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On the solution of a nonlinear fractional integro-differential equation with non-local boundary condition

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Abstract

This work studies the existence and the uniqueness of the solution to a kind of high-order nonlinear fractional integro-differential equations involving Rieman-Liouville fractional derivative. The boundary condition is of integral type which entangles ending point of the domain. First, the unique exact solution is extracted in terms of Green's function for the linear fractional differential equation and then Banach contraction mapping theorem is applied to prove the main result in the case of general nonlinear source term. Furthermore, our main result is demonstrated by an illustrative example to show its legitimacy and applicability.

Keywords: High order differential equations, fractional integro-differential equating, Integral boundary condition, Rieman-liouville derivative, Fixed point theorem
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1 Introduction and the problem formulation

Fractional order differential and integral operators, which are nonlocal in nature have various applications in applied branches such as blood flow problems, anomalous diffusion, spreading of disease, control processes, population dynamics, etc. for instance see [18, 19, 20, 21, 22, 24, 27, 29, 30, 31, 32, 33].

The theory of fixed point to study of the solution to the boundary value problem is a consequential tool and it is played important role in both proving the existence of the solution and obtain it approximately, see [1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 23, 26, 28]. It can be seen a lot interest in researching on the subject of nonlocal nonlinear fractional order in both single-valued and multi-valued boundary value problems in recent years. In the research line of study of the existence and uniqueness results for the boundary value problems based on fixed point theory, finding Green function in a useful way helps to present the unique solution in the linear case. In [8], Cabada and Hamdi have investigated the following fractional nonlocal boundary value problem

$$\begin{cases} {}_0D_t^{\alpha} u + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds. \end{cases}$$

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where $2 < \alpha \leq 3$, $\lambda > 0, \lambda \neq \alpha$ and ${}_0D_t^{\alpha}$ is Rieman-Liouville fractional derivative. Authors in [25] studied very special nonlinear sequential fractional differential equation with respect to nonlocal fractional integral conditions and proved the existence of the unique solution. Also, Cabada and Wang [9] have studied almost the same problem, i.e.

$$\begin{cases} {}^{C}D_t^{\alpha}u + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u^{''}(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds. \end{cases}$$

where, ${}^{C}D_t^{\alpha}$ is Caputo fractional derivative.

In this paper, we investigate the existence of unique solution for a class of high-order fractional differential equations with nonlocal BCs given by:

$$\begin{cases} {}_0D_t^{\alpha}u + \int_0^t K(t,s)f(s,u(s),u'(s),u''(s))ds = -g(t,u(t),u'(t),u''(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds. \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4$, λ is a real number, ${}_0D_t^{\alpha}u$ is the left Rieman-Liouville fractional derivative and, the kernel $K : (0,1) \times (0,1) \to \mathbb{R}$, $g : (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f : (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

2 Preliminaries

This section is devoted to some basic definitions of fractional calculus.

Definition 2.1. Let $\alpha \in \mathbb{R}^+$. The operator ${}_aD_t^{-\alpha}$, defined on $L^1[a,b]$ by

$${}_aD_t^{-\alpha}u(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}u(\tau)d\tau$$

for $t \in [a,b]$, is called the Riemann-Liouville fractional integral of order α.

Definition 2.2. Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$.The operator${}_aD_t^{\alpha}$, defined by

$${}_aD_t^{\alpha}u := D^m({}_aD_t^{-(m-\alpha)}u) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d^m}{dt^m}\right)\int_a^t (t-\tau)^{m-1-\alpha}u(\tau)d\tau$$

is called the Riemann-Liouville fractional differential of order α.

Lemma 2.3. [[10], Lemma 5.2] Let $\alpha > 0$, $\alpha \in \mathbb{N}$ and $m = \lceil \alpha \rceil$. Also assume all the hypotheses of theorem 5.1 in [10] hold. The function $u \in C(0,h]$ is a solution of the differential equation

$${}_0D_t^{\alpha}u(t) = f(t,u(t)),$$

equipped with the initial conditions

$${}_0D_t^{\alpha-k}u(0) = b_k \quad (k = 1,2,...,m-1), \quad \lim_{z\to 0^+} {}_0D_t{}^{-(m-\alpha)}u(z) = b_m$$

if and only if it is a solution of the Volterra integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (x-t)^{\alpha-1}f(t,u(t))dt + \sum_{k=1}^{m}\frac{b_k}{\Gamma(\alpha-k+1)}t^{\alpha-k}. \tag{2.1}$$

3 Solution to the linear equation

We go back to the aimed problem (1.1). As usual, the approach here is to seek solutions as fixed points of an operator defined by using the Green's function corresponding to the linear version of problem, i.e.

$${}_0D_t^{\alpha}u(t) + y(t) = 0, \quad t \in (0,1), \tag{3.1}$$

with respect to the boundary conditions

$$u(0) = u'(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds. \tag{3.2}$$

Theorem 3.1. Let $3 < \alpha \leq 4$. Assume $y \in C[0,1]$, then the problem (3.1)-(3.2) has a unique solution $u \in C^2[0,1]$, given by

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{3.3}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}[\alpha-(1-s)\lambda]t^{\alpha-1}-(\alpha-\lambda)(t-s)^{\alpha-1}}{\Gamma(\alpha)(\alpha-\lambda)}, & 0 \leq s \leq t \leq 1; \\ \frac{(1-s)^{\alpha-1}[\alpha-(1-s)\lambda]t^{\alpha-1}}{\Gamma(\alpha)(\alpha-\lambda)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.4}$$

Proof. By applying Lemma 2.3, we observe that Eq. (3.1) is equivalent to

$$u(t) = {}_0D_t{}^{-\alpha}(-y(t)) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds + \sum_{k=1}^{4} \frac{{}_0D_t^{\alpha-k}u(0)}{\Gamma(\alpha-k+1)} t^{\alpha-k}. \tag{3.5}$$

Set $c_k = \frac{{}_0D_t^{\alpha-k}u(0)}{\Gamma(\alpha-k+1)}$, so

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + c_4t^{\alpha-4}. \tag{3.6}$$

Since $3 < \alpha \leq 4$ and $u(0) = 0$, we should have $c_4 = 0$. Similarly, since $u'(0) = u''(0) = 0$ we have $c_2 = c_3 = 0$. So, Eq. (3.6) deduces to

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1}.$$

Imposing the nonlocal integral condition implies

$$\left[\frac{-1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)ds + c_1\right] = \lambda \int_0^1 u(s)ds, \tag{3.7}$$

then, we have

$$c_1 = \lambda \int_0^1 u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)ds. \tag{3.8}$$

Which yields:

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds + \lambda t^{\alpha-1} \int_0^1 u(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)ds. \tag{3.9}$$

Let us set $A = \int_0^1 u(s)ds$ then, by integrating both sides of the last equality on interval $[0,1]$ with respect to t, we get the following:

$$A = \frac{-1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1}y(s)dsdt + \lambda A \int_0^1 t^{\alpha-1}dt + \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)dsdt,$$

or

$$A = \frac{-1}{\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha}y(s)ds + \frac{\lambda A}{\alpha} + \frac{1}{\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)ds,$$

therefore, A is extracted as

$$A = \frac{1}{\Gamma(\alpha)\,(\alpha-\lambda)}\left[\int_0^1 (1-s)^{\alpha-1}y(s)ds - \int_0^1 (1-s)^{\alpha}y(s)ds\right]. \tag{3.10}$$

Replacing this value to Eq. (3.9), we obtain the following expression for u:

$$\begin{aligned}
u(t) =& \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + \\
& \frac{\lambda t^{\alpha-1}}{\Gamma(\alpha)\,(\alpha-\lambda)}\left[\int_0^1 (1-s)^{\alpha-1}y(s)ds - \int_0^1 (1-s)^{\alpha}y(s)ds\right] \\
=& -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]}{\Gamma(\alpha)\,(\alpha-\lambda)}y(s)ds \\
=& \int_0^t \frac{(1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]t^{\alpha-1} - (\alpha-\lambda)\,(t-s)^{\alpha-1}}{\Gamma(\alpha)\,(\alpha-\lambda)}y(s)ds + \int_t^1 \frac{(1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]t^{\alpha-1}}{\Gamma(\alpha)\,(\alpha-\lambda)}y(s)ds \\
=& \int_0^1 G(t,s)y(s)ds,
\end{aligned} \tag{3.11}$$

□

4 Fixed point iteration

Consider $C^2[0,1]$, the Banach space of all continuously differentiable functions from $[0,1]$ to $\mathbb{R}$, equipped with the usual norm $\|u\| = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$. Replacing $y(t)$ by $\int_0^t K(t,s)f(s,u(s),u'(s),u''(s))ds + g(t,u(t),u'(t),u''(t))$ in Theorem 3.1, an operator $T : C^2[0,1] \to C^2[0,1]$ associated with problem (1.1) can be defined as:

$$\begin{aligned}
Tu(t) =& \int_0^1 G(t,s)\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s))\right]ds :\equiv \\
& -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s))\right]ds \\
& +\mu(t)\int_0^1 (1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]\times\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s))\right]ds,
\end{aligned} \tag{4.1}$$

where

$$\mu(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)\,(\alpha-\lambda)}. \tag{4.2}$$

We verify that, from Theorem 3.1, the fixed points of operator T are exactly the solutions of problem (1.1). So it remains to investigate the fixed points of operator T. Our main tool is Banach's contraction principle.

Theorem 4.1. (Banach's contraction principle[15]) Let D be a nonempty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.

5 Main results

We are now ready to prove the main theorem. For computational convenience, we put

$$\begin{aligned}
R =& \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + \left(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right)\|K\|_\infty \\
& + \left(1 + \frac{\lambda}{\alpha+1}\right)\|\mu\| + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu\|\|K\|_\infty.
\end{aligned} \tag{5.1}$$

Theorem 5.1. Assume the following contraction conditions hold:
(H_1) There exists $L_f > 0$ such that

$$\forall t \in [0,1], u, v, \bar{u}, \bar{v}, \tilde{u}, \tilde{v} \in \mathbb{R}: \\ |f(t,u,\bar{u},\tilde{u}) - f(t,v,\bar{v},\tilde{v})| \leq L_f\big(|u-v| + |\bar{u}-\bar{v} + |\tilde{u}-\tilde{v}|\big). \tag{5.2}$$

(H_2) There exists $L_g > 0$ such that

$$\forall t \in [0,1], u, v, \bar{u}, \bar{v}, \tilde{u}, \tilde{v} \in \mathbb{R}: \\ |g(t,u,\bar{u},\tilde{u}) - g(t,v,\bar{v},\tilde{v})| \leq L_g\big(|u-v| + |\bar{u}-\bar{v} + |\tilde{u}-\tilde{v}|\big). \tag{5.3}$$

Then if $LR < 1$ the BVP (1.1) has a unique solution on the ball $B_r = \{u \in C^2[0,1] \ : \ \|u\| \leq r\}$, where $r \geq \frac{NR}{1-LR}$ with $N_f = \sup_{t\in[0,1]} |f(t,0,0,0)|$, $N_g = \sup_{t\in[0,1]} |g(t,0,0,0)|$, $L = \max\{L_f, L_g\}$ and $N = \max\{N_f, N_g\}$.

Proof. First, we show that $T : B_r \to B_r$ i.e. $T(B_r) \subset B_r$. Take $u \in B_r$, $t \in [0,1]$, this means $\|u\| = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \leq r$. We need to show that $\|Tu\| = \|Tu\|_\infty + \|(Tu)'\|_\infty + \|(Tu)''\|_\infty \leq r$. It is routine to see that

$$\begin{aligned}
|Tu(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \left| \int_0^s K(s,z) f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s)) \right| ds \\
& + \|\mu\|_\infty \int_0^1 |1-s|^{\alpha-1} |\alpha - (1-s)\lambda| \times \left| \int_0^s K(s,z) f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s)) \right| ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \left(s\|K\|_\infty \|f\|_\infty + \|g\|_\infty \right) ds \\
& + \|\mu\|_\infty \int_0^1 |1-s|^{\alpha-1} |\alpha - (1-s)\lambda| \left(s\|K\|_\infty \|f\|_\infty + \|g\|_\infty \right) ds \\
\leq & \frac{1}{\Gamma(\alpha+1)} t^\alpha \|g\|_\infty + \frac{1}{\Gamma(\alpha+2)} t^{\alpha+1} \|K\|_\infty \|f\|_\infty \\
& + \left(1 + \frac{\lambda}{\alpha+1}\right) \|\mu\|_\infty \|g\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)} \right) \|\mu\|_\infty \|K\|_\infty \|f\|_\infty \\
\leq & \|f\|_\infty \left(\frac{1}{\Gamma(\alpha+2)} \|K\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)} \right) \|\mu\|_\infty \|K\|_\infty \right) \\
& + \|g\|_\infty \left(\frac{1}{\Gamma(\alpha+1)} + \left(1 + \frac{\lambda}{\alpha+1}\right) \|\mu\|_\infty \right).
\end{aligned} \tag{5.4}$$

But

$$\begin{aligned}
|f(t,u(t),u'(t),u''(t))| &= |f(t,u(t),u'(t),u''(t)) \pm f(t,0,0,0)| \\
&\leq |f(t,u(t),u'(t),u''(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq L_f(|u(t)| + |u'(t)| + |u''(t)|) + |f(t,0,0,0)| \\
&\leq L_f\|u\| + N_f.
\end{aligned}$$

Take supremum of inequality above

$$\|f\|_\infty \leq L_f\|u\| + N_f \leq L_f r + N_f.$$

In the same way, we can conclude

$$\|g\|_\infty \leq L_g\|u\| + N_g \leq L_g r + N_g.$$

Therefore, the inequality (5.4) results in

$$\begin{aligned}
\|Tu\|_\infty \leq \ & (L_f r + N_f) \left(\frac{1}{\Gamma(\alpha+2)} \|K\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)} \right) \|\mu\|_\infty \|K\|_\infty \right) \\
& + (L_g r + N_g) \left(\frac{1}{\Gamma(\alpha+1)} + \left(1 + \frac{\lambda}{\alpha+1}\right) \|\mu\|_\infty \right),
\end{aligned} \tag{5.5}$$

then

$$\|Tu\|_\infty \le (Lr+N)\left[\frac{1}{\Gamma(\alpha+1)}+\left(1+\frac{\lambda}{\alpha+1}\right)\|\mu\|_\infty+\frac{1}{\Gamma(\alpha+2)}\|K\|_\infty+\left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu\|_\infty\|K\|_\infty\right]. \tag{5.6}$$

Also, on the other hand

$$\begin{aligned}(Tu)'(t) &= \frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right]ds\\ &+\mu'(t)\int_0^1 (1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]\times\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right]ds. \end{aligned}\tag{5.7}$$

So, we have

$$\begin{aligned}|(Tu)'(t)| \le&\frac{1-\alpha}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-2}\times\left|\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right|ds\\ &+\|\mu'\|_\infty\int_0^1 |1-s|^{\alpha-1}\left|\alpha-(1-s)\lambda\right|\times\\ &\left|\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right|ds\\ \le&\frac{1-\alpha}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-2}\left(s\|K\|_\infty\|f\|_\infty+\|g\|_\infty\right)ds\\ &+\|\mu'\|_\infty\int_0^1 |1-s|^{\alpha-1}\left|\alpha-(1-s)\lambda\right|\left(s\|K\|_\infty\|f\|_\infty+\|g\|_\infty\right)ds\\ \le&\frac{1}{\Gamma(\alpha)}t^{\alpha-1}\|g\|_\infty+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\|K\|_\infty\|f\|_\infty\\ &+\left(1+\frac{\lambda}{\alpha+1}\right)\|\mu'\|_\infty\|g\|_\infty+\left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu'\|_\infty\|K\|_\infty\|f\|_\infty\\ \le&\|f\|_\infty\left(\frac{1}{\Gamma(\alpha+1)}\|K\|_\infty+\left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu'\|_\infty\|K\|_\infty\right)\\ &+\|g\|_\infty\left(\frac{1}{\Gamma(\alpha)}+\left(1+\frac{\lambda}{\alpha+1}\right)\|\mu'\|_\infty\right). \end{aligned}\tag{5.8}$$

This yields:

$$\|(Tu)'\|_\infty \le(Lr+N)\left[\frac{1}{\Gamma(\alpha)}+\left(1+\frac{\lambda}{\alpha+1}\right)\|\mu'\|_\infty+\frac{1}{\Gamma(\alpha+1)}\|K\|_\infty+\left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu'\|_\infty\|K\|_\infty\right]. \tag{5.9}$$

In the same way, for the second derivative of T operator after effecting on u, we have

$$\begin{aligned}(Tu)''(t) &= \frac{(1-\alpha)(\alpha-2)}{\Gamma(\alpha)}\times\int_0^t (t-s)^{\alpha-3}\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right]ds\\ &+\mu''(t)\int_0^1 (1-s)^{\alpha-1}\left[\alpha-(1-s)\lambda\right]\times\left[\int_0^s K(s,z)f(z,u(z),u'(z),u''(z))dz+g(s,u(s),u'(s),u''(s))\right]ds, \end{aligned}\tag{5.10}$$

Then, obviously we have

$$
\begin{aligned}
|(Tu)''(t)| &\leq \frac{(1-\alpha)(\alpha-2)}{\Gamma(\alpha)} \times \int_0^t |t-s|^{\alpha-3} \left| \int_0^s K(s,z) f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s)) \right| ds \\
&\quad + \|\mu''\|_\infty \int_0^1 |1-s|^{\alpha-1} |\alpha-(1-s)\lambda| \times \\
&\quad \left| \int_0^s K(s,z) f(z,u(z),u'(z),u''(z))dz + g(s,u(s),u'(s),u''(s)) \right| ds \\
&\leq \frac{(1-\alpha)(\alpha-2)}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-3} \left(s\|K\|_\infty \|f\|_\infty + \|g\|_\infty\right) ds \\
&\quad + \|\mu''\|_\infty \int_0^1 |1-s|^{\alpha-1} |\alpha-(1-s)\lambda| \left(s\|K\|_\infty \|f\|_\infty + \|g\|_\infty\right) ds \\
&\leq \frac{1}{\Gamma(\alpha-1)} t^{\alpha-2} \|g\|_\infty + \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \|K\|_\infty \|f\|_\infty \\
&\quad + \left(1+\frac{\lambda}{\alpha+1}\right) \|\mu''\|_\infty \|g\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)}\right) \|\mu''\|_\infty \|K\|_\infty \|f\|_\infty \\
&\leq \|f\|_\infty \left(\frac{1}{\Gamma(\alpha)} \|K\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)}\right) \|\mu''\|_\infty \|K\|_\infty \right) \\
&\quad + \|g\|_\infty \left(\frac{1}{\Gamma(\alpha-1)} + \left(1+\frac{\lambda}{\alpha+1}\right) \|\mu''\|_\infty \right). \qquad (5.11)
\end{aligned}
$$

Therefore, it is concluded

$$
\begin{aligned}
\|(Tu)''\|_\infty &\leq (Lr+N) \left[\frac{1}{\Gamma(\alpha-1)} + \left(1+\frac{\lambda}{\alpha+1}\right) \|\mu''\|_\infty + \right. \\
&\quad \left. \frac{1}{\Gamma(\alpha)} \|K\|_\infty + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)}\right) \|\mu''\|_\infty \|K\|_\infty \right]. \qquad (5.12)
\end{aligned}
$$

Now, combining Eqs. (5.6), (5.9) and (5.12), we obtain

$$
\begin{aligned}
\|Tu\| &\leq (Lr+N)\big[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + \left(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \right) \|K\|_\infty \\
&\quad + \left(1+\frac{\lambda}{\alpha+1}\right) \|\mu\| + \left(\frac{1}{\alpha+1} + \frac{\lambda}{(\alpha+1)(\alpha+2)}\right) \|\mu\| \|K\|_\infty \big] \\
&= (Lr+N)R. \qquad (5.13)
\end{aligned}
$$

It is sufficient to choose $r \geq (Lr+N)R$, to have $TB_r \subset B_r$. Next, we show that T is a contraction as an operator.

Notice that, for arbitrary $u, v \in C^2[0,1]$, we have

$$\begin{aligned}
&|(Tu)(t)-(Tv)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\Big[\int_0^s |K(s,z)||f(z,u(z),u'(z),u''(z))\\
&-f(z,v(z),v'(z),v''(z))|dz + |g(s,u(s),u'(s),u''(s)) - g(s,v(s),v'(s),v''(s))|\Big]ds\\
&+\|\mu\|_\infty \int_0^1 |1-s|^{\alpha-1}\,|\alpha-(1-s)\lambda| \times\\
&\Big[\int_0^s |K(s,z)||f(z,u(z),u'(z),u''(z)) - f(z,v(z),v'(z),v''(z))|dz +\\
&|g(s,u(s),u'(s),u''(s)) - g(s,v(s),v'(s),v''(s))|\Big]ds\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\left(s\|K\|_\infty L_f\|u-v\| + L_g\|u-v\|\right)ds\\
&+\|\mu\|_\infty \int_0^1 |1-s|^{\alpha-1}\,|\alpha-(1-s)\lambda|\left(s\|K\|_\infty L_f\|u-v\| + L_g\|u-v\|\right)ds\\
\leq\ & \frac{1}{\Gamma(\alpha+1)}t^\alpha L_g\|u-v\| + \frac{1}{\Gamma(\alpha+2)}t^{\alpha+1}\|K\|_\infty L_f\|u-v\|\\
&+\left(1+\frac{\lambda}{\alpha+1}\right)\|\mu\|_\infty L_g\|u-v\| + \left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu\|_\infty\|K\|_\infty L_f\|u-v\|\\
\leq\ & L_f\|u-v\|\left(\frac{1}{\Gamma(\alpha+2)}\|K\|_\infty + \left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu\|_\infty\|K\|_\infty\right)\\
&+L_g\|u-v\|\left(\frac{1}{\Gamma(\alpha+1)} + \left(1+\frac{\lambda}{\alpha+1}\right)\|\mu\|_\infty\right)\\
\leq\ & L\|u-v\|\left[\frac{1}{\Gamma(\alpha+1)} + \left(1+\frac{\lambda}{\alpha+1}\right)\|\mu\|_\infty + \frac{1}{\Gamma(\alpha+2)}\|K\|_\infty + \left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu\|_\infty\|K\|_\infty\right].
\end{aligned} \tag{5.14}$$

Similarly, we can reach to

$$\begin{aligned}
&|(Tu)'(t)-(Tv)'(t)| \leq L\|u-v\|\left[\frac{1}{\Gamma(\alpha)} + \left(1+\frac{\lambda}{\alpha+1}\right)\|\mu'\|_\infty +\right.\\
&\left.\frac{1}{\Gamma(\alpha+1)}\|K\|_\infty + \left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu'\|_\infty\|K\|_\infty\right].
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
&|(Tu)''(t)-(Tv)''(t)| \leq L\|u-v\|\left[\frac{1}{\Gamma(\alpha-1)} + \left(1+\frac{\lambda}{\alpha+1}\right)\|\mu''\|_\infty +\right.\\
&\left.\frac{1}{\Gamma(\alpha)}\|K\|_\infty + \left(\frac{1}{\alpha+1}+\frac{\lambda}{(\alpha+1)(\alpha+2)}\right)\|\mu''\|_\infty\|K\|_\infty\right].
\end{aligned} \tag{5.16}$$

Inequalities (5.14), (5.15) and (5.16), all together, imply

$$\|Tu - Tv\| \leq L\|u-v\|R < \|u-v\|, \tag{5.17}$$

by the assumption of the theorem. Now, using the Banach contraction mapping theorem, the problem (1.1) has a unique solution on interval B_r and the proof is complete. □

6 Illustrative example

Consider the following fractional differential equation

$$\begin{cases} {}_0D_t^{3.5}u + \rho \int_0^t \cos(t-s)\left(\frac{u(s)}{1+u^2(s)} + \sin\left(u'(s)\right) + \cos\left(u''(s)\right) + \cos(s)\right) ds \\ = -\beta\left(\frac{u(t)}{1+\cosh(u(t))} + \arctan\left(u'(t)\right) + \sin^2\left(u''(t)\right) + \exp\left(t^2\right)\right), \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \quad u(1) = 2\int_0^1 u(s)ds. \end{cases} \tag{6.1}$$

As it is seen, $\alpha = 3.5,\ \lambda = 2,\ K(t,s) = \cos(t-s)$

$$f(t, u(t), u'(t), u''(t)) = \rho\left(\frac{u(t)}{1+u^2(t)} + \sin\left(u'(t)\right) + \cos\left(u''(t)\right) + \cos(t)\right),$$

and

$$g(t, u(t), u'(t), u''(t)) = -\beta\left(\frac{u(t)}{1+\cosh\left(u(t)\right)} + \arctan\left(u'(t)\right) + \sin^2\left(u''(t)\right) + \exp\left(t^2\right)\right).$$

It is obviously observed that

$$\begin{aligned}
&|f(t,u(t),u'(t),u''(t)) - f(t,v(t),v'(t),v''(t))| = \\
&\left|\rho\left(\frac{u(t)}{1+u^2(t)} + \sin\left(u'(t)\right) + \cos\left(u''(t)\right) + \cos(t)\right) - \right. \\
&\left.\rho\left(\frac{v(t)}{1+v^2(t)} + \sin\left(v'(t)\right) + \cos\left(v''(t)\right) + \cos(t)\right)\right| = \\
&\left|\rho\left(\frac{u(t)}{1+u^2(t)} - \frac{v(t)}{1+v^2(t)}\right) + \rho\left(\sin(u'(t) - \sin(v'(t)\right) + \rho\left(\cos(u''(t) - \cos(v''(t)\right)\right| \leq \\
&\rho\left(\left|\frac{|u(t)|}{1+u^2(t)} - \frac{|v(t)|}{1+v^2(t)}\right|\right) + \rho\left|\sin(u'(t) - \sin(v'(t)\right| + \rho\left|\cos(u''(t) - \cos(v''(t)\right| \leq \\
&\rho\left|u(t) - v(t)\right| + \rho\left|u'(t) - v'(t)\right| + \rho\left|u''(t) - v''(t)\right| = \\
&\rho\left(\left|u(t) - v(t)\right| + \left|u'(t) - v'(t)\right| + \left|u''(t) - v''(t)\right|\right),
\end{aligned} \tag{6.2}$$

and also

$$\begin{aligned}
&|g(t,u(t),u'(t),u''(t)) - g(t,v(t),v'(t),v''(t))| = \\
&\left|\beta\left(\frac{u(t)}{1+\cosh\left(u(t)\right)} + \arctan\left(u'(t)\right) + \sin^2\left(u''(t)\right) + \exp\left(t^2\right)\right) - \right. \\
&\left.\beta\left(\frac{v(t)}{1+\cosh\left(v(t)\right)} + \arctan\left(v'(t)\right) + \sin^2\left(v''(t)\right) + \exp\left(t^2\right)\right)\right| = \\
&\left|\beta\left(\frac{u(t)}{1+\cosh\left(u(t)\right)} - \frac{v(t)}{1+\cosh\left(v(t)\right)}\right) + \beta\left(\arctan\left(u'(t)\right) - \arctan\left(v'(t)\right)\right)\right. \\
&\left.+\beta\left(\sin^2\left(u''(t)\right) - \sin^2\left(v''(t)\right)\right)\right| \leq \\
&\beta\left(\left|\frac{|u(t)|}{1+\cosh\left(u(t)\right)} - \frac{|v(t)|}{1+\cosh\left(v(t)\right)}\right|\right) + \beta\left|\arctan\left(u'(t)\right) - \arctan\left(v'(t)\right)\right| \\
&+\beta\left|\sin^2\left(u''(t)\right) - \sin^2\left(v''(t)\right)\right| \leq \\
&\beta\left|u(t) - v(t)\right| + \beta\left|u'(t) - v'(t)\right| + \beta\left|u''(t) - v''(t)\right| = \beta\left(\left|u(t) - v(t)\right| + \left|u'(t) - v'(t)\right| + \left|u''(t) - v''(t)\right|\right).
\end{aligned}$$

So, by setting $L_f = \rho$ and $L_g = \beta$ the first and second assumptions of Theorem 5.1 hold. On the other hand,

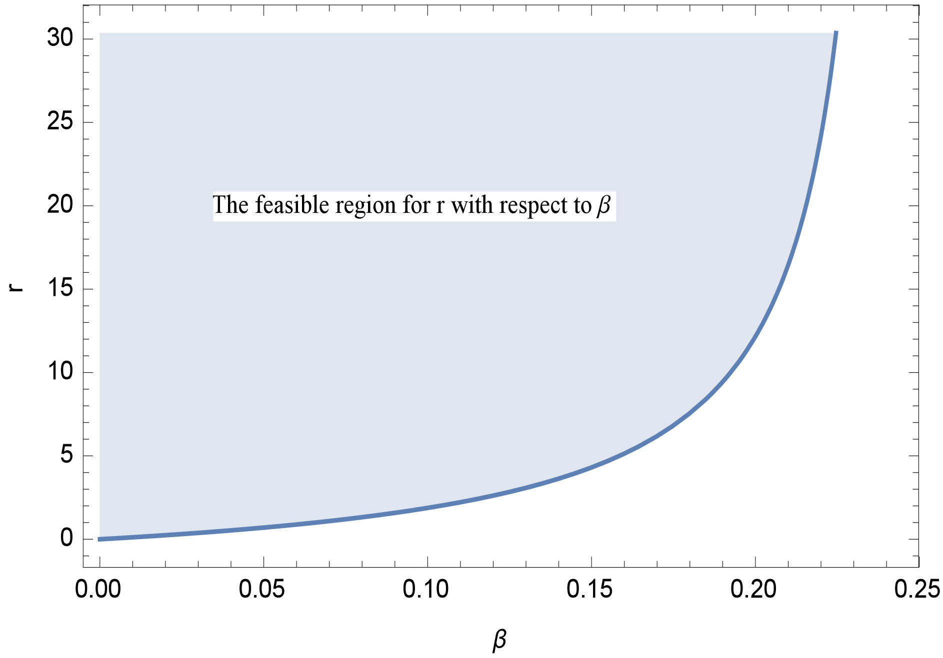

Figure 1: : The feasible region for r with respect to $\beta = \rho$ in the interval $(0, \frac{1}{R})$ for the Example.

routine calculation implies the following results:

$$\|K\|_\infty = 1, \tag{6.3}$$
$$\mu(t) = \frac{16t^{5/2}}{45\sqrt{\pi}}, \tag{6.4}$$
$$\|\mu(t)\| = \|\mu(t)\|_\infty + \|\mu'(t)\|_\infty + \|\mu''(t)\|_\infty = 1.454355370923105, \tag{6.5}$$
$$L = \max\{\rho, \beta\} \tag{6.6}$$
$$N_f = \sup_{t\in[0,1]} |f(t, 0, 0, 0)| = 2\rho, \tag{6.7}$$
$$N_g = \sup_{t\in[0,1]} |g(t, 0, 0, 0)| = e\beta, \tag{6.8}$$
$$N = \max\{2\rho, e\beta\} \tag{6.9}$$
$$R = 4.0865526096446105. \tag{6.10}$$

Therefore, if we choose ρ and β such that $LR = \max\{\rho, \beta\}R < 1$ then the third condition of Theorem 5.1 also holds, so we conclude the existence of the unique solution in the ball $B_r = \{u \in C^2[0, 1] \ : \ \|u\| \leq r\}$, where $r \geq \frac{NR}{1-LR}$. For example, if we set $\rho = \beta = \frac{1}{5}$, so $L = \frac{1}{5}$ and $N = \frac{e}{5}$ then the unique solution belonging to $B_r = \{u \in C^2[0, 1] \ : \ \|u\| \leq 12.16096495225299\}$ is guaranteed. In this example, by letting generally $\rho = \beta$ we have $r \geq \frac{e\beta R}{1-\beta R}$. This issue has been illustrated accurately in Fig. 1.

7 Conclusion

In this paper, we have applied Banach contraction mapping theorem to prove the existence and the uniqueness of the solution to a kind of high-order nonlinear fractional integro-differential equations involving Rieman-Liouville fractional derivative whose boundary condition is of integral type in terms of ending point of the domain. To this aim, we have extracted the unique exact solution in terms of Green's function for the linear fractional differential equation. Moreover, an illustrative example has been given to show legitimacy of the theory. It seems to be a good idea to investigate some similar high-order problems even to derive approximate solution by defining an iteration process to reach fixed point as an exact solution.

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