the family of all functions $h : (0, 1] \to [0, \infty)$ satisfying:

($\mathcal{H}_1$) if $\{a_n\} \subset (0, 1]$, then $\{a_n\} \to 1$ if and only if $\{h(a_n)\} \to 0$;

($\mathcal{H}_2$) if $\{a_n\} \subset (0, 1]$, then $\{a_n\} \to 0$ if and only if $\{h(a_n)\} \to \infty$.

The previous conditions are guaranteed when $h : (0, 1] \to [0, \infty)$ is a strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous (in a broad sense, it is sufficient to assume the continuities of h and h^{-1} on the extremes of the respective domains). For instance, this is the case of the function $h(t) = 1/t - 1$ for all $t \in (0, 1]$. However, the functions in $\mathcal{H}$ need not be continuous nor monotone.

Proposition 3.4. [12] If $h \in \mathcal{H}$, then $h(1) = 0$. Furthermore, $h(t) = 0$ if and only if $t = 1$.

The other main result of the article is the following one.

Theorem 3.5. Let $(X, G, *)$ be a G-complete non-Archimedean G-fuzzy metric space and let $T : X \to X$ be a mapping. Suppose that there exist $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$, and $h \in \mathcal{H}$ such that

$$h(G(Tx, Ty, Tz, \phi(ct))) \leq \psi(h(G(x, y, z, \phi(t)))) \tag{3.1}$$

for all $x, y, z \in X$ and all $t > 0$ for which $G(x, y, z, \phi(t)) > 0$. If there exists $x_0 \in X$ such that $\lim_{t\to\infty} G(x_0, x_0, Tx_0, t) = 1$, then T has at least one fixed point. Additionally, assume that for all $x, y, z \in Fix(T)$ with $x \neq y \neq z$, we have $\lim_{t\to\infty} G(x, y, z, t) = 1$. Then T has a unique fixed point.

Proof . Notice that condition (3.1) implies that if $G(x, y, z, \phi(t)) > 0$, then h must be applicable to $G(Tx, Ty, Tz, \phi(ct))$. Hence $G(Tx, Ty, Tz, \phi(ct)) \in dom\ h = (0, 1]$, which means that

$$G(x, y, z, \phi(t)) > 0 \Rightarrow G(Tx, Ty, Tz, \phi(ct)) > 0. \tag{3.2}$$

By setting $x_1 = Tx_0$, define sequnece $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T, and the existence part of the proof is finished.
On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
Since $\lim_{t\to\infty} G(x_0, x_0, Tx_0, t) = 1$, there exists $t_0 > 0$ such that $G(x_0, x_0, x_1, t_0) = G(x_0, x_0, Tx_0, t_0) > 0$. Moreover, as $\lim_{t\to\infty} \phi(t) = \infty$, it follows that there exists $s_0 \in [0, \infty)$ (we can suppose, without loss of generality, that $s_0 \geq t_0$) such that $\phi(s_0) \geq t_0$. Hence

$$G(x_0, x_0, x_1, \phi(s_0)) \geq G(x_0, x_0, x_1, t_0) > 0.$$

It follows from (3.2) that

$$G(x_1, x_1, x_2, \phi(cs_0)) = G(Tx_0, Tx_0, Tx_1, t_0) > 0,$$

and, by induction, it can be proved that for $G(x_n, x_n, x_{n+1}, \phi(c^n s_0)) > 0$ for all $n \in \mathbb{N}$. If $n, m, r \in \mathbb{N}$ and $r \leq n$, then $c^n s_0 \leq c^r s_0 \leq s_0 \leq s_0/c^m$. Since ϕ and $G(x_n, x_n, x_{n+1}, .)$ are nondecreasing functions, it follows that if $n, m, r \in \mathbb{N}$ and $r \leq n$, then

$$\begin{aligned} 0 < G(x_n, x_n, x_{n+1}, \phi(c^n s_0)) &\leq G(x_n, x_n, x_{n+1}, \phi(c^r s_0)) \\ \leq G(x_n, x_n, x_{n+1}, \phi(s_0)) &\leq G(x_n, x_n, x_{n+1}, \phi(\frac{s_0}{c^m})). \end{aligned} \tag{3.3}$$

We claim that

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}, s) = 1 \quad for\ all\ s > 0. \tag{3.4}$$

To prove it, let $s > 0$ be arbitrary. As $\lim_{r\to\infty} (c^r s_0) = 0$ and ϕ is continuous at $t = 0$, then $\lim_{r\to\infty} \phi(c^r s_0) = \phi(0) = 0$. Since $s > 0$, there exists $r \in \mathbb{N}$ such that $\phi(c^r s_0) \leq s$. Let $n \in \mathbb{N}$ be such that $n > r$. Applying the contractivity condition (3.1) to $x = y = x_n$ and $z = x_{n+1}$, it follows that

$$\begin{aligned} h(G(x_n, x_n, x_{n+1}, \phi(c^r s_0))) &= h(G(Tx_{n-1}, Tx_{n-1}, Tx_n, \phi(c^r s_0))) \\ &\leq \psi(h(G(x_{n-1}, x_{n-1}, x_n, \phi(c^{r-1} s_0)))), \end{aligned} \tag{3.5}$$

where we have used $G(x_{n-1}, x_{n-1}, x_n, \phi(c^{r-1} s_0)) > 0$ by (3.3). Repeating this argument, we find that

$$\begin{aligned} h(G(x_{n-1}, x_{n-1}, x_n, \phi(c^{r-1} s_0))) &= h(G(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}, \phi(c^{r-1} s_0))) \\ &\leq \psi(h(G(x_{n-2}, x_{n-2}, x_{n-1}, \phi(c^{r-2} s_0)))), \end{aligned}$$

where we have used $G(x_{n-2}, x_{n-2}, x_{n-1}, \phi(c^{r-2} s_0)) > 0$ by (3.3). As ψ is nondecreasing, then

$$\begin{aligned} &\psi(h(G(x_{n-1}, x_{n-1}, x_n, \phi(c^{r-1} s_0)))) \leq \\ &\psi^2(h(G(x_{n-2}, x_{n-2}, x_{n-1}, \phi(c^{r-2} s_0)))). \end{aligned} \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we deduce that

$$\begin{aligned} h(G(x_n, x_n, x_{n+1}, \phi(c^r s_0))) &\leq \psi(h(G(x_{n-1}, x_{n-1}, x_n, \phi(c^{r-1} s_0)))) \\ &\leq \psi^2(h(G(x_{n-2}, x_{n-2}, x_{n-1}, \phi(c^{r-2} s_0)))). \end{aligned}$$

Inequality (3.3) permits us to repeat this argument n times, and it follows that

$$\begin{aligned} h(G(x_n, x_n, x_{n+1}, \phi(c^r s_0))) &\leq \psi^n(h(G(x_0, x_0, x_1, \phi(c^{r-2} s_0)))) \\ &= \psi^n(h(G(x_0, x_0, x_1, \phi(\frac{s_0}{c^{n-r}})))), \end{aligned} \tag{3.7}$$

for all $n > r$. As a consequence, $\lim_{n\to\infty} \frac{s_0}{c^{n-r}} = \infty$. Then we have, $\lim_{n\to\infty} \phi(\frac{s_0}{c^{n-r}}) = \infty$. Thus,

$$\lim_{n\to\infty} G(x_0, x_0, x_1, \phi(\frac{s_0}{c^{n-r}}) = 1$$

and this implies that $\lim_{n\to\infty} h(G(x_0, x_0, x_1, \phi(\frac{s_0}{c^{n-r}})) = 0$. As the sequence $\{a_n = h(G(x_0, x_0, x_1, \phi(\frac{s_0}{c^{n-r}}))\} \to 0$ and $h \in \mathcal{H}$, we have $\{\psi^n(a_n)\} \to 0$. By (3.7), we deduce that

$$\lim_{n\to\infty} h(G(x_n, x_n, x_{n+1}, \phi(c^r s_0))) = 0.$$

In particular, as $h \in \mathcal{H}$, condition $(\mathcal{H}_1)$ implies that

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}, \phi(c^r s_0)) = 1.$$

Because $\phi(c^r s_0)) < s$, and $G(x, y, z, t)$ is a nondecreasing function with respect to t, so we have

$$G(x_n, x_n, x_{n+1}, \phi(c^r s_0)) \leq G(x_n, x_n, x_{n+1}, s) \leq 1. \tag{3.8}$$

Taking into account (3.8), we observe that, $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}, s) = 1$ for all $s > 0$, which means that (3.4) holds. Lemma 15 [12] guarantees that $\{x_n\}$ is a G-Cauchy sequence in $(X, G, *)$. As it is G-complete, there exists $x \in X$ such that $\{x_n\} \to x$. We claim that x is the fixed point of T. To prove it, from (1.1) observe that for all $t > 0$ and all $n \in \mathbb{N}$,

$$\begin{aligned} G(x, x, Tx, t) = G(Tx, x, x, t) &\geq G(Tx, x_{n+1}, x_{n+1}, t) * G(x_{n+1}, x, x, t) \\ &= G(Tx, x_{n+1}, x_{n+1}, t) * G(Tx_n, x, x, t). \end{aligned} \tag{3.9}$$

By lemma (1.7),

$$\lim_{n\to\infty} G(x, x, x_{n+1}, t) = 1. \tag{3.10}$$

Let us show that the first factor in (3.9) also converges to 1 when n tends to ∞. Taking into account that ϕ is continuous at $t = 0$, we have $\lim_{s\to 0} \phi(s) = \phi(0) = 0$. Since $t > 0$, there exists $\delta > 0$ such that $\phi(\delta) < t$. Since $\delta/c > 0$, $\phi(\delta/c) > 0$. So, $\lim_{n\to\infty} G(x, x_{n+1}, x_{n+1}, \phi(\frac{\delta}{c})) = 1$. Hence, there exists $n_0 \in \mathbb{N}$ such that $G(x, x_{n+1}, x_{n+1}, \phi(\frac{\delta}{c})) > 0$ for all $n \geq n_0$. Applying the contractivity condition (3.1) to $x = x$ and $y = z = x_{n+1}$ for $n \geq n_0$, we obtain

$$\begin{aligned} h(G(Tx, x_{n+1}, x_{n+1}, \phi(\delta))) &= h(G(Tx, Tx_n, Tx_n, \phi(\delta))) \\ &\leq \psi(G(x, x_n, x_n, \phi(\frac{\delta}{c}))). \end{aligned}$$

Therefore, $\lim_{n\to\infty} G(x, x_{n+1}, x_{n+1}, \phi(\frac{\delta}{c})) = 1$. This implies that $\lim_{n\to\infty} h(G(x, x_{n+1}, x_{n+1}, \phi(\frac{\delta}{c}))) = 0$ and so $\lim_{n\to\infty} \psi(h(G(x, x_{n+1}, x_{n+1}, \phi(\frac{\delta}{c})))) = 0$. Then, we have, $\lim_{n\to\infty} h(G(Tx, x_{n+1}, x_{n+1}, \phi(\delta))) = 0$ and this implies that $\lim_{n\to\infty} G(Tx, x_{n+1}, x_{n+1}, \phi(\delta)) = 1$. Taking into account that $G(Tx, x_{n+1}, x_{n+1}, \phi(\delta)) \leq G(Tx, x_{n+1}, x_{n+1}, t) \leq 1$, we deduce that

$$\lim_{n\to\infty} G(Tx, Tx_n, Tx_n, t) = G(Tx, x_{n+1}, x_{n+1}, t) = 1. \tag{3.11}$$

Letting $n \to \infty$ in (3.9) and using (3.10) and (3.11), we obtain

$$G(Tx, x, x, t) \geq \lim_{n\to\infty} [G(Tx, Tx_n, Tx_n, t) * G(Tx_n, x, x, t)][\lim_{n\to\infty} G(Tx, Tx_n, Tx_n, t)] * [\lim_{n\to\infty} G(Tx_n, x, x, t)] = 1 * 1 = 1.$$

We have just proved that $G(Tx, x, x, t) = 1$ for all $t > 0$, and the axiom (GF-3) guarantees that $Tx = x$, that is, x is the fixed point of T. Next, we study the uniqueness of the fixed point of T. Assume that T has two different fixed points x and y, and we obtain the contradiction $x = y$. By hypothesis, $\lim_{t\to\infty} G(x, y, y, t) = 1$. Then there exists $t_0 > 0$ such that $G(x, y, y, t_0) > 0$. Moreover, there exists $s_0 > 0$ such that $\phi(s_0) > t_0$. Consequently, as ϕ and $G(x, y, z, t)$ are nondecreasing function with respect to t, $G(x, y, z, \phi(s_0)) \geq G(x, y, z, t_0) > 0$. From (3.2), we have $G(x, y, y, \phi(s_0)) = G(Tx, Ty, Ty, \phi(cs_0)) > 0$. By induction, $G(x, y, y, \phi(c^n s_0)) > 0$ for all $n \in \mathbb{N}$. We claim that

$$G(x, y, y, \phi(c^r s_0)) = 1 \quad \text{for all } r \in \mathbb{N}. \tag{3.12}$$

To prove it, let $r \in \mathbb{N}$ be arbitrary and let $n, m \in \mathbb{N}$ be such that $n > r$. As $c^n s_0 \leq c^r s_0 \leq s_0 \leq s_0/c^m$, and ϕ and $G(x, y, z, t)$ are nondecreasing function with respect to t, it follows that if $n, m \in \mathbb{N}$ and $r \leq n$, then

$$0 < G(x, y, y, \phi(c^n s_0)) \leq G(x, y, y, \phi(c^r s_0)) \leq G(x, y, y, \phi(s_0)) \leq G(x, y, y, \phi(\frac{s_0}{c^m})). \tag{3.13}$$

Applying the contractivity condition (3.1) to x and y, it follows that

$$\begin{aligned} h(G(x, y, y, \phi(c^r s_0))) &= h(G(Tx, Ty, Ty, \phi(c^r s_0))) \\ &\leq \psi(h(G(x, y, y, \phi(c^{r-1} s_0)))), \end{aligned} \tag{3.14}$$

where we have used $G(x, y, y, \phi(c^{r-1}s_0)) > 0$ by (3.13). Repeating this argument, we find that

$$\begin{aligned} h(G(x, y, y, \phi(c^{r-1}s_0))) &= h(G(Tx, Ty, Ty, \phi(c^{r-1}s_0))) \\ &\leq \psi(h(G(x, y, y, \phi(c^{r-2}s_0)))), \end{aligned}$$

where we have used $G(x, y, y, \phi(c^{r-2}s_0)) > 0$ by (3.13). As ψ is nondecreasing, we have

$$\psi(h(G(Tx, Ty, Ty, \phi(c^{r-1}s_0)))) \leq \psi^2(h(G(x, y, y, \phi(c^{r-2}s_0)))). \tag{3.15}$$

Combining inequalities (3.14) and (3.15), we obtain

$$\begin{aligned} h(G(x, y, y, \phi(c^r s_0))) &\leq \psi(h(G(Tx, Ty, Ty, \phi(c^{r-1}s_0)))) \\ &\leq \psi^2(h(G(x, y, y, \phi(c^{r-2}s_0)))). \end{aligned}$$

Inequality (3.13) permits us to repeat this argument n times, and it follows that

$$\begin{aligned} h(G(x, y, y, \phi(c^r s_0))) &\leq \psi^n(h(G(x, y, y, \phi(c^{r-n}s_0)))) \\ &= \psi^n(h(G(x, y, y, \phi(\frac{s_0}{c^{n-1}})))), \end{aligned} \tag{3.16}$$

for all $n > r$. As a consequence, $\lim_{n\to\infty} \frac{s_0}{c^{n-r}}$ implies that $\lim_{n\to\infty} \phi(\frac{s_0}{c^{n-r}}) = \infty$. This means that

$$\lim_{n\to\infty} G(x, y, y, \phi(\frac{s_0}{c^{n-r}})) = 1$$

and so, we have $\lim_{n\to\infty} h(G(x, y, t, \phi(\frac{s_0}{c^{n-r}}))) = 0$. As the sequence $\{a_n = h(G(x, y, y, \phi(\frac{s_0}{c^{n-r}})))\} \to 0$ and $h \in \mathcal{H}$, we have $\{\psi^n(a_n)\} \to 0$. By (3.16), we deduce that $h(G(x, y, y, \phi(c^r s_0))) = 0$. In particular, as $h \in \mathcal{H}$, Proposition (3.4) implies that $G(x, y, y, \phi(c^r s_0))) = 1$, which means that (3.12) holds. Next, let us show that $G(x, y, y, t) = 1$ for all $t > 0$. Let $t > 0$ arbitrary. Since $\lim\limits_{n\to\infty}(c^n s_0) = 0$ and $\lim\limits_{n\to\infty} \phi((c^n s_0)) = \phi(0) = 0$, there exists $r \in \mathbb{N}$ such that $\phi(c^r s_0) < t$. Hence $1 = G(x, y, y, \phi(c^r s_0)) \leq G(x, y, y, t) = 1$, so $G(x, y, y, t) = 1$. Varying $t > 0$, we conclude that $x = y$ by virtue of (GF-3), which contradicts the fact that $x \neq y$. As a result, T can only have a unique fixed point. □

Corollary 3.6. Let $(X, G, *)$ be a G-complete non-Archimedean G-fuzzy metric space verifying $\lim\limits_{t\to\infty} G(x, y, z, t) = \infty$ and let $T : X \to X$ be a mapping. Suppose that there exist $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$, and $h \in \mathcal{H}$ such that

$$h(G(Tx, Ty, Tz, \phi(ct))) \leq \psi(h(G(x, y, z, \phi(t)))),$$

for all $x, y, z \in X$ and all $t > 0$ for which $G(x, y, z, \phi(t)) > 0$. If there exists $x_0 \in X$ such that $\lim\limits_{t\to\infty} G(x_0, x_0, Tx_0, t) = 1$, then T has a unique fixed point.

Note that every non-Archimedean G-fuzzy metric space with condition $\lim\limits_{t\to\infty} G(x, y, z, t) = \infty$ is a G-Menger probabilistic metric space and so, by setting $F(x, y, t) := G(x, x, y, t)$, it is a Menger probabilistic metric. So, the following statement trivially follows from Theorem (3.5) by using $h(t) = 1/t - 1$ for all $t \in (0, 1]$.

Corollary 3.7. Theorem (1.1) immediately follows from Theorem (3.5).

Example 3.8. Let $(X, G, *)$ be the non-Archimedean G-fuzzy metric space introduced in Example (2.4) and let $T : X \to X$ be the self-mapping defined by $T(x) = x/2$ for all $x \in X$. Assume that $\psi(t) = \phi(t) = t$ for all $t \in [0, \infty)$, and let $h : (0, 1] \to [0, \infty)$ be an arbitrary strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous (for instance, $h(t) = 1/t - 1$ for all $(0, 1]$, but any other function verifying these properties yields the same result). In this context, the contractivity condition (3.1) is equivalent to

$$\begin{aligned} h(G(Tx, Ty, Tz, \phi(ct))) \leq \psi(h(G(x, y, z, \phi(t)))) \text{ if and only if } & h(G(Tx, Ty, Tz, ct)) \leq h(G(x, y, z, t)) \\ \text{if and only if } & G(Tx, Ty, Tz, ct) \geq G(x, y, z, t). \end{aligned}$$

For all $c \in (0, 1)$, $x, y, z \in X$, such that $x \neq y \neq z$ and for all $t > 0$,

$$\begin{aligned} G(Tx, Ty, Tz, ct) &= G(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, ct) \\ &= \frac{1}{1 + \max\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\}} \\ &\geq \frac{1}{1 + \max\{x, y, z\}}. \end{aligned}$$

Also, if $x = y = z$, it is clear. As a result, the contractivity condition is verified. Hence, Theorem 3.5 guarantees that T has a unique fixed point (which is $x = 0$).

4 Conclusion

In this paper, we proved some fixed point results in non-Archimedean G-fuzzy metric spaces for self-mappings providing γ-contractions and γ-weak contractions. We also presented a more general class of auxiliary functions in the contractivity condition. On the other hand, our results can be extended to other spaces.

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