Int. J. Nonlinear Anal. Appl. 15 (2024) 1, 225–239
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29053.4052

# How many statistical structures and Ricci flat affine connections are there on the tangent bundle?

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(Communicated by Saman Babaie-Kafaki)

**Abstract**

In this paper, we consider the tangent bundle TM of a Riemannian manifold $(M, g)$ with the Sasaki metric $G$ and using the Cauchy-Kowalevski Theorem, we answer the question of how many analytic statistical structures are there on $(\mathrm{TM}, G)$. Also, we study the Ricci tensor of linear affine connections on the tangent bundle TM. In addition, we answer the question of how many Ricci flat affine connections with and without torsion are there on the tangent bundle.

Keywords: Cauchy-Kowalevski Theorem, Sasaki Metrics, Statistical Structures
2020 MSC: 35A10, 35G50, 53B05, 53B20, 53B30

## 1 Introduction

The mathematical scope of information geometry arose in 1945 by C. R. Rao from the idea that using Fisher information, it is possible to define a Riemannian metric in spaces of probability distributions ([14]). This powerful branch of mathematics implements the methods of differential geometry to the extent of probability theory. Information geometry leads us to a geometrical interpretation of probability theory and statistics and enables us to survey the invariant properties of statistical manifolds. It was realized by the works of S. Amari that the differential geometric structure of a statistical manifold can be obtained from divergence functions, giving a Riemannian metric and a pair of affine connections ([3, 4]). Information geometry has many applications in various fields of research. These applications can be found for example in image processing, physics, computer science and machine learning (see for instance [6, 17]).

A statistical manifold is a Riemannian manifold such that each of its points is a probability distribution. Let $\Theta$ be an open subset of $\mathbb{R}^n$. If $S$ is a set of probability density functions on a sample space $\Omega$ with parameter $\theta = (\theta^1, \dots, \theta^n)$ such that

$$S = \left\{ p(x;\theta) \ : \ \int_\Omega p(x;\theta) = 1, \ \ p(x;\theta) > 0, \ \ \theta \in \Theta \subseteq \mathbb{R}^n \right\},$$

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*Received:* November 2022 *Accepted:* March 2023

then $S$ is called a statistical model. The semi-definite Fisher information matrix $g(\theta) = (g_{ij}(\theta))$ is defined on a statistical model $S$ by

$$g_{ij}(\theta) := \int_\Omega \partial_i l_\theta \partial_j l_\theta p(x;\theta)\mathrm{d}x = E_p\left[\partial_i l_\theta \partial_j l_\theta\right],$$

where $l_\theta := \log p(x;\theta)$, $\partial_i = \frac{\partial}{\partial\theta^i}$ and $E_p[f]$ is the expectation of $f(x)$ with respect to $p(x;\theta)$. Equipping $S$ to this metric, $S$ is called an info-manifold or a statistical manifold.

The notion of lifted metrics on the tangent bundle of Riemannian manifolds is widely considered as an interesting field by many mathematicians (see for example [2, 13, 10, 15, 18]). This notion was first introduced by Sasaki in [16] and in recent years his works have generated strong motivation for other mathematicians to study and develop this concept on the tangent bundles of Riemannian manifolds. For example, in [1] the authors introduced the notion of g-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$ as the most general type of lifted metrics on tangent bundles.

The concept of statistical structure is a very important motif in differential geometry, appearing in information geometry and statistic. A statistical structure is a pair $(g, \nabla)$ where $g$ is a metric tensor field and $\nabla$ is a symmetric linear connection such that the cubic tensor field $\mathcal{C} = \nabla g$ is totally symmetric. As an application of such structures, we can mention the theory of equiaffine hypersurfaces in $\mathbb{R}^n$ (see [9], for more details).

Considering an infinite family of geometric objects, this natural question arises that how many such objects live on a Riemannian manifold. Using the Cauchy-Kowalevski Theorem, O. Kowalski and some mathematicians answered such questions ([8, 11, 12]). In fact, the Cauchy-Kowalevski Theorem enables us to answer such questions in case of analytic structures.

In this paper, we consider the tangent bundle of a Riemannian manifold $(M, g)$ with the Sasaki metric and using the Cauchy-Kowalevski Theorem, we count the number of analytic statistical structures on it. In Section 2, we provide some fundamental information on the geometry of tangent bundles with the Sasaki metrics and also, we present the suitable version of Cauchy-Kowalevski Theorem of order one with respect to our consideration. Section 3 contains some definitions and basic information on statistical Riemannian manifolds and statistical structures on the tangent bundle of a Riemannian manifold. In Section 4, we apply the Cauchy-Kowalevski Theorem for counting analytic statistical structures on the tangent bundle TM equipped to the Sasaki metric $G$ and we prove that the set of all analytic statistical structures $(G, \bar{\nabla})$ around the point 0, depends on $\frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n$ arbitrary chosen analytic functions (Christoffel symbols) of $n$ variables, and $\frac{n(n+1)}{2} - 1$ arbitrarily chosen analytic functions $g_{ij}$ of metric $g$ for $(ij) \neq (1, 1)$, of $(n-1)$ variables. In section 5, we compute the Ricci tensor of a linear affine connection on the tangent bundle TM, and then we answer the question of how many Ricci flat affine connections with torsion are there on the tangent bundle. In other words, we prove that the set of all analytic Ricci flat affine connection $\bar{\nabla}$ with torsion on the tangent bundle TM depends on $n^2(8n-4)$ arbitrary chosen analytic functions (Christoffel symbols) of $n$ variables and $4n^2$ analytic functions of $n-1$ variables. Also, we study the problem for the case of torsionless Ricci flat connections, and we show that the family of all analytic linear affine Ricci flat connections without torsion on the tangent bundle TM, depends on $6n^3 - 6n^2 - 2n$ analytic functions of $n$ variables and $3n^2 + n$ analytic functions of $n-1$ variables.

## 2 Preliminaries

In this section, we provide some fundamental information on the geometry of tangent bundles, and then we equip the tangent bundle to the Sasaki metric, which is an important subclass of $g$-natural metric on the tangent bundle. Also, to remain self-contained, we provide the suitable version of Cauchy-Kowalevski Theorem of order one with respect to our consideration.

### 2.1 Cauchy-Kowalevski Theorem

We now present the appropriate case of order one of the Cauchy-Kowalevski Theorem, with the aim of the rest of paper. All coordinate systems supposed to be analytic in this paper.

**Theorem 2.1.** [8] Consider a system of differential equations for unknown functions $U^1\left(x^1,\dots,x^n\right),\dots,U^N\left(x^1,\dots,x^n\right)$ in a neighborhood of $0\in\mathbb{R}^n$ and of the form

$$\frac{\partial U^1}{\partial x^1}=H^1\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),$$
$$\frac{\partial U^2}{\partial x^1}=H^2\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),$$
$$\dots$$
$$\frac{\partial U^N}{\partial x^1}=H^N\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),$$

where $H^i$ , $i=1,\dots,N$, are analytic functions of all variables in a neighborhood of

$$\left(0,\dots,0,\varphi^1(0),\dots,\varphi^N(0),\frac{\partial\varphi^1}{\partial x^2}(0),\dots,\frac{\partial\varphi^1}{\partial x^n}(0),\dots,\frac{\partial\varphi^N}{\partial x^2}(0),\dots,\frac{\partial\varphi^N}{\partial x^n}(0)\right)\in\mathbb{R}^{(N+1)n},$$

for analytic functions $\varphi^1,\dots,\varphi^N$ given in a neighborhood of $0\in\mathbb{R}^{n-1}$.

Then the system has a unique solution $\left(U^1\left(x^1,\dots,x^n\right),\dots,U^N\left(x^1,\dots,x^n\right)\right)$ which is analytic around $0\in\mathbb{R}^n$ and satisfies the initial conditions

$$U^i\left(0,x^2,\dots,x^n\right)=\varphi^i\left(x^2,\dots,x^n\right)\quad\text{for}\quad i=1,\dots,N.$$

The analytic functions $\psi^1,\dots,\psi^n$ defined in a neighborhood of $0\in\mathbb{R}^{n-1}$, appear in the second order Cauchy-Kowalewski Theorem. More precisely, the second order derivatives $\frac{\partial U^1}{\partial x^1\partial x^1},\dots,\frac{\partial U^N}{\partial x^1\partial x^1}$ form the left-hand sides, and on the right-hand sides, we add the first order derivatives $\frac{\partial U^1}{\partial x^1},\dots,\frac{\partial U^N}{\partial x^1}$ and also, the second order derivatives $\frac{\partial U^i}{\partial x^j\partial x^k}$ for $i=1,\dots,N$, $j=1,\dots,n$ and $k=2,\dots,n$ to the set of $H^1,\dots H^N$. As the initial conditions we add the following conditions

$$\frac{\partial U^i}{\partial x^1}\left(0,x^2,\dots,x^n\right)=\psi^i\left(x^2,\dots,x^n\right),$$

for the prescribed functions $\psi^i$, $i=1,\dots,N$. The problem of finding the number of statistical structures has a local nature. Therefore, we look at to these structures in open neighborhoods of $0\in\mathbb{R}^n$. Since such neighborhoods can be implemented by any analytic coordinate system, we quickly choose the canonical one.

### 2.2 Tangent bundle with the Sasaki metric

Let $(M,g)$ be an $n$-dimensional Riemannian manifold, and we denote by $\overset{g}{\nabla}$ its Levi-Civita connection. If $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces concerning $\overset{g}{\nabla}$, then the tangent space $\mathrm{TM}_{(x,y)}$ of the tangent bundle TM at a point $(x,y)$ splits as

$$(\mathrm{TM})_{(x,y)}=\mathcal{H}_{(x,y)}\oplus\mathcal{V}_{(x,y)}.$$

Let $\pi:\mathrm{TM}\to M$ be the natural projection. For a vector $X\in M_x$, the *horizontal lift* of $X$ to $(x,y)\in\mathrm{TM}$, is uniquely determined by the vector $X^h\in\mathcal{H}_{(x,y)}$ such that $\pi_*X^h=X$. Also, the *vertical lift* of a vector $X\in M_x$ is defined by a vector $X^v\in\mathcal{V}_{(x,y)}$ such that $X^v(\mathrm{d}f)=Xf$, for all functions $f$ on $M$. Remark that 1-forms $\mathrm{d}f$ on $M$ are considered as functions on TM (i.e., $(\mathrm{d}f)(x,y)=yf$). Both maps $X\to X^h$ and $X\to X^v$ are isomorphisms between the vector spaces $M_x$ and $\mathcal{H}_{(x,y)}$ and between $M_x$ and $\mathcal{V}_{(x,y)}$ respectively. Moreover, we can write each tangent vector $Z\in(\mathrm{TM})_{(x,y)}$ in the form $Z=X^h+Y^v$, where $X,Y\in M_x$ are uniquely determined vectors.

We now consider the local chart $(x,\mathcal{U})$ on $M$ by $x=(x^1,\dots,x^n)$ where $x^i$'s are smooth functions on $M$ for $i\in\{1,\dots,n\}$. In order to make a local chart on TM, we denote $x^i\circ\pi$ briefly by $x^i$ and we define

$$y^i(X)=X(x^i)=\mathrm{d}x^i(X),\qquad X\in\chi(M),\qquad i\in\{1,\dots,n\}.$$

Considering the local chart $(x^1,\dots,x^n,y^1,\dots,y^n):\pi^{-1}(\mathcal{U})\longrightarrow\mathbb{R}^{2n}$ on TM, it can be checked that if we put $X=X^i\frac{\partial}{\partial x^i}$, then

$$X^h=X^i\frac{\partial}{\partial x^i}-X^jy^k\,\overset{g}{\Gamma}{}^i{}_{jk}\,\frac{\partial}{\partial y^i},\qquad X^v=X^i\frac{\partial}{\partial y^i},$$

where $\overset{g}{\Gamma}{}^i{}_{jk}$'s denote the Christoffel symbols of the Levi-Civita connection $\overset{g}{\nabla}$. The Lie bracket operation of vector fields on the tangent bundle TM is given by (see [7])

$$(i)\ [X^v, Y^v] = 0, \quad (ii)\ [X^h, Y^v] = \left(\overset{g}{\nabla}_X Y\right)^v, \quad (iii)\ [X^h, Y^h] = [X, Y]^h - \left(\overset{g}{\mathrm{R}}(X, Y)y\right)^v,$$

for all vector fields $X$ and $Y$ on $M$ at any point $(x, y)$ in TM, where $\overset{g}{\mathrm{R}}$ is the Riemann curvature tensor of $g$ defined by

$$\overset{g}{\mathrm{R}}(X, Y) = \left[\overset{g}{\nabla}_X, \overset{g}{\nabla}_Y\right] - \overset{g}{\nabla}_{[X,Y]}.$$

We now see how to define the Sasaki metric $G$ on the tangent bundle TM of $(M, g)$. Let $(M, g)$ be a Riemannian manifold. As a natural lift of the Riemannian metric $g$, the Sasaki metric $G$ on the tangent bundle TM is given by

$$\begin{cases} G_{(x,y)}(X^h, Y^h) = g_x(X, Y), \\ G_{(x,y)}(X^h, Y^v) = 0, \\ G_{(x,y)}(X^v, Y^v) = g_x(X, Y), \end{cases} \tag{2.1}$$

for all vector fields $X$ and $Y$ on $M$ at any point $(x, y)$ in TM. Considering the natural basis $\left\{\frac{\partial}{\partial x^i}|_{(x,y)}, \frac{\partial}{\partial y^i}|_{(x,y)}\right\}_{i=1}^n$ of $(\mathrm{TM})_{(x,y)}$, it can be verified that $\mathcal{H}_{(x,y)}$ could be spanned by the set $\left\{\frac{\delta}{\delta x^i}|_{(x,y)}\right\}_{i=1}^n$, where

$$\frac{\delta}{\delta x^i}|_{(x,y)} \ = \ \frac{\partial}{\partial x^i}|_{(x,y)} - \ y^k \ \overset{g}{\Gamma}{}^j{}_{ki}(x) \ \frac{\partial}{\partial y^j}|_{(x,y)}.$$

If we put

$$\delta y^i = \mathrm{d}y^i + \overset{g}{\Gamma}{}^j{}_{ki}(x)\ \mathrm{d}x^j,$$

then the set $\left\{\mathrm{d}x^i, \delta y^i\right\}_{i=1}^n$ is the dual basis for $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}_{i=1}^n$.

**Remark 2.2.** From now on, we use $\partial_i$, $\partial_{\bar{i}}$ and $\delta_i$ instead of $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^i}$ and $\frac{\delta}{\delta x^i}$ respectively, to simplify notations.

Taking into account Remark 2.2, we can express (2.1) by

$$\begin{cases} G_{(x,y)}(\delta_i, \delta_i) = g_{ij}|_x, \\ G_{(x,y)}(\delta_i, \partial_{\bar{j}}) = 0, \\ G_{(x,y)}(\partial_{\bar{i}}, \partial_{\bar{j}}) = g_{ij}|_x. \end{cases} \tag{2.2}$$

Therefore, the Sasaki metric $G$ is presented by

$$G(x, y) = g_{ij}(x, y)\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}(x, y)\delta y^i \otimes \delta y^j.$$

Also, in view of Remark 2.2, the Lie bracket operation of the basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}_{i=1}^n$ on the tangent bundle TM is given by

$$\begin{cases} \left[\delta_i, \partial_{\bar{j}}\right] = \overset{g}{\Gamma}{}^k{}_{ji}\partial_{\bar{k}}, \\ \left[\delta_i, \delta_j\right] = y^r \overset{g}{\mathrm{R}}{}^k{}_{ijr}\partial_{\bar{k}}, \\ \left[\partial_{\bar{i}}, \partial_{\bar{j}}\right] = 0, \end{cases} \tag{2.3}$$

where $\overset{g}{\Gamma}{}^i{}_{jk}$'s denote the Christoffel symbols of the Levi-Civita connection $\overset{g}{\nabla}$ and $\overset{g}{\mathrm{R}}$ is the Riemann curvature tensor of $g$.

## 3 Statistical structures on the tangent bundle

This section contains some definitions and basic information on statistical Riemannian manifolds and statistical structures on the tangent bundle of a Riemannian manifold. Also, we answer the question of how many affine connections without torsion are there on the tangent bundle.

### 3.1 Statistical manifold

A statistical manifold is a triple $(M, g, \nabla)$ where $g$ is a Riemannian metric on manifold $M$ and $\nabla$ is a symmetric linear connection such that the cubic tensor field $\mathcal{C} = \nabla g$ is totally symmetric, i.e., the following Codazzi equations

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X),$$

for all vector fields $X$, $Y$, and $Z$ on the Riemannian manifold $M$ hold. The cubic tensor field $\mathcal{C}$ in local coordinate has the following expression.

$$\mathcal{C}(\partial_i, \partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) - g(\nabla_{\partial_i}\partial_j, \partial_k) - g(\nabla_{\partial_j}\partial_i, \partial_k),$$

hence, we get

$$\mathcal{C}_{ijk} = \partial_k(g_{ij}) - \Gamma^h_{ik} g_{jh} - \Gamma^h_{jk} g_{ih}, \qquad \mathcal{C}_{ijk} = \mathcal{C}_{jki} = \mathcal{C}_{kij},$$

where $\Gamma^i_{jk}$'s are the Christoffel symbols of $\nabla$. Therefore, every statistical manifold $(M, g, \nabla)$ naturally corresponds to a totally symmetric covariant tensor field $\mathcal{C}$ of degree 3. Conversely, if triple $(M, g, \mathcal{C})$ is a semi-Riemannian manifold with a totally symmetric covariant tensor field $\mathcal{C}$ of degree 3, then the linear connection $\nabla$ defined by $\nabla = \overset{g}{\nabla} - \frac{A}{2}$ is torsionless and also, satisfies $\nabla g = \mathcal{C}$, where the tensor field $A$ is determined by

$$g(A(X)Y, Z) = \mathcal{C}(X, Y, Z).$$

Hence, the triple $(M, g, \nabla)$ becomes a statistical manifold. Therefore, existing a statistical structure $(g, \nabla)$ on $M$ is equivalent to equip a structure $(g, \mathcal{C})$ consisting of a semi-Riemannian metric $g$ and a totally symmetric trilinear $\mathcal{C}$.

### 3.2 Statistical structures on the tangent bundle

Suppose that $\bar{\nabla}$ is a linear connection on the tangent bundle $(\mathrm{TM}, G)$ of a Riemannian manifold $(M, g)$, where $G$ is the Sasakian lift of metric $g$. Concerning $\{\delta_i, \partial_{\bar{i}}\}$, the linear connection $\bar{\nabla}$ has the following expression

$$\begin{aligned} \bar{\nabla}_{\delta_i}\delta_j = \bar{\Gamma}^k_{ij}\ \delta_k + \bar{\Gamma}^{\bar{k}}_{ij}\ \partial_{\bar{k}}, \quad \bar{\nabla}_{\delta_i}\partial_{\bar{j}} = \bar{\Gamma}^k_{i\bar{j}}\ \delta_k + \bar{\Gamma}^{\bar{k}}_{i\bar{j}}\ \partial_{\bar{k}}, \\ \bar{\nabla}_{\partial_{\bar{i}}}\delta_j = \bar{\Gamma}^k_{\bar{i}j}\ \delta_k + \bar{\Gamma}^{\bar{k}}_{\bar{i}j}\ \partial_{\bar{k}}, \quad \bar{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}} = \bar{\Gamma}^k_{\bar{i}\bar{j}}\ \delta_k + \bar{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}}\ \partial_{\bar{k}}, \end{aligned}$$

where $\bar{\Gamma}^C_{AB}$, for $A, B, C \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, are smooth functions on TM. In [5], the first author et al., equipped the tangent bundle of a statistical manifold with the Sasaki metric, and classified all the statistical connections living on it. We now report the following propositions.

**Proposition 3.1 ([5]).** Let $\bar{\nabla}$ be a symmetric linear connection on $(\mathrm{TM}, G)$. Symmetry of $\bar{\nabla}$ has the following locally alternative:

$$\bar{\Gamma}^{\bar{k}}_{i\bar{j}} - \bar{\Gamma}^{\bar{k}}_{\bar{j}i} = \overset{g}{\Gamma}{}^k{}_{ij}, \tag{3.1}$$

$$\bar{\Gamma}^k_{i\bar{j}} = \bar{\Gamma}^k_{\bar{j}i}, \tag{3.2}$$

$$\bar{\Gamma}^{\bar{k}}_{ij} - \bar{\Gamma}^{\bar{k}}_{ji} = -y^r \overset{g}{R}{}^k{}_{ijr}, \tag{3.3}$$

$$\bar{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ji}, \tag{3.4}$$

$$\bar{\Gamma}^k_{\bar{i}\bar{j}} = \bar{\Gamma}^k_{\bar{j}\bar{i}}, \tag{3.5}$$

$$\bar{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = \bar{\Gamma}^{\bar{k}}_{\bar{j}\bar{i}}. \tag{3.6}$$

**Proposition 3.2 ([5]).** Let $(M, g, \nabla)$ be a statistical manifold. Then, $\bar{\nabla}$ is a statistical connection on $(\mathrm{TM}, G)$ if and only if

$$(\bar{\Gamma}^{r}_{ik} - \Gamma^{r}_{ik})g_{rj} = (\bar{\Gamma}^{r}_{jk} - \Gamma^{r}_{jk})g_{ri}, \tag{3.7}$$
$$\bar{\Gamma}^{\bar{r}}_{ij}g_{rk} = \bar{\Gamma}^{r}_{\bar{k}j}g_{ri}, \tag{3.8}$$
$$\bar{\Gamma}^{r}_{i\bar{k}}g_{jr} - y^m \overset{g}{R}_{ijmk} = \bar{\Gamma}^{r}_{j\bar{k}}g_{ri}, \tag{3.9}$$
$$(\bar{\Gamma}^{\bar{r}}_{i\bar{k}} - \overset{g}{\Gamma}{}^{r}{}_{ik})g_{jr} = \bar{\Gamma}^{r}_{\bar{j}\bar{k}}g_{ri}, \tag{3.10}$$
$$\bar{\Gamma}^{\bar{r}}_{\bar{j}i}g_{rk} = \bar{\Gamma}^{\bar{r}}_{\bar{k}i}g_{rj}, \tag{3.11}$$
$$\bar{\Gamma}^{\bar{r}}_{\bar{i}\bar{k}}g_{rj} = \bar{\Gamma}^{\bar{r}}_{\bar{j}\bar{k}}g_{ri}, \tag{3.12}$$

where $\overset{g}{\mathrm{R}}_{ijmk} = \overset{g}{\mathrm{R}}{}^{r}{}_{ijm}g_{rk}$.

We finish this section with the following theorem, which is a corollary of Proposition 3.1.

**Theorem 3.3.** Let $(M, g)$ be an $n$-dimensional manifold. The set of all affine connections without torsion on the tangent bundle TM depends locally on $2n^2(2n + 1)$ arbitrary functions of $n$ variables.

**Proof .** Taking into account Proposition 3.1, each symmetry conditions (3.1) and (3.2) determines $n^3$ functions. Also, each conditions (3.3)-(3.6) determines $\frac{n^2(n+1)}{2}$ Christoffel symbols. Therefore, in view of these symmetry conditions, the connection is given by $2n^3 + 4\frac{n^2(n+1)}{2} = 2n^2(2n + 1)$ functions of $n$ variables. □

## 4 How many are analytic statistical structures on the tangent bundle?

In this section, we apply the Cauchy-Kowalevski Theorem for counting analytic statistical structures on the tangent bundle TM equipped to the Sasakian metric $G$. The metric tensor field $g$ is unknown in the following theorem. However, the Cauchy-Kowalevski Theorem enables us to choose it arbitrarily at the point 0. We present the following lemmas.

**Lemma 4.1.** Let $(M, g)$ be a Riemannian manifold and $\nabla$ be a linear torsionless connection on $M$. The pair $(g, \nabla)$ is a statistical structure if and only if

$$(\nabla g)(\partial_i, \partial_j, \partial_k) = (\nabla g)(\partial_j, \partial_i, \partial_k), \tag{4.1}$$

for every $i, j, k = 1, \dots, n$ with $i < j$ and $i \le k$.

**Proof .** The symmetry of metric $g$ implies the symmetry for the last two arguments of $\nabla g$. Suppose that (4.1) holds for every $i, j, k = 1, \dots, n$ with $i < j$ and $i \le k$. We choose $i, j, k = 1, \dots, n$ such that $i < j$ and $k < i$. So $k < j$ and we have

$$\begin{aligned}(\nabla g)(\partial_i, \partial_j, \partial_k) &= (\nabla g)(\partial_i, \partial_k, \partial_j) = (\nabla g)(\partial_k, \partial_i, \partial_j)\\ &= (\nabla g)(\partial_k, \partial_j, \partial_i) = (\nabla g)(\partial_j, \partial_k, \partial_i) = (\nabla g)(\partial_j, \partial_i, \partial_k).\end{aligned}$$

□

Similar to the Lemma 4.1, we establish the truthfulness of the following.

**Lemma 4.2.** Let $(M, g)$ be a Riemannian manifold and TM be its tangent bundle equipped to the Sasaki metric $G$. Let $\bar{\nabla}$ be a linear torsionless connection on TM. The pair $(G, \bar{\nabla})$ satisfies the Codazzi equation on the set $\{\delta_r\}_{r=1}^{n}$ if and only if $(\nabla G)(\delta_i, \delta_j, \delta_k) = (\nabla G)(\delta_j, \delta_i, \delta_k)$, for all $i, j, k = 1, \dots, n$ with $i < j$ and $i \le k$.

**Remark 4.3.** In the following theorem, according to the Cauchy-Kowalevski Theorem, we suppose that the metric tensor field $g$ has the identity matrix $(g_{ij})$ at point 0. More precisely, in the following theorem each $g_{ij}$ appears by $g_{ij}(0) = \delta_{ij}$, where $\delta$ is notation for Kronecker delta.

**Theorem 4.4.** The set of all analytic statistical structures $(G, \bar{\nabla})$ around the point 0, depends on $\frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n$ arbitrary chosen analytic functions of $n$ variables, such that one of these functions is $g_{11}$ and other $\frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n - 1$ functions are some Christoffel symbols of $\bar{\nabla}$, and $\frac{n(n+1)}{2} - 1$ arbitrarily chosen analytic functions $g_{ij}$ for $(ij) \neq (1,1)$, of $(n-1)$ variables.

**Proof .** Let $\bar{\nabla}$ be a linear connection on $(\mathrm{TM}, G)$. According to Proposition 3.2, $\bar{\nabla}$ is a statistical connection if and only if (3.7)-(3.12) satisfy. Applying the Codazzi equations for the couple of $(g, \nabla)$ yields

$$\partial_i g_{jk} - \partial_j g_{ki} = \Gamma^r_{ik} g_{rj} - \Gamma^r_{jk} g_{ri}. \tag{4.2}$$

Also, implementing the Codazzi equations on the pair of $(G, \bar{\nabla})$, we get

$$\partial_i g_{jk} - \bar{\Gamma}^r_{ij} g_{rk} - \bar{\Gamma}^r_{ik} g_{rj} = \partial_j g_{ki} - \bar{\Gamma}^r_{jk} g_{ri} - \bar{\Gamma}^r_{ji} g_{rk}. \tag{4.3}$$

Taking into account (4.2), (4.3), Lemma 4.1 and Lemma 4.2 the equation (3.7) is equivalent to

$$\partial_i g_{jk} - \bar{\Gamma}^r_{ik} g_{rj} = \partial_j g_{ki} - \bar{\Gamma}^r_{jk} g_{ri}, \tag{4.4}$$

for $i, j, k = 1, \ldots, n$ with $i < j$ and $i \leq k$. We now consider the system of equations (4.4) for the indices $1, j, k$, where $1 \leq k \leq j$. So we have

$$\partial_1 g_{jk} = \partial_j g_{1k} + \bar{\Gamma}^r_{1k} g_{rj} - \bar{\Gamma}^r_{jk} g_{r1}, \tag{4.5}$$

The system (4.5) is our Cauchy-Kowalevski system of equations. There exist $\frac{n(n+1)}{2} - 1$ equations in system (4.5). The part $-1$ is related to $\partial_1 g_{11}$. In fact, the Cauchy-Kowalevski theorem enables us to prescribe all functions $g_{jk}$ at the point 0. We choose these functions such that the metric $(g_{ij})(0)$ has the identity one form. Moreover, $g_{11}$ can be chosen arbitrarily modulo the assumption that $g_{11}(0) = 1$. Considering the system (4.4) for the indices $1, j, k$, where $1 < k < j$, we get

$$\partial_1 g_{jk} - \bar{\Gamma}^r_{ik} g_{rj} = \partial_j g_{k1} - \bar{\Gamma}^r_{jk} g_{ri}. \tag{4.6}$$

Taking into account (3.4), we have $\bar{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ji}$. Now using $g_{jk} = g_{kj}$, (4.5) leads us to the following.

$$\partial_j g_{1k} + \bar{\Gamma}^r_{1k} g_{jr} = \partial_k g_{1j} + \bar{\Gamma}^r_{1j} g_{kr}, \tag{4.7}$$

for $1 < k < j \leq n$. The system (4.7) contains $(n-2) + (n-3) + \ldots + 1 = \frac{(n-1)(n-2)}{2}$ equations. We assign the unique pair $(k, j)$ with $k < j$ to each equation of the system (4.7). So we have a bijective correspondence between the set of equations of the system (4.7) and the set of pairs $(k, j)$ of integers with $1 < k < j \leq n$. Therefore, the system (4.7) can be ordered using the reverse lexical ordering for the set of pairs $(k, j)$. More precisely, $(k_1, j_1) \leq (k, j)$ if and only if $j < j_1$ or $j = j_1$ and $k \leq k_1$. We now consider the system (4.4) for indices $i, j, k$ where all indices are different from 1. First, we suppose that two indices $i$ and $k$ are equal. Hence, we have

$$\partial_i g_{ji} - \bar{\Gamma}^r_{ii} g_{jr} = \partial_j g_{ii} - \bar{\Gamma}^r_{ji} g_{ir}, \tag{4.8}$$

where $i = 2, \ldots, n$ and $j \in \{2, \ldots, n\} \setminus \{i\}$. Therefore, we have $(n-1)(n-2)$ equations here. We assign the unique pair $(j, i)$ of indices to each equation of the system (4.8) and we obtain a bijective correspondence between the set of equations of system (4.8) and the set of pairs $(j, i)$ for $i = 2, \ldots, n$ and $j \in \{2, \ldots, n\} \setminus \{i\}$. Employing reverse lexical ordering for pairs $(i, j)$, we order the equations of system (4.8). We now pay attention to system (4.4) for all remaining indices $i, j, k$ such that $2 \leq i < j \leq n$ and $k \in \{2, \ldots, n\} \setminus \{i, j\}$ and $i \leq k$. So the system (4.4) gives

$$\sum_{i=2}^{n-2} (n-i)(n-i-1) = \sum_{r=1}^{n-2} r(r-1) = \frac{n^3 - 6n^2 + 11n - 6}{3}$$

equations

$$\partial_i g_{jk} - \bar{\Gamma}^r_{ik} g_{jl} = \partial_j g_{ik} - \bar{\Gamma}^r_{jk} g_{il}, \tag{4.9}$$

for $2 \leq i < j \leq n$ and $k \in \{2, \ldots, n\} \setminus \{i, j\}$ and $i \leq k$. We assign the unique triple $(i, j, k)$ of indices to each equation of system (4.9). We now have a bijective correspondence between the set of equations of system (4.9) and the set of triple $(i, j, k)$ of integers for $2 \leq i < j \leq n$ and $k \in \{2, \ldots, n\} \setminus \{i, j\}$ and $i \leq k$. Using reverse lexical ordering for triple $(i, j, k)$, we can order the equations of system (4.9). We now consider the ordered system of equations with unknown Christoffel symbols including the ordered systems (4.7), (4.8) and (4.9) in the sequence (4.7), (4.8), (4.9). We denote this system by $\Lambda$. In the system $\Lambda$, our approach is to begin with the first equation and continue to the last one to determine one Christoffel symbol and replace it into the all next equations of the system $\Lambda$ as well as into the Cauchy-Kowalevski system. It is noticeable that at each step, equations in the Cauchy-Kowalevski system will change. We determine $\bar{\Gamma}^j_{1k}$ from the subsystem (4.7) of ordered system $\Lambda$. Also, using the subsystem (4.8) of $\Lambda$, we determine Christoffel symbols $\bar{\Gamma}^j_{ii}$ and the subsystem (4.9) gives $\bar{\Gamma}^j_{ik}$. Notice that our system of differential equations at each step, remains a Cauchy-Kowalevski system. Moreover, in some neighborhoods of 0, the coefficient in front of the symbol determined in a sequential step is non-zero and hence, we can determine this symbol from the equation. With the purpose of determining Christoffel symbols, we suppose that the metric $g$ has the identity one matrix at the point 0. Hence, the ordered system $\Lambda$ at the point 0 is of the following form.

$$\begin{cases} \partial_j g_{1k} + \bar{\Gamma}^j_{1k} = \partial_k g_{1j} + \bar{\Gamma}^k_{1j}, & \text{for } \ 1 < k < j \leq n, \\ \partial_i g_{ji} - \bar{\Gamma}^j_{ii} = \partial_j g_{ii} - \bar{\Gamma}^i_{ji}, & \text{for } \ i = 2, \ldots, n; \ \ j \in \{2, \ldots, n\} \setminus \{i\}, \\ \partial_i g_{jk} - \bar{\Gamma}^j_{ik} = \partial_j g_{ik} - \bar{\Gamma}^i_{jk}, & \text{for } \ 2 \leq i < j \leq n; \ k \neq j; \ \ i \leq k, \end{cases} \tag{4.10}$$

In the system (4.10), each Christoffel symbol that we are going to determine it, appears only once. The coefficient in front of a Christoffel symbol that we wish to determine it at a step is non-zero in some neighborhoods of 0. Therefore, it is possible to determine this Christoffel symbol in a neighborhood of 0.

Now by solving the Cauchy-Kowalevski system, and then going back to algebraic system and starting from the last to first equation, we completely determine the set of desired Christoffel symbols. To summarize, we now present a brief statement of the main points of this proof. We have the system of algebraic equations o unknowns Christoffel symbols, including the system (4.7) for $1 < k < j \leq n$, the system (4.8) for $i = 2, \ldots, n; \ j \in \{2, \ldots, n\} \setminus \{i\}$, and the system (4.9) for $2 \leq i < j \leq n; \ k \in \{2, \ldots, n\} \setminus \{i, j\}; \ i \leq k$. At the point 0, the system is of form (4.10). The coefficients of the system have a matrix of maximal rank at 0 and so it is around 0. Therefore, around the point 0, the system has an analytic solution depending on

$$\frac{n^2(n+1)}{2} - \frac{(n-1)(n-2)}{2} - (n-1)(n-2) - \frac{n^3 - 6n^2 + 11n - 6}{3} = \frac{n^3 + 6n^2 + 5n - 6}{6}$$

arbitrarily chosen analytic arguments. We now consider (3.8)-(3.12), to determine all remaining Christoffel symbols of the statistical structure $(G, \bar{\nabla})$ on the tangent bundle TM. We have the following system of equations.

$$\begin{cases} \bar{\Gamma}^{\bar{r}}_{ij} g_{rk} = \bar{\Gamma}^{r}_{\bar{k}j} g_{ri}, \\ \bar{\Gamma}^{r}_{i\bar{k}} g_{jr} - y^m \overset{g}{R}_{ijmk} = \bar{\Gamma}^{r}_{j\bar{k}} g_{ri}, \\ (\bar{\Gamma}^{\bar{r}}_{i\bar{k}} - \overset{g}{\Gamma}{}^r{}_{ik}) g_{jr} = \bar{\Gamma}^{r}_{\bar{j}\bar{k}} g_{ri}, \\ \bar{\Gamma}^{\bar{r}}_{\bar{j}i} g_{rk} = \bar{\Gamma}^{\bar{r}}_{\bar{k}i} g_{rj}, \\ \bar{\Gamma}^{\bar{r}}_{\bar{i}\bar{k}} g_{rj} = \bar{\Gamma}^{\bar{r}}_{\bar{j}\bar{k}} g_{ri}. \end{cases} \tag{4.11}$$

Taking into account Remark 4.3, the system of equations (4.11) at the point 0 is of following form.

$$\bar{\Gamma}^{\bar{k}}_{ij} = \bar{\Gamma}^{i}_{\bar{k}j}, \tag{4.12}$$

$$\bar{\Gamma}^{j}_{i\bar{k}} - y^m \overset{g}{R}_{ijmk} = \bar{\Gamma}^{i}_{j\bar{k}}, \tag{4.13}$$

$$\bar{\Gamma}^{\bar{j}}_{i\bar{k}} - \overset{g}{\Gamma}{}^j{}_{ik} = \bar{\Gamma}^{i}_{\bar{j}\bar{k}}, \tag{4.14}$$

$$\bar{\Gamma}^{\bar{k}}_{\bar{j}i} = \bar{\Gamma}^{\bar{j}}_{\bar{k}i}, \tag{4.15}$$

$$\bar{\Gamma}^{\bar{j}}_{\bar{i}\bar{k}} = \bar{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}}. \tag{4.16}$$

Because of the symmetry of (4.12), this equation determines $n^3$ Christoffel symbols. Similarly, (4.14) determines $n^3$ functions (Christoffel symbols). Moreover, each equation (4.13), (4.15) and (4.16) specifies $\frac{n^2(n+1)}{2}$ Christoffel symbols. Hence, the system of equations consisting of (4.12)-(4.16) completely determine

$$2n^3 + \frac{3n^2(n+1)}{2} = \frac{n^2(7n+3)}{2}$$

Christoffel symbols. Therefore, around the point 0, the system of equations consisting of (4.10), (4.12)-(4.16) has an analytic solution depending on

$$\frac{n^2(7n+3)}{2} + \frac{n^3+6n^2+5n-6}{6} = \frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n - 1 \tag{4.17}$$

arbitrarily chosen analytic arguments. In conclusion, the set of all analytic statistical structures $(G, \bar\nabla)$ around the point 0, depends on $\frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n$ arbitrary chosen analytic functions of $n$ variables, such that one of these functions is $g_{11}$ and other $\frac{11}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n - 1$ functions are some Christoffel symbols of $\bar\nabla$, and $\frac{n(n+1)}{2} - 1$ arbitrarily chosen analytic functions $g_{ij}$ for $(ij) \neq (1,1)$, of $(n-1)$ variables. □

## 5 How many are linear Ricci flat affine connections on the tangent bundle?

In this section, we answer the question of how many Ricci flat affine connections with torsion are there on the tangent bundle. Also, we study the problem for the case of torsionless Ricci flat connections.

### 5.1 Ricci flat connections on the tangent bundle

Let $\bar\nabla$ be a linear connection on the tangent bundle TM of an $n$-dimensional Riemannian manifold $(M, g)$. Denoting by $\overline{\mathrm{Ric}}$ the Ricci tensor corresponding to the linear connection $\bar\nabla$, we investigate the condition $\overline{\mathrm{Ric}} = 0$ on the tangent bundle TM. Let $\bar{\mathrm{R}}$ be the curvature tensor field corresponding the linear connection $\bar\nabla$. Considering the standard basis $\left\{\delta_i|_{(x,y)}, \partial_{\bar i}|_{(x,y)}\right\}_{i=1}^n$ of $(\mathrm{TM})_{(x,y)}$, we present the following proposition.

**Proposition 5.1.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\bar\nabla$ be a linear connection on TM. The curvature tensor field $\bar{\mathrm{R}}$ corresponding $\bar\nabla$ on the tangent bundle TM is given by

$$\begin{aligned}\bar{\mathrm{R}}(\delta_i,\delta_j)\delta_k ={}& \bar\Gamma^r_{jk}\bar\Gamma^s_{ir}\delta_s + \bar\Gamma^r_{jk}\bar\Gamma^{\bar s}_{ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{jk}\bar\Gamma^s_{i\bar r}\delta_s + \bar\Gamma^{\bar r}_{jk}\bar\Gamma^{\bar s}_{i\bar r}\partial_{\bar s} - \bar\Gamma^r_{ik}\bar\Gamma^s_{jr}\delta_s - \bar\Gamma^r_{ik}\bar\Gamma^{\bar s}_{jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{ik}\bar\Gamma^s_{j\bar r}\delta_s - \bar\Gamma^{\bar r}_{ik}\bar\Gamma^{\bar s}_{j\bar r}\partial_{\bar s}\\ &+ \delta_i(\bar\Gamma^r_{jk})\delta_r + \delta_i(\bar\Gamma^{\bar r}_{jk})\partial_{\bar r} - \delta_j(\bar\Gamma^r_{ik})\delta_r - \delta_j(\bar\Gamma^{\bar r}_{ik})\partial_{\bar r} + y^p \overset{g}{\mathrm{R}}{}^q{}_{ijp}(\bar\Gamma^s_{\bar q k}\delta_s + \bar\Gamma^{\bar s}_{\bar q k}\partial_{\bar s}),\end{aligned}$$

$$\begin{aligned}\bar{\mathrm{R}}(\delta_i,\delta_j)\partial_{\bar k} ={}& \bar\Gamma^r_{j\bar k}\bar\Gamma^s_{ir}\delta_s + \bar\Gamma^r_{j\bar k}\bar\Gamma^{\bar s}_{ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{j\bar k}\bar\Gamma^s_{i\bar r}\delta_s + \bar\Gamma^{\bar r}_{j\bar k}\bar\Gamma^{\bar s}_{i\bar r}\partial_{\bar s} - \bar\Gamma^r_{i\bar k}\bar\Gamma^s_{jr}\delta_s - \bar\Gamma^r_{i\bar k}\bar\Gamma^{\bar s}_{jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{i\bar k}\bar\Gamma^s_{j\bar r}\delta_s - \bar\Gamma^{\bar r}_{i\bar k}\bar\Gamma^{\bar s}_{j\bar r}\partial_{\bar s}\\ &+ \delta_i(\bar\Gamma^r_{j\bar k})\delta_r + \delta_i(\bar\Gamma^{\bar r}_{j\bar k})\partial_{\bar r} - \delta_j(\bar\Gamma^r_{i\bar k})\delta_r - \delta_j(\bar\Gamma^{\bar r}_{i\bar k})\partial_{\bar r} + y^p \overset{g}{\mathrm{R}}{}^q{}_{ijp}(\bar\Gamma^s_{\bar q\bar k}\delta_s + \bar\Gamma^{\bar s}_{\bar q\bar k}\partial_{\bar s}),\end{aligned}$$

$$\begin{aligned}\bar{\mathrm{R}}(\delta_i,\partial_{\bar j})\delta_k ={}& \bar\Gamma^r_{\bar jk}\bar\Gamma^s_{ir}\delta_s + \bar\Gamma^r_{\bar jk}\bar\Gamma^{\bar s}_{ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{\bar jk}\bar\Gamma^s_{i\bar r}\delta_s + \bar\Gamma^{\bar r}_{\bar jk}\bar\Gamma^{\bar s}_{i\bar r}\partial_{\bar s} - \bar\Gamma^r_{ik}\bar\Gamma^s_{\bar jr}\delta_s - \bar\Gamma^r_{ik}\bar\Gamma^{\bar s}_{\bar jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{ik}\bar\Gamma^s_{\bar j\bar r}\delta_s - \bar\Gamma^{\bar r}_{ik}\bar\Gamma^{\bar s}_{\bar j\bar r}\partial_{\bar s}\\ &+ \delta_i(\bar\Gamma^r_{\bar jk})\delta_r + \delta_i(\bar\Gamma^{\bar r}_{\bar jk})\partial_{\bar r} - \partial_{\bar j}(\bar\Gamma^r_{ik})\delta_r - \partial_{\bar j}(\bar\Gamma^{\bar r}_{ik})\partial_{\bar r} - \overset{g}{\Gamma}{}^q{}_{ji}(\bar\Gamma^s_{\bar q k}\delta_s + \bar\Gamma^{\bar s}_{\bar q k}\partial_{\bar s}),\end{aligned}$$

$$\begin{aligned}\bar{\mathrm{R}}(\delta_i,\partial_{\bar j})\partial_{\bar k} ={}& \bar\Gamma^r_{\bar j\bar k}\bar\Gamma^s_{ir}\delta_s + \bar\Gamma^r_{\bar j\bar k}\bar\Gamma^{\bar s}_{ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{\bar j\bar k}\bar\Gamma^s_{i\bar r}\delta_s + \bar\Gamma^{\bar r}_{\bar j\bar k}\bar\Gamma^{\bar s}_{i\bar r}\partial_{\bar s} - \bar\Gamma^r_{i\bar k}\bar\Gamma^s_{\bar jr}\delta_s - \bar\Gamma^r_{i\bar k}\bar\Gamma^{\bar s}_{\bar jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{i\bar k}\bar\Gamma^s_{\bar j\bar r}\delta_s - \bar\Gamma^{\bar r}_{i\bar k}\bar\Gamma^{\bar s}_{\bar j\bar r}\partial_{\bar s}\\ &+ \delta_i(\bar\Gamma^r_{\bar j\bar k})\delta_r + \delta_i(\bar\Gamma^{\bar r}_{\bar j\bar k})\partial_{\bar r} - \partial_{\bar j}(\bar\Gamma^r_{i\bar k})\delta_r - \partial_{\bar j}(\bar\Gamma^{\bar r}_{i\bar k})\partial_{\bar r} - \overset{g}{\Gamma}{}^q{}_{ji}(\bar\Gamma^s_{\bar q\bar k}\delta_s + \bar\Gamma^{\bar s}_{\bar q\bar k}\partial_{\bar s}),\end{aligned}$$

$$\begin{aligned}\bar{\mathrm{R}}(\partial_{\bar i},\partial_{\bar j})\delta_k ={}& \bar\Gamma^r_{\bar jk}\bar\Gamma^s_{\bar ir}\delta_s + \bar\Gamma^r_{\bar jk}\bar\Gamma^{\bar s}_{\bar ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{\bar jk}\bar\Gamma^s_{\bar i\bar r}\delta_s + \bar\Gamma^{\bar r}_{\bar jk}\bar\Gamma^{\bar s}_{\bar i\bar r}\partial_{\bar s} - \bar\Gamma^r_{\bar ik}\bar\Gamma^s_{\bar jr}\delta_s - \bar\Gamma^r_{\bar ik}\bar\Gamma^{\bar s}_{\bar jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{\bar ik}\bar\Gamma^s_{\bar j\bar r}\delta_s - \bar\Gamma^{\bar r}_{\bar ik}\bar\Gamma^{\bar s}_{\bar j\bar r}\partial_{\bar s}\\ &+ \partial_{\bar i}(\bar\Gamma^r_{\bar jk})\delta_r + \partial_{\bar i}(\bar\Gamma^{\bar r}_{\bar jk})\partial_{\bar r} - \partial_{\bar j}(\bar\Gamma^r_{\bar ik})\delta_r - \partial_{\bar j}(\bar\Gamma^{\bar r}_{\bar ik})\partial_{\bar r},\end{aligned}$$

$$\begin{aligned}\bar{\mathrm{R}}(\partial_{\bar i},\partial_{\bar j})\partial_{\bar k} ={}& \bar\Gamma^r_{\bar j\bar k}\bar\Gamma^s_{\bar ir}\delta_s + \bar\Gamma^r_{\bar j\bar k}\bar\Gamma^{\bar s}_{\bar ir}\partial_{\bar s} + \bar\Gamma^{\bar r}_{\bar j\bar k}\bar\Gamma^s_{\bar i\bar r}\delta_s + \bar\Gamma^{\bar r}_{\bar j\bar k}\bar\Gamma^{\bar s}_{\bar i\bar r}\partial_{\bar s} - \bar\Gamma^r_{\bar i\bar k}\bar\Gamma^s_{\bar jr}\delta_s - \bar\Gamma^r_{\bar i\bar k}\bar\Gamma^{\bar s}_{\bar jr}\partial_{\bar s} - \bar\Gamma^{\bar r}_{\bar i\bar k}\bar\Gamma^s_{\bar j\bar r}\delta_s - \bar\Gamma^{\bar r}_{\bar i\bar k}\bar\Gamma^{\bar s}_{\bar j\bar r}\partial_{\bar s}\\ &+ \partial_{\bar i}(\bar\Gamma^r_{\bar j\bar k})\delta_r + \partial_{\bar i}(\bar\Gamma^{\bar r}_{\bar j\bar k})\partial_{\bar r} - \partial_{\bar j}(\bar\Gamma^r_{\bar i\bar k})\delta_r - \partial_{\bar j}(\bar\Gamma^{\bar r}_{\bar i\bar k})\partial_{\bar r},\end{aligned}$$

where $\overset{g}{\Gamma}{}^i{}_{jk}$'s denote the Christoffel symbols of the Levi-Civita connection $\overset{g}{\nabla}$ and $\overset{g}{\mathrm{R}}$ is the Riemann curvature tensor of $g$.

**Proof .** Using (2.3) and the definition

$$\bar{\mathrm{R}}(X,Y)Z = \bar\nabla_X \bar\nabla_Y Z - \bar\nabla_Y \bar\nabla_X Z - \bar\nabla_{[X,Y]} Z,$$

the truthfulness of the assertion is verified. □ We now calculate the Ricci curvature tensor $\overline{\mathrm{Ric}}$ corresponding to the linear connection $\bar\nabla$ on the tangent bundle TM. The Ricci tensor $\overline{\mathrm{Ric}}$ is defined to be the trace:

$$\overline{\mathrm{Ric}}(X,Y) = \mathrm{trace}\left(W \mapsto \bar{\mathrm{R}}(W,X)Y\right). \tag{5.1}$$

Using Proposition 5.1 and (5.1), we compute the Ricci curvature tensor $\overline{\mathrm{Ric}}$ as follows.

$$\begin{aligned}\overline{\mathrm{Ric}}(\delta_i,\delta_j) &= \bar\Gamma^{r}_{ij}\bar\Gamma^{k}_{kr} + \bar\Gamma^{\bar r}_{ij}\bar\Gamma^{k}_{k\bar r} + \bar\Gamma^{r}_{ij}\bar\Gamma^{\bar k}_{\bar k r} + \bar\Gamma^{\bar r}_{ij}\bar\Gamma^{\bar k}_{\bar k\bar r} - \bar\Gamma^{r}_{kj}\bar\Gamma^{k}_{ir} - \bar\Gamma^{\bar r}_{kj}\bar\Gamma^{k}_{i\bar r} - \bar\Gamma^{r}_{\bar kj}\bar\Gamma^{\bar k}_{ir} - \bar\Gamma^{\bar r}_{\bar kj}\bar\Gamma^{\bar k}_{i\bar r}\\ &+ \delta_k(\bar\Gamma^{k}_{ij}) - \delta_i(\bar\Gamma^{k}_{kj}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{ij}) - \delta_i(\bar\Gamma^{\bar k}_{\bar kj}) + \overset{g}{\Gamma}{}^{q}{}_{ki}\bar\Gamma^{\bar k}_{\bar qj} + y^r \overset{g}{\mathrm{R}}{}^{s}{}_{kir}\bar\Gamma^{k}_{\bar sj},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\delta_i,\partial_{\bar j}) &= \bar\Gamma^{r}_{i\bar j}\bar\Gamma^{k}_{kr} + \bar\Gamma^{\bar r}_{i\bar j}\bar\Gamma^{k}_{k\bar r} + \bar\Gamma^{r}_{i\bar j}\bar\Gamma^{\bar k}_{\bar k r} + \bar\Gamma^{\bar r}_{i\bar j}\bar\Gamma^{\bar k}_{\bar k\bar r} - \bar\Gamma^{r}_{k\bar j}\bar\Gamma^{k}_{ir} - \bar\Gamma^{\bar r}_{k\bar j}\bar\Gamma^{k}_{i\bar r} - \bar\Gamma^{r}_{\bar k\bar j}\bar\Gamma^{\bar k}_{ir} - \bar\Gamma^{\bar r}_{\bar k\bar j}\bar\Gamma^{\bar k}_{i\bar r}\\ &+ \delta_k(\bar\Gamma^{k}_{i\bar j}) - \delta_i(\bar\Gamma^{k}_{k\bar j}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{i\bar j}) - \delta_i(\bar\Gamma^{\bar k}_{\bar k\bar j}) + \overset{g}{\Gamma}{}^{q}{}_{ki}\bar\Gamma^{\bar k}_{\bar q\bar j} + y^r \overset{g}{\mathrm{R}}{}^{s}{}_{kir}\bar\Gamma^{k}_{\bar s\bar j},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_{\bar i},\delta_j) &= \bar\Gamma^{r}_{\bar ij}\bar\Gamma^{k}_{kr} + \bar\Gamma^{\bar r}_{\bar ij}\bar\Gamma^{k}_{k\bar r} + \bar\Gamma^{r}_{\bar ij}\bar\Gamma^{\bar k}_{\bar k r} + \bar\Gamma^{\bar r}_{\bar ij}\bar\Gamma^{\bar k}_{\bar k\bar r} - \bar\Gamma^{r}_{kj}\bar\Gamma^{k}_{\bar ir} - \bar\Gamma^{\bar r}_{kj}\bar\Gamma^{k}_{\bar i\bar r} - \bar\Gamma^{r}_{\bar kj}\bar\Gamma^{\bar k}_{\bar ir}\\ &- \bar\Gamma^{\bar r}_{\bar kj}\bar\Gamma^{\bar k}_{\bar i\bar r} + \delta_k(\bar\Gamma^{k}_{\bar ij}) - \partial_{\bar i}(\bar\Gamma^{k}_{kj}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{\bar ij}) - \partial_{\bar i}(\bar\Gamma^{\bar k}_{\bar kj}) - \overset{g}{\Gamma}{}^{q}{}_{ik}\bar\Gamma^{k}_{\bar qj},\end{aligned} \tag{5.4}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_{\bar i},\partial_{\bar j}) &= \bar\Gamma^{r}_{\bar i\bar j}\bar\Gamma^{k}_{kr} + \bar\Gamma^{\bar r}_{\bar i\bar j}\bar\Gamma^{k}_{k\bar r} + \bar\Gamma^{r}_{ij}\bar\Gamma^{\bar k}_{\bar k r} + \bar\Gamma^{\bar r}_{ij}\bar\Gamma^{\bar k}_{\bar k\bar r} - \bar\Gamma^{r}_{\bar k\bar j}\bar\Gamma^{k}_{ir} - \bar\Gamma^{\bar r}_{\bar k\bar j}\bar\Gamma^{k}_{i\bar r} - \bar\Gamma^{r}_{kj}\bar\Gamma^{\bar k}_{\bar ir}\\ &- \bar\Gamma^{\bar r}_{kj}\bar\Gamma^{\bar k}_{\bar i\bar r} + \delta_k(\bar\Gamma^{k}_{\bar i\bar j}) - \delta_i(\bar\Gamma^{k}_{\bar k\bar j}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{ij}) - \partial_{\bar i}(\bar\Gamma^{\bar k}_{kj}) - \overset{g}{\Gamma}{}^{q}{}_{ki}\bar\Gamma^{k}_{\bar q\bar j}.\end{aligned} \tag{5.5}$$

We now suppose that $\overline{\mathrm{Ric}}(\delta_i,\delta_j)=0$. Taking into account (5.2), we have

$$\begin{aligned}\delta_k(\bar\Gamma^{k}_{ij}) - \delta_i(\bar\Gamma^{k}_{kj}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{ij}) - \delta_i(\bar\Gamma^{\bar k}_{\bar kj}) &= -\bar\Gamma^{r}_{ij}\bar\Gamma^{k}_{kr} - \bar\Gamma^{\bar r}_{ij}\bar\Gamma^{k}_{k\bar r} - \bar\Gamma^{r}_{ij}\bar\Gamma^{\bar k}_{\bar k r} - \bar\Gamma^{\bar r}_{ij}\bar\Gamma^{\bar k}_{\bar k\bar r}\\ &= \bar\Gamma^{r}_{kj}\bar\Gamma^{k}_{ir} + \bar\Gamma^{\bar r}_{kj}\bar\Gamma^{k}_{i\bar r} + \bar\Gamma^{r}_{\bar kj}\bar\Gamma^{\bar k}_{ir} + \bar\Gamma^{\bar r}_{\bar kj}\bar\Gamma^{\bar k}_{i\bar r} - \overset{g}{\Gamma}{}^{q}{}_{ki}\bar\Gamma^{\bar k}_{\bar qj} - y^r \overset{g}{\mathrm{R}}{}^{s}{}_{kir}\bar\Gamma^{k}_{\bar sj}.\end{aligned} \tag{5.6}$$

Denoting sum of all terms on the right-hand side of (5.6) by $\bar\Lambda_{ij}$, we rewrite this equation as

$$\delta_k(\bar\Gamma^{k}_{ij}) - \delta_i(\bar\Gamma^{k}_{kj}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{ij}) - \delta_i(\bar\Gamma^{\bar k}_{\bar kj}) = \bar\Lambda_{ij}. \tag{5.7}$$

For $\overline{\mathrm{Ric}}(\delta_i,\partial_{\bar j})=0$, using (5.3) we get

$$\begin{aligned}\delta_k(\bar\Gamma^{k}_{i\bar j}) - \delta_i(\bar\Gamma^{k}_{k\bar j}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{i\bar j}) - \delta_i(\bar\Gamma^{\bar k}_{\bar k\bar j}) &= -\bar\Gamma^{r}_{i\bar j}\bar\Gamma^{k}_{kr} - \bar\Gamma^{\bar r}_{i\bar j}\bar\Gamma^{k}_{k\bar r} - \bar\Gamma^{r}_{i\bar j}\bar\Gamma^{\bar k}_{\bar k r} - \bar\Gamma^{\bar r}_{i\bar j}\bar\Gamma^{\bar k}_{\bar k\bar r} + \bar\Gamma^{r}_{k\bar j}\bar\Gamma^{k}_{ir}\\ &+ \bar\Gamma^{\bar r}_{k\bar j}\bar\Gamma^{k}_{i\bar r} + \bar\Gamma^{r}_{\bar k\bar j}\bar\Gamma^{\bar k}_{ir} + \bar\Gamma^{\bar r}_{\bar k\bar j}\bar\Gamma^{\bar k}_{i\bar r} - \overset{g}{\Gamma}{}^{q}{}_{ki}\bar\Gamma^{\bar k}_{\bar q\bar j} - y^r \overset{g}{\mathrm{R}}{}^{s}{}_{kir}\bar\Gamma^{k}_{\bar s\bar j}.\end{aligned} \tag{5.8}$$

We denote the sum of all terms on the right-hand side of (5.8) by $\bar\Lambda_{i\bar j}$ and rewrite (5.8) as

$$\delta_k(\bar\Gamma^{k}_{i\bar j}) - \delta_i(\bar\Gamma^{k}_{k\bar j}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{i\bar j}) - \delta_i(\bar\Gamma^{\bar k}_{\bar k\bar j}) = \bar\Lambda_{i\bar j}. \tag{5.9}$$

Taking into account (5.4), the case $\overline{\mathrm{Ric}}(\partial_{\bar i},\delta_j)=0$ is equivalent to

$$\begin{aligned}\delta_k(\bar\Gamma^{k}_{\bar ij}) - \partial_{\bar i}(\bar\Gamma^{k}_{kj}) + \partial_{\bar k}(\bar\Gamma^{\bar k}_{\bar ij}) - \partial_{\bar i}(\bar\Gamma^{\bar k}_{\bar kj}) &= -\bar\Gamma^{r}_{\bar ij}\bar\Gamma^{k}_{kr} - \bar\Gamma^{\bar r}_{\bar ij}\bar\Gamma^{k}_{k\bar r} - \bar\Gamma^{r}_{\bar ij}\bar\Gamma^{\bar k}_{\bar k r} - \bar\Gamma^{\bar r}_{\bar ij}\bar\Gamma^{\bar k}_{\bar k\bar r} + \bar\Gamma^{r}_{kj}\bar\Gamma^{k}_{\bar ir}\\ &+ \bar\Gamma^{\bar r}_{kj}\bar\Gamma^{k}_{\bar i\bar r} + \bar\Gamma^{r}_{\bar kj}\bar\Gamma^{\bar k}_{\bar ir} + \bar\Gamma^{\bar r}_{\bar kj}\bar\Gamma^{\bar k}_{\bar i\bar r} + \overset{g}{\Gamma}{}^{q}{}_{ik}\bar\Gamma^{k}_{\bar qj}.\end{aligned} \tag{5.10}$$

We denote by $\bar{\Lambda}_{\bar{i}j}$ the right-hand side of (5.10). Rewriting this equation we get

$$\delta_k(\bar{\Gamma}^k_{\bar{i}j}) - \partial_{\bar{i}}(\bar{\Gamma}^k_{kj}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{\bar{i}j}) - \partial_{\bar{i}}(\bar{\Gamma}^{\bar{k}}_{\bar{k}j}) = \bar{\Lambda}_{\bar{i}j}. \tag{5.11}$$

Also, by means of (5.5) for the case $\overline{\mathrm{Ric}}(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$, we obtain

$$\begin{aligned}\delta_k(\bar{\Gamma}^k_{\bar{i}\bar{j}}) - \delta_i(\bar{\Gamma}^k_{k\bar{j}}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{ij}) - \partial_{\bar{i}}(\bar{\Gamma}^{\bar{k}}_{kj}) = &-\bar{\Gamma}^r_{\bar{i}\bar{j}}\bar{\Gamma}^k_{kr} - \bar{\Gamma}^{\bar{r}}_{\bar{i}\bar{j}}\bar{\Gamma}^k_{k\bar{r}} - \bar{\Gamma}^r_{ij}\bar{\Gamma}^{\bar{k}}_{\bar{k}r} - \bar{\Gamma}^{\bar{r}}_{ij}\bar{\Gamma}^{\bar{k}}_{\bar{k}\bar{r}} + \bar{\Gamma}^r_{\bar{k}\bar{j}}\bar{\Gamma}^k_{ir} \\ &+ \bar{\Gamma}^{\bar{r}}_{\bar{k}\bar{j}}\bar{\Gamma}^k_{i\bar{r}} + \bar{\Gamma}^r_{kj}\bar{\Gamma}^{\bar{k}}_{\bar{i}r} + \bar{\Gamma}^{\bar{r}}_{kj}\bar{\Gamma}^{\bar{k}}_{\bar{i}\bar{r}} + \overset{g}{\Gamma}{}^q{}_{ki}\bar{\Gamma}^k_{\bar{q}\bar{j}}.\end{aligned} \tag{5.12}$$

The sum of all terms on the right-hand side of (5.12) is denoted by $\bar{\Lambda}_{\bar{i}\bar{j}}$ and hence, we have

$$\delta_k(\bar{\Gamma}^k_{\bar{i}\bar{j}}) - \delta_i(\bar{\Gamma}^k_{k\bar{j}}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{ij}) - \partial_{\bar{i}}(\bar{\Gamma}^{\bar{k}}_{kj}) = \bar{\Lambda}_{\bar{i}\bar{j}}. \tag{5.13}$$

We now investigate the condition

$$\overline{\mathrm{Ric}}(\bar{X}, \bar{Y}) = 0, \tag{5.14}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary tangent vectors in $(\mathrm{TM})_{(x,y)}$ with the standard basis $\{\delta_i|_{(x,y)}, \partial_{\bar{i}}|_{(x,y)}\}_{i=1}^n$ and we show that the Cauchy-Kowalevski Theorem is applicable to this system of equations. According to (5.7), (5.9), (5.11) and (5.13), the Ricci curvature tensor $\overline{\mathrm{Ric}}$ vanishes if and only if the following system of equations for $i, j = 1, \ldots, n$ holds.

$$\delta_k(\bar{\Gamma}^k_{ij}) - \delta_i(\bar{\Gamma}^k_{kj}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{ij}) - \delta_i(\bar{\Gamma}^{\bar{k}}_{\bar{k}j}) = \bar{\Lambda}_{ij}, \tag{5.15}$$

$$\delta_k(\bar{\Gamma}^k_{i\bar{j}}) - \delta_i(\bar{\Gamma}^k_{k\bar{j}}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{i\bar{j}}) - \delta_i(\bar{\Gamma}^{\bar{k}}_{\bar{k}\bar{j}}) = \bar{\Lambda}_{i\bar{j}}, \tag{5.16}$$

$$\delta_k(\bar{\Gamma}^k_{\bar{i}j}) - \partial_{\bar{i}}(\bar{\Gamma}^k_{kj}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{\bar{i}j}) - \partial_{\bar{i}}(\bar{\Gamma}^{\bar{k}}_{\bar{k}j}) = \bar{\Lambda}_{\bar{i}j}, \tag{5.17}$$

$$\delta_k(\bar{\Gamma}^k_{\bar{i}\bar{j}}) - \delta_i(\bar{\Gamma}^k_{k\bar{j}}) + \partial_{\bar{k}}(\bar{\Gamma}^{\bar{k}}_{ij}) - \partial_{\bar{i}}(\bar{\Gamma}^{\bar{k}}_{kj}) = \bar{\Lambda}_{\bar{i}\bar{j}}. \tag{5.18}$$

Now we consider (5.15) and rewrite this equation into the more suitable form

$$\left[\left(\bar{\Gamma}^1_{ij}\right)_1 + \cdots + \left(\bar{\Gamma}^n_{ij}\right)_n\right] - \left[\left(\bar{\Gamma}^1_{1j}\right)_i + \cdots + \left(\bar{\Gamma}^n_{nj}\right)_i\right] + \partial_{\bar{k}}\left(\bar{\Gamma}^{\bar{k}}_{ij}\right) - \delta_i\left(\bar{\Gamma}^{\bar{k}}_{\bar{k}j}\right) = \bar{\Lambda}_{ij}, \tag{5.19}$$

where the derivative $\delta_i$ is denoted by the bottom index $i$. For $i = 1$ and $j = 1, \ldots, n$, we keep the derivatives $\left(\bar{\Gamma}^n_{nj}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{1j}$, and move it to the right-hand side. For $i > 1$ and $j = 1, \ldots, n$, we keep the derivatives $\left(\bar{\Gamma}^1_{ij}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{ij}$, and move it to the right-hand side. So we obtain the following system

$$\begin{cases} \left(\bar{\Gamma}^n_{nj}\right)_1 = \bar{\Lambda}_{1j} - \bar{\Lambda}'_{1j}, & j = 1, \ldots, n, \\ \left(\bar{\Gamma}^1_{ij}\right)_1 = \bar{\Lambda}_{ij} - \bar{\Lambda}'_{ij}, & i = 2, \ldots, n, \quad j = 1, \ldots, n. \end{cases} \tag{5.20}$$

The system of equations (5.20) is our Cauchy-Kowalevski system. It can be verified that the first derivatives which are on the left-hand sides of the system (5.20), are not present in any terms $\bar{\Lambda}'_{ij}$ in the right-hand sides. Therefore, using the Cauchy-Kowalevski Theorem, we can choose all Christoffel symbols, except those whose derivatives appear in the left-hand sides of the system (5.20), as arbitrary functions and determine the other Christoffel symbols. Hence, the $n^2$ Christoffel symbols appear by solving the system (5.20) by the Cauchy–Kowalevski Theorem.

Here, we consider (5.16) and rewrite this equation into the following form.

$$\left[\left(\bar{\Gamma}^1_{i\bar{j}}\right)_1 + \cdots + \left(\bar{\Gamma}^n_{i\bar{j}}\right)_n\right] - \left[\left(\bar{\Gamma}^1_{1\bar{j}}\right)_i + \cdots + \left(\bar{\Gamma}^n_{n\bar{j}}\right)_i\right] + \partial_{\bar{k}}\left(\bar{\Gamma}^{\bar{k}}_{i\bar{j}}\right) - \delta_i\left(\bar{\Gamma}^{\bar{k}}_{\bar{k}\bar{j}}\right) = \bar{\Lambda}_{i\bar{j}}. \tag{5.21}$$

For $i = 1$ and $j = 1, \ldots, n$, we keep the derivatives $\left(\bar{\Gamma}^n_{n\bar{j}}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{1\bar{j}}$, and move it to the right-hand side. For $i > 1$ and $j = 1, \ldots, n$, we keep the derivatives $\left(\bar{\Gamma}^1_{i\bar{j}}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{i\bar{j}}$, and move it to the right-hand side. So we have the following system.

$$\begin{cases} \left(\bar{\Gamma}^{n}_{n\bar{j}}\right)_1 = \bar{\Lambda}_{1\bar{j}} - \bar{\Lambda}'_{1\bar{j}}, & j = 1, \dots, n, \\ \left(\bar{\Gamma}^{1}_{i\bar{j}}\right)_1 = \bar{\Lambda}_{i\bar{j}} - \bar{\Lambda}'_{i\bar{j}}, & i = 2, \dots, n, \quad j = 1, \dots, n. \end{cases} \tag{5.22}$$

The system of equations (5.22) will be our Cauchy-Kowalevski system. Note that the first derivatives which are on the left-hand sides of the system (5.22), are not present in any terms $\bar{\Lambda}'_{i\bar{j}}$ and $\bar{\Lambda}'_{ij}$ in the right-hand sides. Therefore, using the Cauchy-Kowalevski Theorem, we can choose all Christoffel symbols, except those whose derivatives appear in the left-hand sides of the system (5.22), as arbitrary functions and determine the other Christoffel symbols. Therefore, the $n^2$ Christoffel symbols appear by solving the system (5.22) by the Cauchy–Kowalevski Theorem.

We now consider (5.17) and rewrite it into the following form.

$$\left[\left(\bar{\Gamma}^{1}_{\bar{i}j}\right)_1 + \dots + \left(\bar{\Gamma}^{n}_{\bar{i}j}\right)_n\right] - \partial_{\bar{i}}\left(\bar{\Gamma}^{k}_{kj}\right) + \partial_{\bar{k}}\left(\bar{\Gamma}^{\bar{k}}_{\bar{i}j}\right) - \partial_{\bar{i}}\left(\bar{\Gamma}^{\bar{k}}_{\bar{k}j}\right) = \bar{\Lambda}_{\bar{i}j}. \tag{5.23}$$

For $i, j = 1, \dots, n$, we keep the derivatives $\left(\bar{\Gamma}^{1}_{\bar{i}j}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{\bar{i}j}$, and move it to the right-hand side. Hence, we have the following system.

$$\left(\bar{\Gamma}^{1}_{\bar{i}j}\right)_1 = \bar{\Lambda}_{\bar{i}j} - \bar{\Lambda}'_{\bar{i}j}, \qquad i, j = 1, \dots, n. \tag{5.24}$$

The system of equations (5.24) is our Cauchy-Kowalevski system. It can be checked that the first derivatives which are in the left-hand sides of the system (5.24), are not present in any terms $\bar{\Lambda}'_{\bar{i}j}$ and $\bar{\Lambda}'_{i\bar{j}}$ and $\bar{\Lambda}'_{ij}$ in the right-hand sides. Therefore, using the Cauchy-Kowalevski Theorem, we can choose all Christoffel symbols, except those whose derivatives appear in the left-hand sides of the system (5.24), as arbitrary functions and determine the other Christoffel symbols. Therefore, the $n^2$ Christoffel symbols appear by solving the system (5.24) by means of the Cauchy–Kowalevski Theorem.

Finally, we consider (5.18) and rewrite that into the following form.

$$\left[\left(\bar{\Gamma}^{1}_{\bar{i}\bar{j}}\right)_1 + \dots + \left(\bar{\Gamma}^{n}_{\bar{i}\bar{j}}\right)_n\right] - \left[\left(\bar{\Gamma}^{1}_{\bar{1}\bar{j}}\right)_i + \dots + \left(\bar{\Gamma}^{n}_{\bar{n}\bar{j}}\right)_i\right] + \partial_{\bar{k}}\left(\bar{\Gamma}^{\bar{k}}_{ij}\right) - \partial_{\bar{i}}\left(\bar{\Gamma}^{\bar{k}}_{kj}\right) = \bar{\Lambda}_{\bar{i}\bar{j}}. \tag{5.25}$$

For $i = 1$ and $j = 1, \dots, n$, we keep the derivatives $\left(\bar{\Gamma}^{n}_{\bar{n}\bar{j}}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{\bar{1}\bar{j}}$, and move it to the right-hand side. For $i > 1$ and $j = 1, \dots, n$, we keep the derivatives $\left(\bar{\Gamma}^{1}_{\bar{i}\bar{j}}\right)_1$ in the left-hand side of the corresponding equation, and denote the sum of all remaining terms in the left-hand side of the corresponding equation by $\bar{\Lambda}'_{\bar{i}\bar{j}}$, and move it to the right-hand side. Therefore, we have the following system.

$$\begin{cases} \left(\bar{\Gamma}^{n}_{\bar{n}\bar{j}}\right)_1 = \bar{\Lambda}_{\bar{1}\bar{j}} - \bar{\Lambda}'_{\bar{1}\bar{j}}, & j = 1, \dots, n, \\ \left(\bar{\Gamma}^{1}_{\bar{i}\bar{j}}\right)_1 = \bar{\Lambda}_{\bar{i}\bar{j}} - \bar{\Lambda}'_{\bar{i}\bar{j}}, & i = 2, \dots, n, \quad j = 1, \dots, n. \end{cases} \tag{5.26}$$

The system of equations (5.26) is our Cauchy-Kowalevski system. It is easy to check that the first derivatives which are in the left-hand sides of the system (5.26), are not present in any terms $\bar{\Lambda}'_{\bar{i}\bar{j}}$ and $\bar{\Lambda}'_{\bar{i}j}$ and $\bar{\Lambda}'_{i\bar{j}}$ and $\bar{\Lambda}'_{ij}$ in the right-hand sides. Therefore, using the Cauchy-Kowalevski Theorem, we can choose all Christoffel symbols, except those whose derivatives appear in the left-hand sides of the system (5.26), as arbitrary functions and determine the other Christoffel symbols. Therefore, the $n^2$ Christoffel symbols appear by solving the system (5.26) by means of the Cauchy–Kowalevski Theorem.

**Theorem 5.2.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold. The set of all analytic Ricci flat affine connection $\bar{\nabla}$ with torsion on the tangent bundle TM depends on $8n^3 - 4n^2$ arbitrary chosen analytic functions (Christoffel symbols) of $n$ variables and $4n^2$ analytic functions of $n - 1$ variables.

**Proof .** We have the following system of equations consisting of (5.20), (5.22), (5.24) and (5.26) as follows.

$$\begin{cases} \begin{cases} \left(\bar{\Gamma}^n_{nj}\right)_1 = \bar{\Lambda}_{1j} - \bar{\Lambda}'_{1j}, & j = 1,\dots,n, \\ \left(\bar{\Gamma}^1_{ij}\right)_1 = \bar{\Lambda}_{ij} - \bar{\Lambda}'_{ij}, & i = 2,\dots,n, \quad j = 1,\dots,n; \end{cases} \\ \begin{cases} \left(\bar{\Gamma}^n_{n\bar{j}}\right)_1 = \bar{\Lambda}_{1\bar{j}} - \bar{\Lambda}'_{1\bar{j}}, & j = 1,\dots,n, \\ \left(\bar{\Gamma}^1_{i\bar{j}}\right)_1 = \bar{\Lambda}_{i\bar{j}} - \bar{\Lambda}'_{i\bar{j}}, & i = 2,\dots,n, \quad j = 1,\dots,n; \end{cases} \\ \begin{cases} \left(\bar{\Gamma}^1_{\bar{i}j}\right)_1 = \bar{\Lambda}_{\bar{i}j} - \bar{\Lambda}'_{\bar{i}j}, & i,j = 1,\dots,n; \end{cases} \\ \begin{cases} \left(\bar{\Gamma}^n_{\bar{n}\bar{j}}\right)_1 = \bar{\Lambda}_{\bar{1}\bar{j}} - \bar{\Lambda}'_{\bar{1}\bar{j}}, & j = 1,\dots,n, \\ \left(\bar{\Gamma}^1_{\bar{i}\bar{j}}\right)_1 = \bar{\Lambda}_{\bar{i}\bar{j}} - \bar{\Lambda}'_{\bar{i}\bar{j}}, & i = 2,\dots,n, \quad j = 1,\dots,n. \end{cases} \end{cases} \tag{5.27}$$

Note that the first derivatives which are in the left-hand sides of the system (5.27), are not present in any terms $\bar{\Lambda}'_{\bar{i}\bar{j}}$ and $\bar{\Lambda}'_{\bar{i}j}$ and $\bar{\Lambda}'_{i\bar{j}}$ and $\bar{\Lambda}'_{ij}$ in the right-hand sides. The $4n^2$ Christoffel symbols are determined from the system of equations (5.27). Therefore, we can choose arbitrarily $8n^3 - 4n^2$ functions (Christoffel symbols) of $n$ variables. The $4n^2$ Christoffel symbols of $n-1$ variables appear by solving the systems (5.20), (5.22), (5.24) and (5.26) by the Cauchy–Kowalevski Theorem. □

We now describe all real analytic Ricci flat linear connections without torsion. We have the following theorem.

**Theorem 5.3.** Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ Riemannian manifold. The family of all analytic linear affine Ricci flat connections without torsion on the tangent bundle TM, depends on $6n^3 - 6n^2 - 2n$ analytic functions of $n$ variables and $3n^2 + n$ analytic functions of $n-1$ variables.

**Proof .** It is obvious that this problem is equivalent to finding all solutions of the system consisting of the system (5.27) and the following system of equations obtained from Proposition 3.1

$$\bar{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ji}, \tag{5.28}$$

$$\bar{\Gamma}^k_{i\bar{j}} = \bar{\Gamma}^k_{\bar{j}i}, \tag{5.29}$$

$$\bar{\Gamma}^k_{\bar{i}\bar{j}} = \bar{\Gamma}^k_{\bar{j}\bar{i}}, \tag{5.30}$$

$$\bar{\Gamma}^{\bar{k}}_{i\bar{j}} = \overset{g}{\Gamma}{}^k{}_{ij} + \bar{\Gamma}^{\bar{k}}_{\bar{j}i}, \tag{5.31}$$

$$\bar{\Gamma}^{\bar{k}}_{ij} = -y^r \overset{g}{R}{}^k{}_{ijr} + \bar{\Gamma}^{\bar{k}}_{ji}, \tag{5.32}$$

$$\bar{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = \bar{\Gamma}^{\bar{k}}_{\bar{j}\bar{i}}. \tag{5.33}$$

First, we study the condition (5.28). Using the symmetry condition $\bar{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ji}$ we have

$$\begin{aligned} \bar{\Gamma}^{k+1}_{k\ k+1} &= -\sum_{i=1}^{k-1} \bar{\Gamma}^i_{ki} - \sum_{i=k+2}^{n} \bar{\Gamma}^i_{ki} + \sum_{i=1}^{k-1} \bar{\Gamma}^i_{ik} + \sum_{i=k+1}^{n} \bar{\Gamma}^i_{ik}, \qquad k = 1,\dots,n-1, \\ \bar{\Gamma}^{n-1}_{n\ n-1} &= -\sum_{i=1}^{n-2} \bar{\Gamma}^i_{ni} + \sum_{i=1}^{n-1} \bar{\Gamma}^i_{in}. \end{aligned} \tag{5.34}$$

Since $n \geq 3$, the $n$ Christoffel symbols in the left-hand sides of system (5.34) are not present in the left-hand side of the $4n^2$ equations of system (5.27). Hence, by means of the symmetry condition (5.28), we can determine $n$ Christoffel symbols.

Now we consider the condition (5.29). We get the following relations.

$$\begin{aligned} \bar{\Gamma}^{k+1}_{k\ \overline{k+1}} &= -\sum_{i=1}^{k-1} \bar{\Gamma}^i_{k\bar{i}} - \sum_{i=k+2}^{n} \bar{\Gamma}^i_{k\bar{i}} + \sum_{i=1}^{k-1} \bar{\Gamma}^i_{\bar{i}k} + \sum_{i=k+1}^{n} \bar{\Gamma}^i_{\bar{i}k}, \qquad k = 1,\dots,n-1, \\ \bar{\Gamma}^{n-1}_{n\ \overline{n-1}} &= -\sum_{i=1}^{n-2} \bar{\Gamma}^i_{n\bar{i}} + \sum_{i=1}^{n-1} \bar{\Gamma}^i_{\bar{i}n}. \end{aligned} \tag{5.35}$$

Note that the $n$ Christoffel symbols in the left-hand sides of system (5.35) are not present in the left-hand side of the $4n^2$ equations of system (5.27). Also, using the symmetry condition (5.29), the $n^2 - n$ Christoffel symbols $(\bar{\Gamma}^1_{i\bar{j}})$ of system (5.24) can be obtained from the $n(n-1) = n^2 - n$ Christoffel symbols $(\bar{\Gamma}^1_{\bar{i}j})$ of the second line of system (5.22). Therefore, by means of the symmetry condition (5.29), we can determine $n + (n^2 - n) = n^2$ Christoffel symbols.

We now study the condition (5.30). Using the symmetry condition $\bar{\Gamma}^k_{\bar{i}\bar{j}} = \bar{\Gamma}^k_{\bar{j}\bar{i}}$ we have

$$\begin{aligned} \bar{\Gamma}^{k+1}_{\bar{k}\,\overline{k+1}} &= -\sum_{i=1}^{k-1} \bar{\Gamma}^i_{\bar{k}\bar{i}} - \sum_{i=k+2}^{n} \bar{\Gamma}^i_{\bar{k}\bar{i}} + \sum_{i=1}^{k-1} \bar{\Gamma}^i_{\bar{i}\bar{k}} + \sum_{i=k+1}^{n} \bar{\Gamma}^i_{\bar{i}\bar{k}}, \qquad k = 1, \dots, n-1, \\ \bar{\Gamma}^{n-1}_{\bar{n}\,\overline{n-1}} &= -\sum_{i=1}^{n-2} \bar{\Gamma}^i_{\bar{n}\bar{i}} + \sum_{i=1}^{n-1} \bar{\Gamma}^i_{\bar{i}\bar{n}}. \end{aligned} \tag{5.36}$$

Since $n \geq 3$, the $n$ Christoffel symbols in the left-hand sides of system (5.36) are not present in the left-hand side of the $4n^2$ equations of system (5.27). So, by means of the symmetry condition (5.30), we can determine $n$ Christoffel symbols.

Now we study the conditions (5.31), (5.32) and (5.33). According to the proof of Theorem 3.3, these three conditions determine $n^3 + 2\frac{n^2(n+1)}{2} = 2n^3 + n^2$ Christoffel symbols. Also, we note that the Christoffel symbols in the left-hand sides of equations of (5.31), (5.32) and (5.33) are not present in the left-hand side of the $4n^2$ equations of system (5.27).

We substitute the $n$ equations of system (5.34), the $n$ equations of system (5.35), the $n^2 - n$ equations $(\bar{\Gamma}^1_{\bar{i}j})$, the $n$ equations of system (5.36) and the $2n^3 + n^2$ equations of system (5.31), (5.32) and (5.33) , into the $4n^2$ equations of system (5.27). We obtain the following system of equations.

$$\left\{ \begin{array}{l} \left\{ \begin{array}{ll} \left(\bar{\Gamma}^n_{nj}\right)_1 = \bar{\bar{\Lambda}}_{1j} - \bar{\bar{\Lambda}}'_{1j}, & j = 1, \dots, n, \\ \left(\bar{\Gamma}^1_{ij}\right)_1 = \bar{\bar{\Lambda}}_{ij} - \bar{\bar{\Lambda}}'_{ij}, & i = 2, \dots, n, \quad j = 1, \dots, n; \end{array} \right. \\ \left\{ \begin{array}{ll} \left(\bar{\Gamma}^n_{n\bar{j}}\right)_1 = \bar{\bar{\Lambda}}_{1\bar{j}} - \bar{\bar{\Lambda}}'_{1\bar{j}}, & j = 1, \dots, n, \\ \left(\bar{\Gamma}^1_{i\bar{j}}\right)_1 = \bar{\bar{\Lambda}}_{i\bar{j}} - \bar{\bar{\Lambda}}'_{i\bar{j}}, & i = 2, \dots, n, \quad j = 1, \dots, n; \end{array} \right. \\ \left\{ \left(\bar{\Gamma}^1_{\bar{i}1}\right)_1 = \bar{\bar{\Lambda}}_{\bar{i}1} - \bar{\bar{\Lambda}}'_{\bar{i}1}, \quad i = 1, \dots, n; \right. \\ \left\{ \begin{array}{ll} \left(\bar{\Gamma}^n_{\bar{n}\bar{j}}\right)_1 = \bar{\bar{\Lambda}}_{\bar{1}\bar{j}} - \bar{\bar{\Lambda}}'_{\bar{1}\bar{j}}, & j = 1, \dots, n, \\ \left(\bar{\Gamma}^1_{\bar{i}\bar{j}}\right)_1 = \bar{\bar{\Lambda}}_{\bar{i}\bar{j}} - \bar{\bar{\Lambda}}'_{\bar{i}\bar{j}}, & i = 2, \dots, n, \quad j = 1, \dots, n; \end{array} \right. \end{array} \right. \tag{5.37}$$

where $\bar{\bar{\Lambda}}$ and $\bar{\bar{\Lambda}}'$ are $\bar{\Lambda}$ and $\bar{\Lambda}'$ respectively, after the substitutions. It can be checked that the first derivatives which are in the left-hand sides of the system (5.37) are not present in the right-hand sides. Now we can choose $8n^3 - 4n^2 - n - (n + (n^2 - n)) - n - (2n^3 + n^2) = 6n^3 - 6n^2 - 2n$ Christoffel symbols not present in the left-hand sides of (5.37) and of (5.34), (5.35), (5.36) and of system (5.31), (5.32) and (5.33), as arbitrary analytic functions. Then $3n^2 + n$ analytic functions of $n - 1$ variables appear by solving the system (5.37) using the Cauchy-Kowalevski Theorem. □

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