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Convergence theorems by monotone hybrid algorithms for a family of generalized nonexpansive mappings and maximal monotone operators

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Abstract

Finding a zero of a maximal monotone operator is known as one of the most impressive problems which are associated with convex analysis and mathematical optimization. Akin to this is solving the fixed point problems of the class of nonexpansive mappings, which constitutes an important part of nonlinear operators with fascinating applications in several areas such as signal processing and image restoration. This study presents a monotone hybrid algorithm for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a family of a generalized nonexpansive mapping in a Banach space. Suitable conditions under which the algorithm converges strongly are established.

Keywords: Generalized, Maximal monotone, Nonexpansive, Retraction; Algorithm
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1 Introduction

Let E be a real Banach space with the dual space E^* and let $A \subset E \times E^*$ be a maximal monotone operator. If $0 \in Ax$, then x is called a zero of A. The problem of finding such a point is known as one of the most impressive problems which are associated with convex analysis and mathematical optimization (See, e.g., [11, 10, 15, 20]). Such problems have applications in economics, science and engineering and indeed, they have connection with variational inequality problems. The equivalence of variational inequalities to fixed point problems is well known (See, e.g., [16, 3, 24, 22]). Let K be a nonempty closed convex subset of E and $T : K \to K$ be a self mapping of K. The set of fixed points of T will be denoted by $F(T) := \{x : Tx = x\}$. A self mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in K,$$

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and a mapping $T : K \to E$ is said to be generalized nonexpansive provided $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x) \text{ for all } x \in K \text{ and } p \in F(T).$$

The class of nonexpansive mappings constitutes an important part of nonlinear operators. Signal processing and image restoration are classical examples of where the iterative processes on the class of nonexpansive mappings have been applied (see, e.g., [6, 4]). For a nonexpansive self-mapping T in a Hilbert space H, Qin and Su [18] presented a monotone hybrid method:

$$\begin{cases} x_1 = x \in K, K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \|u - u_n\| \leq \|u - x_n\|\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{K_n \cap Q_n} x, \end{cases} \tag{1.1}$$

and established its strong convergence under appropriate control conditions. Klin-eam et al. [12], extended the above result by considering a family of generalized nonexpansive mappings in a Banach space E and presented a monotone hybrid iterative method as

$$\begin{cases} x_1 = x \in K, K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n)T_n x_n, \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \phi(u, u_n) \leq \phi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases} \tag{1.2}$$

where J denotes the duality mapping on E, $R_{K_n \cap Q_n}$ is the sunny nonexpansive retraction from K onto $K_n \cap Q_n$, $\{T_n\}$ is defined from $T : K \to E$ by

$$T_n x = \alpha_n x + (1-\alpha_n) Tx \tag{1.3}$$

and

$$T_n x = \alpha_n Tx + (1-\alpha_n) Gx, \tag{1.4}$$

$x \in K$ and $\{\alpha_n\} \subset (0,1)$, while T and G are generalized nonexpansive mappings.

Being motivated by the previous studies monotone hybrid algorithms and the class of generalized nonexpansive mappings, this paper will present a new monotone hybrid algorithm. This study considers the family of generalized nonexpansive mappings in a Banach space and finds a common element of their fixed point set and the zero point set of a maximal monotone operator. The study demonstrates the proof of a strong convergence theorem for a proposed monotone hybrid algorithm to a common element of the zero point set of a maximal monotone operator and the fixed point set of a family of generalized nonexpansive mappings in a Banach space.

2 Preliminaries

Let E be a real Banach space with the dual space E^* and $S(E) := \{x \in E : \|x\| = 1\}$. The norm $\|.\|$ of E is said to be Gâteaux differentiable provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$ with $\|x\| = \|y\| = 1$. In such a case, E is said to be smooth. In addition, E is said to be uniformly smooth if it is smooth and the limit (2.1) is attained uniformly for each $x, y \in S(E)$. The modulus of convexity of a Banach space E, $\delta_E : (0,2] \to [0,1]$ is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| > \epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for all $x, y \in E$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$. It is well known that a space E is uniformly smooth if and only if E^* is uniformly convex. The sets of all positive integers and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \|x\|\} \ \forall \ x \in E.$$

J is known to be uniformly norm-to-norm continuous on bounded sets of E if E is uniformly smooth. For a given Banach space E, let $A \subset E \times E^*$ be a multi-valued operator. A is said to be monotone if for all $(x, x^*), (y, y^*) \in A$,

$$\langle x - y, x^* - y^* \rangle \geq 0,$$

and it is said to be maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone mapping. For a maximal monotone operator A, the set $A^{-1}(0) := \{x \in E : Ax = 0\}$ is closed and convex. According to a result of Rockafellar [21], in a given strictly convex, smooth and reflexive Banach space E, A is said to be maximum monotone if it is monotone and the range of $(J + rA)$ is all of E^* for all $r > 0$.

Definition 2.1. For a given smooth Banach space E, define the function $\varphi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$. In a Hilbert space, it is expressed as $\phi(x, y) = \|x - y\|^2 \geq 0$. The following identity holds for all $x, y, z \in E$:

(i) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

(ii) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2 \langle x - z, Jz - Jy \rangle$,

(iii) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle x - y, Jy \rangle \leq \|x\| \|Jx - Jy\| + \|x - y\| \|y\|$.

Definition 2.2. Resolvent: Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone mapping. Given $r > 0$ and $x \in E$, then there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. Thus one can define a single-valued mapping $J_r : E \to D(A)$ by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\} ,$$

which is called the resolvent of A. $J_r x$ consists of one point and for all $r > 0$, $A^{-1}(0) = F(J_r)$, where $F(J_r)$ is the set of fixed points of J_r. Also, for all $r > 0$ and $x \in E$, the Yosida approximation $A_r : C \to E^*$ is defined by

$$A_r x = \frac{1}{r}(J - JJ_r)x.$$

For all $r > 0$ and $x \in E$, the following hold (See, for example, [13, 9])

(i) $\phi(p, J_r x) + \phi(J_r x, x) \leq \phi(p, x)$ for all $p \in A^{-1}(0)$.

(ii) $(J_r x, A_r x) \in A$.

Definition 2.3. Metric projection: Let K be a nonempty closed convex subset of a Hilbert space H. A mapping $P_K : H \to K$ of H onto K satisfying

$$\|x - P_K x\| = \min_{y \in K} \|x - y\|,$$

is called the metric projection. This set is known to be a singleton. The metric projection has the important property that; for $x \in H$ and $x_0 \in K, x_0 = P_K x$ if and only if

$$\langle x - x_0, x_0 - y \rangle \geq 0 \ \forall \ y \in K.$$

Definition 2.4. Retraction: Let K be nonempty subset of a Banach space E. A mapping $R : E \to K$ is called sunny if

$$R(Rx + \alpha(x - Rx)) = Rx,$$

for all $x \in E$ and all $\alpha \geq 0$. If $Rx = x$ for all $x \in K$, it is also called a retraction. A retraction which is also sunny and nonexpansive is called a sunny nonexpansive retraction. If E is a smooth Banach space, the sunny nonexpansive retraction of E onto K is denoted by R_K. K is said to be a sunny generalized nonexpansive retract of E provided that there exists a sunny generalized nonexpansive retraction R from E onto K.

The following results on sunny generalized nonexpansive retraction will be needed and for their proof, see [9, 14].

Lemma 2.5. Let K be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_K be a retraction of E onto K. Then R_K is sunny and generalized nonexpansive if and only if

$$\langle x - R_K x, JR_K x - Jy\rangle \geq 0$$

for each $x \in E$ and $y \in K$.

Lemma 2.6. Let K be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto K and let $(x, z) \in E \times K$. Then the following hold:

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in K$;

(ii) $\phi(x, R_K y) + \phi(R_K y, y) \leq \phi(x, y)$.

Lemma 2.7. Let E be a smooth, strictly convex and reflexive Banach space and let K be a nonempty closed subset of E. Then the following are equivalent:

(i) K is a sunny generalized nonexpansive retract of E;

(ii) K is a generalized nonexpansive retract of E;

(iii) JK is closed and convex.

The following results are well known results and will be applied to establish the main results.

Lemma 2.8. Let E be a uniformly convex and smooth Banach space and let $\{u_n\}$ and $\{v_n\}$ be two sequences in E such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \varphi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$ (See [11]).

Lemma 2.9. Let E be a uniformly convex and smooth Banach space and let $d > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_d(0)$, where $B_d(0) = \{z \in E : \|z\| \leq d\}$ (See, for example, [11]).

Lemma 2.10. Let E be a uniformly convex Banach space and let $d > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|)$$

for all $x, y \in B_d(0)$ and $\alpha \in [0, 1]$, where $B_d(0) = \{w \in E : \|w\| \leq d\}$ (See, for example, [26]).

Lemma 2.11. Let E be a smooth and strictly convex Banach space, let $p \in E$ and let $\{\alpha_i\}_i^m \subset (0, 1)$ with $\sum_i^m \alpha_i = 1$. If $\{\alpha_i\}_i^m$ is a finite sequence in E such that

$$\phi\left(p, J^{-1}\left(\sum_i^m \alpha_i J z_i\right)\right) = \phi(p, z_i),$$

then $z_1 = z_2 = ... = z_m$(See, for example, [8]).

3 Main Results

Lemma 3.1. Let E be a strictly convex, smooth, and reflexive Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For each $\lambda > 0$, let $J_\lambda : E \to E$ be the resolvent of A for λ. Then J_λ is a generalized nonexpansive mapping.

Proof . Let $x \in E, y \in F(J_\lambda)$ and $\lambda > 0$. Since A is maximal monotone, recall that $A^{-1}(0) = F(J_\lambda)$. Apply Definition 2.2(i) to have

$$\phi(y, J_\lambda x) + \phi(J_\lambda x, x) \leq \phi(y, x) \text{ for all } y \in A^{-1}(0).$$

By Definition 2.1(i), $\phi(J_\lambda x, x) \geq 0$. Consequently

$$\phi(y, J_\lambda x) \leq \phi(y, x).$$

□

Theorem 3.2. Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K. For all $\lambda > 0$, let $J_\lambda : E \to E$ denote the resolvent which is associated with a maximal monotone mapping $A \subset E \times E^*$. Let T and G be closed generalized nonexpansive mappings from K to E with $\Gamma = \{T, G\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. For each $n \in N$, define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in K, \ K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J S_n R_K\left(J_{\lambda_n} x_n\right)\right), \\ v_n = J^{-1}\left(\gamma_n J u_n + (1-\gamma_n) J S_n R_K\left(J_{\lambda_n} x_n\right)\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \phi(u, v_n) \leq \phi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n\rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E and $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n from K into E is given by

$$S_n x = J^{-1}\left(\alpha_n J T x + (1-\alpha_n) J G x\right), \tag{3.1}$$

for all $x \in K$. Suppose that the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}(1-\beta_n) > 0$ and $\lim_{n\to\infty} \gamma = 1$, while $\{\lambda_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)} x$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from K onto $F(\Gamma) \cap A^{-1}(0)$.

Proof . Step 1: It can be easily shown that S_n is a generalized nonexpansive mapping for each $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} F(S_n) = F(\Gamma)$. Indeed, observe that

$$F(\Gamma) = F(T) \cap F(G) \subset \bigcap_{n=1}^{\infty} F(S_n). \tag{3.2}$$

Therefore, for $p \in F(\Gamma)$ and $x \in K$,

$$\begin{aligned} \phi\left(p, S_n x\right) &= \phi\left(p, J^{-1}\left(\alpha_n J T x + (1-\alpha_n) J G x\right)\right) \\ &= \|p\|^2 - 2\left\langle p, \alpha_n J T x + (1-\alpha_n) J G x\right\rangle + \|\alpha_n J T x + (1-\alpha_n) J G x\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, JTx\rangle - 2(1-\alpha_n)\langle p, JGx\rangle + \alpha_n \|Tx\|^2 + (1-\alpha_n)\|Gx\|^2 \\ &= \alpha_n \phi\left(p, Tx\right) + (1-\alpha_n)\phi\left(p, Gx\right) \\ &\leq \alpha_n \phi\left(p, x\right) + (1-\alpha_n)\phi\left(p, x\right) \\ &= \phi\left(p, x\right). \end{aligned}$$

Therefore, S_n is a generalized nonexpansive. Moreover, for $q \in \bigcap_{n=1}^{\infty} F(S_n)$,

$$\begin{aligned} \phi(p, q) &= \phi\left(p, S_n q\right) \\ &= \phi\left(p, J^{-1}\left(\alpha_n J T q + (1-\alpha_n) J G q\right)\right) \\ &= \|p\|^2 - 2\left\langle p, \alpha_n J T q + (1-\alpha_n) J G q\right\rangle + \|\alpha_n J T q + (1-\alpha_n) J G q\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \|p\|^2 - 2\alpha_n \langle p, JTq\rangle - 2(1-\alpha_n)\langle p, JGq\rangle + \alpha_n\|Tq\|^2 + (1-\alpha_n)\|Gq\|^2 \\
&= \alpha_n\phi(p,Tq) + (1-\alpha_n)\phi(p,Gq) \\
&\leq \alpha_n\phi(p,q) + (1-\alpha_n)\phi(p,q) \\
&= \phi(p,q),
\end{aligned}
$$

which shows clearly that

$$\phi\left(p, J^{-1}\left(\alpha_n JTq + (1-\alpha_n) JGq\right)\right) = \alpha_n\phi(p,Tq) + (1-\alpha_n)\phi(p,Gq) = \phi(p,q).$$

Apply Lemma 2.11 to get $Tq = Gq$, which implies that $q = S_nq = Tq = Gq$. Therefore, $F(T_n) \subset F(\Gamma)$ for all $n \in \mathbb{N}$. Consequently, $\bigcap_{n=1}^{\infty} F(S_n) = F(\Gamma)$.

Step 2: To verify the closedness and convexity of K_n and Q_n for all $n \in \mathbb{N}$. It is known from their definitions that K_n is closed and Q_n is closed and convex for each $n \in \mathbb{N}$. To show that K_n is convex, observe that

$$\phi(u, v_n) \leq \phi(u, x_n)$$

implies that for all $u \in K_n$,

$$\|x_n\|^2 - \|v_n\|^2 - 2\langle u, Jx_n - Jv_n\rangle \geq 0,$$

which is affine in u, and thus K_n is convex. So for all $n \in \mathbb{N}$, $K_n \cap Q_n \subset E$ is closed and convex.

Step 3: We are to demostrate that $F(\Gamma)\cap A^{-1}(0) \subset K_n \cap Q_n$. We set $y_n = R_K(J_{r_n}x_n)$ and for $p \in F(\Gamma)\cap A^{-1}(0)$,

$$
\begin{aligned}
\phi(p,u_n) &= \phi\left(p, J^{-1}\left(\beta_n Jx_n + (1-\beta_n)JS_ny_n\right)\right) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JS_ny_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JS_ny_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, JS_ny_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|S_ny_n\|^2 \\
&= \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,S_ny_n) \\
&\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,y_n) \ \text{(by generalized nonexpansive property of } S_n) \qquad (3.3)\\
&= \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,R_K(J_{r_n}x_n)) \\
&\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,J_{r_n}x_n) \ \text{(by the property of } R_K) \\
&\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,x_n) \ \text{(y gbeneralized nonexpansive property of } J_{r_n}) \\
&= \varphi(p,x_n).
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
\phi(p,v_n) &= \phi\left(p, J^{-1}\left(\gamma_n Ju_n + (1-\gamma_n)JS_ny_n\right)\right) \\
&= \|p\|^2 - 2\langle p, \gamma_n Ju_n + (1-\gamma_n)JS_ny_n\rangle + \|\gamma_n Ju_n + (1-\gamma_n)JS_ny_n\|^2 \\
&\leq \|p\|^2 - 2\gamma_n\langle p, Ju_n\rangle - 2(1-\gamma_n)\langle p, JS_ny_n\rangle + \gamma_n\|u_n\|^2 + (1-\gamma_n)\|S_ny_n\|^2 \\
&= \gamma_n\phi(p,u_n) + (1-\gamma_n)\varphi(p,S_ny_n) \\
&\leq \gamma_n\phi(p,u_n) + (1-\gamma_n)\phi(p,y_n) \\
&= \gamma_n\phi(p,u_n) + (1-\gamma_n)\phi(p,R_K(J_{r_n}u_n)) \\
&\leq \gamma_n\phi(p,u_n) + (1-\gamma_n)\phi(p,J_{r_n}u_n) \\
&\leq \gamma_n\phi(p,u_n) + (1-\gamma_n)\phi(p,u_n) \\
&\leq \gamma_n\phi(p,x_n) + (1-\gamma_n)\phi(p,x_n) \\
&= \varphi(p,x_n).
\end{aligned}
$$

This is a justification that $p \in K_n$ for all $n \in \mathbb{N}$, consequently $F(\Gamma)\cap A^{-1}(0) \subset K_n$. To use induction to show that $F(\Gamma)\cap A^{-1}(0) \subset Q_n$ for all $n \in \mathbb{N}$, observe that by definition, for $n = 1, F(\Gamma)\cap A^{-1}(0) \subset K = K_0 \cap Q_0$. It is known that J is one-to-one, therefore $J(K_n \cap Q_n) = JK_n \cap JQ_n$, which is closed convex. By Lemma 2.7, $K_n \cap Q_n$

is a sunny generalized nonexpansive retract of E. For some $i \in \mathbb{N}$, assume that $F(\Gamma) \cap A^{-1}(0) \subset K_{i-1} \cap Q_{i-1}$. Since $x_i = R_{K_{i-1} \cap Q_{i-1}} z$, applying Lemma 2.5 leads to

$$\langle x - x_i, Jx_i - Jz \rangle \geq 0,$$

for all $z \in K_{i-1} \cap Q_{i-1}$. So, it can be stated that

$$\langle x - x_i, Jx_i - Jz \rangle \geq 0, \ \forall \ z \in F(\Gamma) \cap A^{-1}(0) \tag{3.4}$$

as it is known that $F(\Gamma) \cap A^{-1}(0) \subset K_{i-1} \cap Q_{i-1}$. By the inequality (3.4) and from the definition of Q_n, it can be deduced that $F(\Gamma) \cap A^{-1}(0) \subset Q_i$ and thus $F(\Gamma) \cap A^{-1}(0) \subset Q_n$ for all $n \in \mathbb{N}$. So, $F(\Gamma) \cap A^{-1}(0) \subset K_n \cap Q_n$ for all $n \in \mathbb{N}$, which justifies that $\{x_n\}$ is well defined.

Step 4: It is shown here that as $n \to \infty$, $x_n \to R_{F(\Gamma) \cap A^{-1}(0)} x$. By the definition of Q_n, one can have $x_n = R_{Q_n} x$. Therefore by Lemma 2.6(ii),

$$\phi(x, x_n) = \phi(x, R_{Q_n} x) \leq \phi(x, u) - \phi(R_{Q_n} x, u) \leq \phi(x, u),$$

for all $F(\Gamma) \cap A^{-1}(0) \subset Q_n$. Therefore, $\{\phi(x, x_n)\}$ is bounded. Furthermore, by the definition of ϕ, it can be deduced that $\{x_n\}, \{u_n\}$ and $\{v_n\}$ are bounded. This shows that the limit of $\{\varphi(x, x_n)\}$ exists. Given a positive integer j, it can be obtained from $x_n = R_{Q_n} x$ for each $n \in \mathbb{N}$ such that

$$\varphi(x_n, x_{n+j}) = \phi(R_{Q_n} x, x_{n+j}) \leq \phi(x, x_{n+j}) - \phi(x, R_{Q_n} x) \leq \phi(x, x_{n+j}) - \phi(x, x_n),$$

which results in

$$\lim_{n \to \infty} \phi(x_n, x_{n+j}) \quad = \quad 0. \tag{3.5}$$

According to Lemma 2.9, there exists a strictly increasing, convex and continuous function $g : [0, 2r] \to [0, \infty)$, such that for $i, j \in \mathbb{N}$ with $j > i$,

$$g(\|x_j - x_i\|) \leq \phi(x_j, x_i) \leq \phi(x_j, x_0) - \phi(x_i, x_0).$$

A deduction from the property of g is that $\{x_n\}$ is Cauchy. Therefore, there exists $w \in K$ such that $x_n \to w$. Consider $x_{n+1} = R_{K_n \cap Q_n} x \in K_n$ and also from the definition of K_n, we have

$$\phi(x_{n+1}, x_n) - \phi(x_{n+1}, v_n) \quad \geq \quad 0, \ \forall \ n \in \mathbb{N}. \tag{3.6}$$

According to (3.5) and (3.6), one has that $\lim_{n \to \infty} \phi(x_{n+1}, x_n) = \lim_{n \to \infty} \phi(x_{n+1}, v_n) = 0$. Since E is uniformly convex and smooth, apply Lemma 2.8 to have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_{n+1} - v_n\| \quad = \quad 0, \tag{3.7}$$

therefore

$$\lim_{n \to \infty} \|x_n - v_n\| \quad = \quad 0. \tag{3.8}$$

Due to the norm-to-norm uniform continuity of the duality mapping J on bounded sets, one can have that

$$\lim_{n \to \infty} \|Jx_{n+1} - Jx_n\| = \lim_{n \to \infty} \|Jx_{n+1} - Jv_n\| = \|Jx_n - Jv_n\| = 0. \tag{3.9}$$

From (3.3), we see that

$$\phi(p, y_n) \geq \frac{1}{(1 - \beta_n)} (\phi(p, u_n) - \beta_n \phi(p, x_n)).$$

Recall that $y_n := R_K\left(J_{r_n}x_n\right)$, thus,

$$\begin{aligned}
\phi\left(y_n, x_n\right) &= \phi\left(R_K\left(J_{r_n}x_n\right), x_n\right) \leq \phi\left(p, x_n\right) - \phi\left(p, y_n\right) \ \text{(by Lemma 2.6 (ii),)} \\
&\leq \phi\left(p, x_n\right) - \frac{1}{(1-\beta_n)}\left(\phi\left(p, u_n\right) - \beta_n\phi\left(p, x_n\right)\right) \\
&= \frac{1}{(1-\beta_n)}\left(\phi\left(p, x_n\right) - \phi\left(p, u_n\right)\right) \\
&= \frac{1}{(1-\beta_n)}\left(\|x_n\|^2 - \|u_n\|^2 - 2\left\langle p, Jx_n - Ju_n\right\rangle\right) \\
&\leq \frac{1}{(1-\beta_n)}\left(\left|\|x_n\|^2 - \|u_n\|^2\right| + 2\left|\left\langle p, Jx_n - Ju_n\right\rangle\right|\right) \\
&\leq \frac{1}{(1-\beta_n)}\left(\left|\|x_n\| - \|u_n\|\right|\left(\|x_n\| + \|u_n\|\right) + 2\|p\|\|Jx_n - Ju_n\|\right) \\
&\leq \frac{1}{(1-\beta_n)}\left(\|x_n - u_n\|\left(\|x_n\| + \|u_n\|\right) + 2\|p\|\|Jx_n - Ju_n\|\right).
\end{aligned}$$

Using (3.8) and (3.9), $\lim_{n\to\infty}\phi\left(y_n, x_n\right) = 0$. Then by Lemma 2.8,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.10}$$

Additionally, observe that

$$\begin{aligned}
\|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \beta_nJx_n - (1-\beta_n)JS_ny_n\| \\
&= \|(1-\beta_n)\left(Jx_{n+1} - JS_ny_n\right) - \beta_n\left(Jx_n - Jx_{n+1}\right)\| \\
&\geq (1-\beta_n)\|Jx_{n+1} - JS_ny_n\| - \beta_n\|Jx_n - Jx_{n+1}\|.
\end{aligned}$$

So

$$\|Jx_{n+1} - JS_ny_n\| \leq \frac{1}{(1-\beta_n)}\left(\|Jx_{n+1} - Ju_n\| + \beta_n\|Jx_n - Jx_{n+1}\|\right).$$

Since it is given that $\liminf_{n\to\infty}(1-\beta_n) > 0$ and by considering (3.8), one can have that

$$\lim_{n\to\infty}\|Jx_{n+1} - JS_ny_n\| = 0.$$

Recall that J^{-1} is norm-to-norm uniformly continuous on bounded sets. Thus

$$\lim_{n\to\infty}\|x_{n+1} - S_ny_n\| = 0. \tag{3.11}$$

Notice that

$$\|x_n - S_ny_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_ny_n\|,$$

which by (3.7) and (3.11), results in

$$\lim_{n\to\infty}\|x_n - S_ny_n\| = 0. \tag{3.12}$$

Similarly, observe that

$$\|y_n - S_ny_n\| \leq \|y_n - x_n\| + \|x_n - S_ny_n\|,$$

by using (3.10) and (3.12), we have

$$\lim_{n\to\infty}\|y_n - S_ny_n\| = 0. \tag{3.13}$$

Due to norm-to-norm uniform continuity of the duality mapping J on bounded sets and by (3.13),

$$\lim_{n\to\infty}\|Jy_n - JS_ny_n\| = 0.$$

Since $\{y_n\}$ is bounded, $\{Ty_n\}$ and $\{Gy_n\}$ are bounded. Let $r = \max\{\sup_n \|y_n\|, \sup_n \|Ty_n\|, \sup_n \|Gy_n\|\}$. Therefore, there exists $r > 0$ with $B_r(0) = \{z \in E : \|z\| \le r\}$ and $\{y_n\}, \{Ty_n\}, \{Gy_n\} \subset B_r(0)$. According to Lemma 2.10, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for $p \in \bigcap_{n=1}^{\infty} F(S_n)$,

$$\begin{aligned}
\phi(p, S_n y_n) &= \phi\left(p, J^{-1}(\alpha_n JTy_n + (1-\alpha_n) JGy_n)\right) \\
&= \|p\|^2 - 2\langle p, \alpha_n JTy_n + (1-\alpha_n) JGy_n\rangle + \|\alpha_n JTy_n + (1-\alpha_n) JGy_n\|^2 \\
&\le \|p\|^2 - 2\alpha_n \langle p, JTy_n\rangle - 2(1-\alpha_n)\langle p, JGy_n\rangle + \alpha_n \|Ty_n\|^2 \\
&\quad + (1-\alpha_n)\|Gy_n\|^2 - \alpha_n(1-\alpha_n) g(\|Ty_n - Gy_n\|) \\
&= \alpha_n \phi(p, Ty_n) + (1-\alpha_n)\phi(p, Gy_n) - \alpha_n(1-\alpha_n) g(\|Ty_n - Gy_n\|) \\
&\le \alpha_n \phi(p, y_n) + (1-\alpha_n)\phi(p, y_n) - \alpha_n(1-\alpha_n) g(\|Ty_n - Gy_n\|) \\
&= \phi(p, y_n) - \alpha_n(1-\alpha_n) g(\|Ty_n - Gy_n\|).
\end{aligned}$$

In view of this,

$$\alpha_n(1-\alpha_n) g(\|Ty_n - Gy_n\|) \le \phi(p, y_n) - \phi(p, S_n y_n). \tag{3.14}$$

Let $\{\|Ty_{n_i} - Gy_{n_i}\|\}$ be any subsequent set of $\{\|Ty_n - Gy_n\|\}$. Since $\{y_{n_i}\}$ is known to be bounded, there exists a subsequent set $\left\{y_{n'_j}\right\}$ of $\{y_{n_i}\}$ such that

$$\lim_{j\to\infty} \phi\left(p, y_{n'_j}\right) = \limsup_{i\to\infty} \phi(p, y_{n_i}) = 0.$$

Using Definition 2.1 ((ii) and (iii)) leads to

$$\begin{aligned}
\phi\left(p, y_{n'_j}\right) &= \phi\left(p, S_{n'_j} y_{n'_j}\right) + \phi\left(S_{n'_j} y_{n'_j}, y_{n'_j}\right) + 2\left\langle p - S_{n'_j} y_{n'_j}, JS_{n'_j} y_{n'_j} - Jy_{n'_j}\right\rangle \\
&\le \phi\left(p, S_{n'_j} y_{n'_j}\right) + \|S_{n'_j} y_{n'_j}\| \|JS_{n'_j} y_{n'_j} - Jy_{n'_j}\| \\
&\quad + \|S_{n'_j} y_{n'_j} - y_{n'_j}\| \|y_{n'_j}\| + 2\|p - S_{n'_j} y_{n'_j}\| \|JS_{n'_j} y_{n'_j} - Jy_{n'_j}\|.
\end{aligned} \tag{3.15}$$

Consequently,

$$c = \liminf_{j\to\infty} \phi(p, y_{n_j}) = \liminf_{j\to\infty} \phi\left(p, S_{n'_j} y_{n'_j}\right).$$

On the other hand, $\varphi(p, S_n y_n) \le \varphi(p, y_n)$ results in

$$\limsup_{j\to\infty} \phi\left(p, S_{n'_j} y_{n'_j}\right) = \limsup_{j\to\infty} \phi(p, y_{n_j}) = c,$$

hence

$$\lim_{j\to\infty} \phi(p, y_{n_j}) = \lim_{j\to\infty} \phi\left(p, S_{n'_j} y_{n'_j}\right) = c.$$

Since it is already given that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, the deduction from (3.14) is that $\lim_{j\to\infty} g\left(\|Ty_{n'_j} - Gy_{n'_j}\|\right) = 0$. The properties of the function g lead to the deduction that $\lim_{j\to\infty} \|Ty_{n'_j} - Gy_{n'_j}\| = 0$, and for that reason

$$\lim_{n\to\infty} \|Ty_n - Gy_n\| = 0.$$

Considering that

$$\|y_n - Ty_n\| \le \|y_n - S_n y_n\| + \|S_n y_n - Ty_n\| = \|y_n - S_n y_n\| + (1-\alpha_n)\|Gy_n - Ty_n\|,$$

leads to $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. In a similar manner, $\lim_{n\to\infty} \|y_n - Gy_n\| = 0$. Hence,

$$\lim_{n\to\infty}\|y_n - Sy_n\| = 0 \quad \forall\, S \in \Gamma. \tag{3.16}$$

By (3.10) and (3.16),

$$\|x_n - Sy_n\| \le \|x_n - y_n\| + \|y_n - Sy_n\| \to 0 \text{ as } n \to \infty.$$

Since $x_n \to w$ and by (3.10), one can deduce that $y_n \to w$. It is known that S is closed since the elements of the set Γ are closed, and moreover $y_n \to w$, thus w is a fixed point of S. The next task is to show that $w \in A^{-1}(0)$. Given that E is uniformly smooth, from (3.10), one can have that

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0.$$

For $\lambda_n \ge a$, one can have that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\|Jx_n - Jy_n\| = 0.$$

As a consequence of this,

$$\lim_{n\to\infty}\|A_{\lambda_n}x_n\| = \lim_{n\to\infty}\frac{1}{\lambda_n}\|Jx_n - Jy_n\| = 0.$$

For $(z, z^*) \in A$, the monotonicity of A gives that

$$\langle z - w_n, z^* - A_{\lambda_n}x_n\rangle \ge 0 \text{ for all } n \in \mathbb{N}.$$

As $n \to \infty$, this results in

$$\langle z - w, z^*\rangle \ge 0.$$

Given that A is maximal monotone confirms that $w \in A^{-1}(0)$. Lastly, it is required to show that $w = R_{F(\Gamma)\cap A^{-1}(0)}x$. Apply Lemma 2.6 to get

$$\phi\left(w, R_{F(\Gamma)\cap A^{-1}(0)}x\right) + \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}x, x\right) \le \phi\left(w, x\right).$$

Since $x_{n+1} = R_{K_n\cap Q_n}x$ and $w \in F(\Gamma)\cap A^{-1}(0) \subset K_n \cap Q_n$, by Lemma 2.6,

$$\phi\left(R_{F(\Gamma)\cap A^{-1}(0)}x, x_{n+1}\right) + \phi\left(x_{n+1}, x\right) \le \phi\left(R_{F(\Phi)\cap A^{-1}(0)}x, x\right).$$

From the definition of ϕ, one has that $\phi\left(w, x\right) \le \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}x, x\right)$ and $\phi\left(w, x\right) \ge \varphi\left(R_{F(\Gamma)\cap A^{-1}(0)}x, x\right)$, thus, $\phi\left(w, x\right) = \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}x, x\right)$. Thus, since $R_{F(T)\cap A^{-1}(0)}x$ is unique, it can be concluded that $w = R_{F(\Gamma)\cap A^{-1}(0)}x$. □

The proof of the following results can be deduced from the main result of this paper, which is Theorem 3.2.

Corollary 3.3. Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K. For all $\lambda > 0$, let $J_\lambda : E \to E$ denote the resolvent which is associated with a maximal monotone mapping $A \subset E \times E^*$. Let T and G be closed generalized nonexpansive mappings from K to E with $\Gamma = \{T, G\}$ such that $F(\Gamma)\cap A^{-1}(0) \ne \emptyset$. For each $n \in N$, define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in K,\ K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n)JS_nR_K\left(J_{\lambda_n}x_n\right)\right), \\ K_n = \{u \in K_{n-1}\cap Q_{n-1} : \phi(u, v_n) \le \phi(u, x_n)\} \\ Q_n = \{u \in K_{n-1}\cap Q_{n-1} : \langle x_n - u, Jx - Jx_n\rangle \ge 0\} \\ x_{n+1} = R_{K_n\cap Q_n}x, \end{cases}$$

where $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n from K into E is given by (3.1) and J is the duality mapping on E. Suppose that the real sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}(1-\beta_n) > 0$ and $\{\lambda_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)}x$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from K onto $F(\Gamma)\cap A^{-1}(0)$.

Proof . By letting $\gamma_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, the desired result follows. □

Corollary 3.4. Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K. For all $\lambda > 0$, let $J_\lambda : E \to E$ denote the resolvent which is associated with a maximal monotone mapping $A \subset E \times E^*$. Let T be a closed generalized nonexpansive mapping from K to E and such that $F(T) \cap A^{-1}(0) \neq \emptyset$. For each $n \in N$, define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in K, \ K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) JTR_K\left(J_{\lambda_n} x_n\right)\right), \\ v_n = J^{-1}\left(\gamma_n J u_n + (1-\gamma_n) JTR_K\left(J_{\lambda_n} x_n\right)\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \phi(u, v_n) \leq \phi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E. Suppose that the real sequence $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\lim_{n\to\infty} \gamma = 1$, and $\{\lambda_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}(0)} x$, where $R_{F(T)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from K onto $F(T) \cap A^{-1}(0)$.

Proof . By letting $\alpha_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, it is obvious that $\{S_n\} = \{T\}$. Then the desired result follows. □

The main result of this paper generalizes the result below which is in the framework of Hilbert spaces.

Corollary 3.5. Let K be a nonempty closed convex subset of Hilbert space H and $P_K : H \to K$ be a metric projection from H onto K. For all $\lambda > 0$, let $J_\lambda : H \to H$ denote the resolvent which is associated with a maximal monotone mapping $A \subset H \times H$. Let T and G be closed generalized nonexpansive mappings from K to H with $\Gamma = \{T, G\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. For each $n \in N$, define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in K, \ K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n) S_n R_K\left(J_{\lambda_n} x_n\right), \\ v_n = \gamma_n u_n + (1-\gamma_n) S_n R_K\left(J_{\lambda_n} x_n\right), \\ K_n = \{y \in K_{n-1} \cap Q_{n-1} : \|y - u_n\| \leq \|y - x_n\|\} \\ Q_n = \{y \in K_{n-1} \cap Q_{n-1} : \langle x_n - y, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{K_n \cap Q_n} x, \end{cases}$$

where $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n from K into H is given by (3.1). Suppose that the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}(1-\beta_n) > 0$ and $\lim_{n\to\infty} \gamma = 1$, while $\{\lambda_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(\Gamma)\cap A^{-1}(0)} x$, where $P_{F(\Gamma)\cap A^{-1}(0)}$ is the metric projection from K onto $F(\Gamma) \cap A^{-1}(0)$.

Proof . Recall that in a Hilbert space, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$ and J is the identity mapping. Therefore, the desired result readily follows from Theorem 3.2. □

Conclusion

Many problems in machine learning, signal processing and image recovery can be modeled as contructing zeros of a maximal monotone operator and finding the fixed point problems of the class of nonexpansive mappings. Most of the proposed algorithms in the literature are for either the class of nonexpansive mappings (See e.g., [2, 1, 17, 25, 19]) or monotone type mappings (See e.g., [5, 7, 23]). In this paper, a monotone hybrid algorithm is presented for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a family of generalized nonexpansive mappings in a Banach space. Moreover, a strong convergence result is established under suitable conditions. The parameters in the main theorem which satisfy the stated conditions are $\{\alpha_n\} = \left\{\frac{1}{2} + \frac{1}{5n}\right\}$, $\{\beta_n\} = \left\{\frac{2}{3} - \frac{1}{2n}\right\}$ and $\{\beta_n\} = \left\{1 - \frac{1}{5+n}\right\}$.

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