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Numerical solution of three-dimensional Volterra-Hammerstein integral equations by hybrid of block-pulse and Legendre polynomials

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Abstract

Our work proposes a new numerical method for finding the solution of three-dimensional Volterra-Hammerstein integral equations by using three-dimensional hybrid block-pulse functions and Legendre polynomials. Our integral equation is converted to a system of nonlinear equations. An error bound for the suggested method is established. Eventually, some numerical examples illustrate that our method is feasible and efficient.

Keywords: three-dimensional Volterra-Hammerstein integral equations, hybrid functions, collocation points, numerical Solution
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1 Introduction

An influential tool for modeling and solving engineering and science problems is multidimensional differential and integral equations [2, 13]. We consider three-dimensional hybrid block-pulse functions and Legendre polynomials for solving Volterra-Hammerstein integral equation

$$w(x,y,z) = v(z,y,z) + \int_0^x \int_0^y \int_0^z k(x,y,z,p,q,r)\, \psi(p,q,r,w(p,q,r))\, drdqdp, \tag{1.1}$$

where $w(x,y,z)$ is unknown and defined on $\Lambda = [0,1) \times [0,1) \times [0,1)$, functions v, k, and ψ are specified functions that ψ is defined on $\Lambda \times (-\infty, +\infty)$. Valuable properties of hybrid functions is the main reason that are applied to solving integral equations[5]. For instance, different choices for the number of subintervals and the degree of the polynomials can lead to more accurate results. Volterra-Hammerstein integral equations are used in fields like electromagnetism, communication theory, and potential theory [1, 12].The existence conclusions for the solution of two-dimensional form of (1.1) have been considered in [3].

The authors in [14] have solved system of Volterra-Hammerstein integral equations by the three-dimensional block-pulse functions. Some Numerical techniques of finding approximate solution of Volterra-Hammerstein integral equations have been investigated in[11, 10].

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In [8] nonlinear mixed Volterra–Fredholm integral equations have been solved by three-dimensional block-pulse functions. In [6] Bernstein's approximation has been used to solve three-dimensional integral equations. Nonlinear mixed integral equations have been solved by triangular functions in [7].

2 Existence and uniqueness of solution

We want to find sufficient condition for uniqueness of the solution of the three dimensional volterra-Hammerstein integral equation. Now, we suppose that

$$U(x,y,z,p,q,r,\omega(p,q,r)) = k(x,y,z,p,q,r)\psi(p,q,r,\omega(p,q,r)).$$

Consider integral equation on the complete metric space of real-valued functions $(C(\Theta),d)$ where

$$d(\omega_1,\omega_2) = \sup\{|\omega_1(p,q,r)-\omega_2(p,q,r)|\colon (p,q,r)\in\Theta\},$$

with $\Theta = [0,1]\times[0,1]\times[0,1]$.

Theorem 2.1. Assume that v and U are continuous functions on Θ and $\Theta\times\Theta\times\mathbb{R}$, respectively. Let $B<1$ be a nonnegative constant where

$$|U(x,y,z,p,q,r,\omega_1(p,q,r)) - U(x,y,z,p,q,r,\omega_2(p,q,r))| \leqslant B|\omega_1(p,q,r)-\omega_2(p,q,r)|,$$

Then (1.1) has a unique solution on Θ.

Proof . Take the sequence

$$w_{n+1}(x,y,z) = v(z,y,z) + \int_0^x\int_0^y\int_0^z k(x,y,z,p,q,r)\,\psi(p,q,r,w_n(p,q,r))\,drdqdp,$$

where $n=1,2,\ldots$. We have

$$\begin{aligned}|w_{n+1}(x,y,z)-w_n(x,y,z)| = &\leqslant \int_0^x\int_0^y\int_0^z |U(x,y,z,p,q,r,\omega_{n+1}(p,q,r)) - U(x,y,z,p,q,r,\omega_n(p,q,r))|drdqdp\\ &\leqslant \int_0^1\int_0^1\int_0^1 |\omega_{n+1}(p,q,r)-\omega_n(p,q,r)|drdqdp\\ &\leqslant Bd(\omega_n,\omega_{n-1}).\end{aligned}$$

Thus $d(w_{n+1},w_n)\leqslant Bd(\omega_n,\omega_{n-1})\leqslant B^{n-1}d(w_2,w_1)$. Weierstrass M-Test and $0\leqslant B<1$ implies that the following series

$$\sum_{n=1}^{\infty}(\omega_{n+1}(p.q.r)-\omega_n(p,q,r)),$$

is uniformly and absolutely convergent on Θ. We know that

$$\omega_n(p,q,r) = \omega_1(p,q,r) + \sum_{i=1}^{n-1}(\omega_{i+1}(p.q.r)-\omega_i(p,q,r)),$$

since $(C(\Theta),d)$ is a complete metric space, we have a unique solution $w\in C(\Theta)$ that

$$\lim_{n\to\infty}\omega_n(p,q,r) = \omega(p,q,r),$$

which gives

$$\begin{aligned}\omega(p,q,r) &= \lim_{n\to\infty}\omega_{n+1}(p,q,r)\\ &= \lim_{n\to\infty}[v(z,y,z) + \int_0^x\int_0^y\int_0^z U(x,y,z,p,q,r,w_n(p,q,r))drdqdp]\\ &= v(z,y,z) + \int_0^x\int_0^y\int_0^z U(x,y,z,p,q,r,\lim_{n\to\infty}w_n(p,q,r))drdqdp\\ &= v(z,y,z) + \int_0^x\int_0^y\int_0^z U(x,y,z,p,q,r,w(p,q,r))drdqdp.\end{aligned}$$

Consequently, unique solution w is given by

$$\omega(z,y,z) = v(z,y,z) + \int_0^x \int_0^y \int_0^z U(x,y,z,p,q,r,w(p,q,r))drdqdp.$$

□

3 Properties of three-dimensional hybrid functions

Hybrid of block-pulse and Legendre polynomials on Λ are defined as follows:

$$\phi_{\alpha_1\beta_1\alpha_2\beta_2,\alpha_3\beta_3}(x,y,z) = \begin{cases} L_{\beta_1}(2Nx-2\alpha_1+1)\, L_{\beta_2}(2Ny-2\alpha_2+1)\, L_{\beta_3}(2Nz-2\alpha_3+1) & ,(x,y,z)\in I \\ 0 & ,elsewhere \end{cases},$$

where $\alpha_1,\alpha_2,\alpha_3 = 1,2,\dots,N$, $\beta_1,\beta_2,\beta_3 = 0,1,\dots,M-1$,$M$ and N are positive integers, and the interval I is defined by $I = \left[\frac{\alpha_1-1}{N},\frac{\alpha_1}{N}\right)\times\left[\frac{\alpha_2-1}{N},\frac{\alpha_2}{N}\right)\times\left[\frac{\alpha_3-1}{N},\frac{\alpha_3}{N}\right)$. Here L_{β_1}, L_{β_2}, and L_{β_3} are the Legendre Polynomials defined on $[-1,1]$ and we have

$$L_0(x)=1, \qquad L_1(x)=x, \qquad L_{m+1}(x)=\frac{2m+1}{m+1}xL_m(x)-\frac{m}{m+1}L_{m-1}(x),\ m=1,2,3,\dots.$$

Orthogonality of three-dimensional hybrid of block-pulse and Legendre polynomials can be detected by

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 \phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x,y,z)\,\phi_{\gamma_1\delta_1\gamma_2\delta_2\gamma_3\delta_3}(x,y,z)\,dzdydx \\ &= \begin{cases}\frac{1}{N^3(2\beta_1+1)(2\beta_2+1)(2\beta_3+1)} & ,\alpha_1=\gamma_1,\alpha_2=\gamma_2,\alpha_3=\gamma_3,\beta_1=\delta_1,\beta_2=\delta_2,\beta_3=\delta_3 \\ 0 & ,elsewhere\end{cases}.\end{aligned}$$

The set of all quadratically integrable and measurable functions is denoted by $Y = L^2(\Lambda)$. The norm is defined by

$$\|w\|_2 = \langle\, w,w\,\rangle^{\frac12} = \Big(\int_0^1\int_0^1\int_0^1 |w(x,y,z)|^2 dzdydx\Big)^{\frac12}.$$

Now we consider

$$\begin{aligned}Y_{N,M} =& sapn\big\{\phi_{101010},\phi_{101011},\dots,\phi_{10101(M-1)},\phi_{101020},\phi_{101021},\dots,\phi_{10102(M-1)},\dots \\ &,\phi_{N(M-1)N(M-1)N0},\phi_{N(M-1)N(M-1)N1},\dots,\phi_{N(M-1)N(M-1)N(M-1)}\big\},\end{aligned}$$

$Y_{N,M}$ is finite dimensional and a subspace of the Hilbert space Y, so if $w\in Y$ then there exists $w_{N,M}\in Y_{N,M}$ such that

$$\|w-w_{N,M}\|_2 = \inf_{v\in Y_{N,M}} \|w-v\|_2\,,$$

which yields

$$\begin{aligned}w(x,y,z) \simeq w_{N,M}(x,y,z) &= \sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x,y,z) \\ &= W^T\Phi(x,y,z), \qquad (3.1)\end{aligned}$$

where

$$\begin{aligned}W =& \big[w_{101010},w_{101011},\dots,w_{10101(M-1)},w_{101020},w_{101021},\dots,w_{10102(M-1)},\dots, \\ & w_{N(M-1)N(M-1)N0},w_{N(M-1)N(M-1)N1},\dots,w_{N(M-1)N(M-1)N(M-1)}\big]^T,\end{aligned}$$

$$\begin{aligned}\Phi(x,y,z) =& \big[\phi_{101010}(x,y,z),\dots,\phi_{10101(M-1)}(x,y,z),\phi_{101020}(x,y,z),\dots,\phi_{10102(M-1)}(x,y,z),\dots, \\ & \phi_{N(M-1)N(M-1)N0}(x,y,z),\dots,\phi_{N(M-1)N(M-1)N(M-1)}(x,y,z)\big]^T,\end{aligned}$$

and hybrid coefficients can be uniquely detected by

$$w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = \frac{\langle w, \phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\rangle}{\langle \phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}, \phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\rangle} \ .$$

In the space $L^2(\Lambda \times \Lambda)$, we define the norm

$$\|k\|_2 = \langle\, k, k \,\rangle^{\frac{1}{2}} = \Big(\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 |k(x,y,z,p,q,r)|^2 drdqdpdzdydx\Big)^{\frac{1}{2}}.$$

and we can expand the function k into three-dimensional hybrid functions

$$k(x,y,z,p,q,r) \simeq \Phi^T(x,y,z)K\Phi(p,q,r),$$

where K is an N^3M^3–square matrix whose entries are given by

$$k_{i,j} = \frac{\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 k(x,y,z,p,q,r)\Phi_{(i)}(x,y,z)\Phi_{(j)}(p,q,r)drdqdpdzdydx}{(\int_0^1\int_0^1\int_0^1 |\Phi_{(i)}(x,y,z)|^2 dzdydx)(\int_0^1\int_0^1\int_0^1 |\Phi_{(j)}(p,q,r)|^2 drdqdp)},$$

in which the i-th entry of the vector $\Phi(x,y,z)$ is shown by $\Phi_{(i)}(x,y,z)$.

4 Solving technique

We apply three-dimensional hybrid functions to determine the solution of three-dimensional Volterra -Hammerstein integral equations. We approximate the unknown function $w(x,y,z)$ by

$$w_{N,M}(x,y,z) = \sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x,y,z), \tag{4.1}$$

where the coefficients $w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ should be determined. Substituting (4.1) in (1.1) gives

$$\Big[\sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3,\beta_3}\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3,\beta_3}(x,y,z)\Big]$$
$$-\int_0^x\int_0^y\int_0^z k(x,y,z,p,q,r)\psi(p,q,r,w_{N,M}(p,q,r))drdqdp \simeq v(x,y,z).$$

In the linear case, the above equation can be written as

$$\sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\Big[\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x,y,z)$$
$$-\int_0^x\int_0^y\int_0^z k(x,y,z,p,q,r)\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p,q,r)\Big] \simeq v(x,y,z).$$

By introducing residual function$R_{N,M}(x,y,z)$, we get

$$R_{N,M}(x,y,z) = \Big[\sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x,y,z)\Big]$$
$$-\int_0^x\int_0^y\int_0^z k(x,y,z,p,q,r)\psi(p,q,r,w_{N,M}(p,q,r))drdqdp - v(x,y,z). \tag{4.2}$$

Now we collocate (4.2) in N^3M^3 Newton-Cotes nodes as

$$x_i = \frac{2i-1}{2NM}, y_j = \frac{2j-1}{2NM}, z_l = \frac{2l-1}{2NM},\ \ i,j,l = 1,2,\ldots,NM. \tag{4.3}$$

Thus $R_{N,M}(x_i, y_j, z_l) = 0$ for $i, j, l = 1, 2, \ldots, NM$ which gives

$$\Big[\sum_{\alpha_1=1}^{N}\sum_{\beta_1=0}^{M-1}\sum_{\alpha_2=1}^{N}\sum_{\beta_2=0}^{M-1}\sum_{\alpha_3=1}^{N}\sum_{\beta_3=0}^{M-1} w_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x_i, y_j, z_l)\Big]$$
$$-\int_0^{x_i}\int_0^{y_j}\int_0^{z_l} k(x_i, y_j, z_l, p, q, r)\psi(p, q, r, w_{N,M}(p, q, r))drdqdp$$
$$= v(x_i, y_j, z_l).$$

Obviously in the linear case we have a system of N^3M^3 linear equations in N^3M^3 variables, which can be solved by common methods. In general, we have a nonlinear system of N^3M^3 equations.

5 Error estimation

Here, we attention to the error involved in approximating $w(x, y, z)$ by series expansion of three-dimensional hybrid functions and find a bound for it. For this purpose, we consider $\Lambda = \bigcup_{1\leqslant\alpha_1,\alpha_2,\alpha_3\leqslant N} \Lambda_{\alpha_1\alpha_2\alpha_3}$ where $\Lambda_{\alpha_1\alpha_2\alpha_3} = \big[\frac{\alpha_1-1}{N}, \frac{\alpha_1}{N}\big) \times \big[\frac{\alpha_2-1}{N}, \frac{\alpha_2}{N}\big) \times \big[\frac{\alpha_3-1}{N}, \frac{\alpha_3}{N}\big)$ and

$$w(x, y, z) = \sum_{\alpha_1=1}^{N}\sum_{\alpha_2=1}^{N}\sum_{\alpha_3=1}^{N} w_{\alpha_1\alpha_2\alpha_3}(x, y, z),$$

where $w_{\alpha_1\alpha_2\alpha_3}$ is the restriction of w to $\Lambda_{\alpha_1\alpha_2\alpha_3}$. We also consider

$$\begin{aligned}Z_{\alpha_1\alpha_2\alpha_3} = sapn\{&\phi_{\alpha_10\alpha_20\alpha_30}(x, y, z), \phi_{\alpha_10\alpha_20\alpha_31}(x, y, z), \ldots, \phi_{\alpha_10\alpha_20\alpha_3(M-1)}(x, y, z)\\ &, \phi_{\alpha_10\alpha_21\alpha_30}(x, y, z), \phi_{\alpha_10\alpha_21\alpha_31}(x, y, z), \ldots, \phi_{\alpha_10\alpha_21\alpha_3(M-1)}(x, y, z), \ldots\\ &, \phi_{\alpha_1(M-1)\alpha_2(M-1)\alpha_30}(x, y, z), \phi_{\alpha_1(M-1)\alpha_2(M-1)\alpha_31}(x, y, z)\ldots, \phi_{\alpha_1(M-1)\alpha_2(M-1)\alpha_3(M-1)}(x, y, z)\},\end{aligned}$$

that $\alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N$. We assume that $W^T_{\alpha_1\alpha_2\alpha_3}\Phi_{\alpha_1\alpha_2\alpha_3}(x, y, z)$ is the best approximation to $w_{\alpha_1\alpha_2\alpha_3}(x, y, z)$ in $Z_{\alpha_1\alpha_2\alpha_3}$, where

$$\begin{aligned}W_{\alpha_1\alpha_2\alpha_3} = \big[&w_{\alpha_10\alpha_20\alpha_30}, w_{\alpha_10\alpha_20\alpha_31}, \ldots, w_{\alpha_10\alpha_20\alpha_3(M-1)}, w_{\alpha_10\alpha_21\alpha_30}, w_{\alpha_10\alpha_21\alpha_31}, \ldots, w_{\alpha_10\alpha_21\alpha_3(M-1)}, \ldots\\ &, w_{\alpha_1(M-1)\alpha_2(M-1)\alpha_30}, w_{\alpha_1(M-1)\alpha_2(M-1)\alpha_31}, \ldots, w_{\alpha_1(M-1)\alpha_2(M-1)\alpha_3(M-1)}\big]^T,\end{aligned}$$

$$\begin{aligned}\Phi_{\alpha_1\alpha_2\alpha_3}(x, y, z) = \big[&\phi_{\alpha_10\alpha_20\alpha_30}(x, y, z), \ldots, \phi_{\alpha_10\alpha_20\alpha_3(M-1)}(x, y, z), \phi_{\alpha_10\alpha_21\alpha_30}(x, y, z), \ldots, \phi_{\alpha_10\alpha_21\alpha_3(M-1)}(x, y, z), \ldots\\ &, \phi_{\alpha_1(M-1)\alpha_2(M-1)\alpha_30}(x, y, z)\ldots, \phi_{\alpha_1(M-1)\alpha_2(M-1)\alpha_3(M-1)}(x, y, z)\big]^T,\end{aligned}$$

for $\alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N$. The above assumptions will be used for obtaining following theorem.

Theorem 5.1. Suppose that $w_{N,M}(x, y, z) = W^T\phi(x, y, z)$ be the series expansion of three-dimensional hybrid functions for real-valued function w specified by (3.1). If w is sufficiently smooth on every subinterval $\Lambda_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N)$, then there is a constant ξ such that

$$\|w - w_{N,M}\|_2 \leqslant \frac{\xi}{2^{2M-1}N^M M!}.$$

Proof . Let

$$w = \sum_{\alpha_1=1}^{N}\sum_{\alpha_2=1}^{N}\sum_{\alpha_3=1}^{N} w_{\alpha_1\alpha_2\alpha_3},$$

and suppose that $P_{(M-1)\alpha_1\alpha_2\alpha_3}$ be the interpolating polynomial for $w_{\alpha_1\alpha_2\alpha_3}$ relative to the nodes $x_i, y_j, z_l, i, j, l = 0, 1, \ldots, M-1$ where x_i, y_j, z_l are the zeros of shifted Chebyshev polynomials of degree $M-1$ in the intervals $\big[\frac{\alpha_1-1}{N}, \frac{\alpha_1}{N}\big)$, $\big[\frac{\alpha_2-1}{N}, \frac{\alpha_2}{N}\big)$, and $\big[\frac{\alpha_3-1}{N}, \frac{\alpha_3}{N}\big)$, respectively. Now we have [4]

$$\begin{aligned}w_{\alpha_1\alpha_2\alpha_3}(x, y, z) - P_{(M-1)\alpha_1\alpha_2\alpha_3}(x, y, z) &= \frac{\partial^M w(\sigma, y, z)}{\partial x^M}\frac{\prod_{i=0}^{M-1}(x - x_i)}{M!} + \frac{\partial^M w(x, \tau, z)}{\partial y^M}\frac{\prod_{j=0}^{M-1}(y - y_j)}{M!}\\ &+ \frac{\partial^M w(x, y, \mu)}{\partial z^M}\frac{\prod_{l=0}^{M-1}(z - z_l)}{M!} - \frac{\partial^{3M} w(\sigma', \tau', \mu')}{\partial x^M\partial y^M\partial z^M}\frac{\prod_{i=0}^{M-1}(x - x_i)}{M!}\frac{\prod_{j=0}^{M-1}(y - y_j)}{M!}\frac{\prod_{l=0}^{M-1}(z - z_l)}{M!},\end{aligned} \tag{5.1}$$

where $\sigma, \sigma' \in \left[\frac{\alpha_1-1}{N}, \frac{\alpha_1}{N}\right)$, $\tau, \tau' \in \left[\frac{\alpha_2-1}{N}, \frac{\alpha_2}{N}\right)$ and $\mu, \mu' \in \left[\frac{\alpha_3-1}{N}, \frac{\alpha_3}{N}\right)$. Now we take $\eta = \max\{\eta_1, \eta_2, \eta_3, \eta_4\}$ at which

$$\eta_1 = \max\left\{ \sup_{(x,y,z)\in\Lambda_{\alpha_1\alpha_2\alpha_3}} \left|\frac{\partial^M w(x,y,z)}{\partial x^M}\right| \;\middle|\; \alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N \right\},$$

$$\eta_2 = \max\left\{ \sup_{(x,y,z)\in\Lambda_{\alpha_1\alpha_2\alpha_3}} \left|\frac{\partial^M w(x,y,z)}{\partial y^M}\right| \;\middle|\; \alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N \right\},$$

$$\eta_3 = \max\left\{ \sup_{(x,y,z)\in\Lambda_{\alpha_1\alpha_2\alpha_3}} \left|\frac{\partial^M w(x,y,z)}{\partial z^M}\right| \;\middle|\; \alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N \right\},$$

$$\eta_4 = \max\left\{ \sup_{(x,y,z)\in\Lambda_{\alpha_1\alpha_2\alpha_3}} \left|\frac{\partial^{3M} w(x,y,z)}{\partial x^M \partial y^M \partial z^M}\right| \;\middle|\; \alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, N \right\}.$$

Using (5.1) and taking estimation of Chebyshev interpolation nodes follows that

$$\begin{aligned}
&\left|w_{\alpha_1\alpha_2\alpha_3} - P_{(M-1)\alpha_1\alpha_2\alpha_3}(x,y,z)\right| \\
&\leqslant \eta_1\left(\frac{1}{2N}\right)^M \frac{1}{M!2^{M-1}} + \eta_2\left(\frac{1}{2N}\right)^M \frac{1}{M!2^{M-1}} + \eta_3\left(\frac{1}{2N}\right)^M \frac{1}{M!2^{M-1}} + \eta_4\left(\frac{1}{2N}\right)^{3M} \frac{1}{M!^3 2^{3M-3}} \\
&\leqslant \frac{\eta}{2^{2M-1}N^M M!}\left(3 + \frac{1}{2^{4M-2}N^{2M}(M!)^2}\right) \leqslant \frac{\xi}{2^{2M-1}N^M M!},
\end{aligned} \tag{5.2}$$

where $\xi = 4\eta$. Since $W^T_{\alpha_1\alpha_2\alpha_3}\Phi_{\alpha_1\alpha_2\alpha_3}(x,y,z) \in Z_{\alpha_1\alpha_2\alpha_3}$ is the best approximation to $w_{\alpha_1\alpha_2\alpha_3}$ therefore (5.2) gives

$$\begin{aligned}
\left\|w_{\alpha_1\alpha_2\alpha_3} - W^T_{\alpha_1\alpha_2\alpha_3}\Phi_{\alpha_1\alpha_2,\alpha_3}\right\|_2^2 &\leqslant \left\|w_{\alpha_1\alpha_2\alpha_3} - P_{(M-1)\alpha_1\alpha_2\alpha_3}\right\|_2^2 \\
&= \int_0^1\int_0^1\int_0^1 \left|w_{\alpha_1\alpha_2\alpha_3}(x,y,z) - P_{(M-1)\alpha_1\alpha_2\alpha_3}(x,y,z)\right|^2 dzdydx \\
&\leqslant \int_{\frac{\alpha_1-1}{N}}^{\frac{\alpha_1}{N}}\int_{\frac{\alpha_2-1}{N}}^{\frac{\alpha_2}{N}}\int_{\frac{\alpha_3-1}{N}}^{\frac{\alpha_3}{N}} \left(\frac{\xi}{2^{2M-1}N^M M!}\right)^2 dzdydx = \frac{1}{N^3}\left(\frac{\xi}{2^{2M-1}N^M M!}\right)^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left\|w - W^T\Phi\right\|_2^2 &\leqslant \sum_{\alpha_1=1}^N\sum_{\alpha_2=1}^N\sum_{\alpha_3=1}^N \left\|w_{\alpha_1\alpha_2\alpha_3} - W^T_{\alpha_1\alpha_2\alpha_3}\Phi_{\alpha_1\alpha_2\alpha_3}\right\|_2^2 \\
&\leqslant \left(\frac{\xi}{2^{2M-1}N^M M!}\right)^2.
\end{aligned}$$

We extract square root from each side and replace $W^T\Phi$ by $w_{N,M}$, we take

$$\|w - w_{N,M}\|_2 \leqslant \frac{\xi}{2^{2M-1}N^M M!}.$$

□

Now we try to obtain the error estimation of (1.1). Assume that $\widetilde{w}$ be the approximate solution by the proposed method and $w(x,y,z)$ be the exact solution. Let

$$G = \sup\nolimits_{0\leqslant x,y,z,p,q,r<1}\left|k\left(x,y,z,p,q,r\right)\right| < \infty \tag{5.3}$$

and ψ fulfills a Lipschitz condition where

$$\left|\psi\left(x,y,z,\widetilde{w}(p,q,r)\right) - \psi\left(x,y,z,w(p,q,r)\right)\right| \leqslant S\left|\widetilde{w}(p,q,r) - w(p,q,r)\right|. \tag{5.4}$$

Suppose that $r(x,y,z)$ be the error in approximating $w(x,y,z)$ by $\widetilde{w}(x,y,z)$ so

$$\widetilde{w}\left(x,y,z\right) = v\left(z,y,z\right) + \int_0^x\int_0^y\int_0^z k\left(x,y,z,p,q,r\right)\psi\left(p,q,r,\widetilde{w}(p,q,r)\right)drdqdp + r\left(x,y,z\right). \tag{5.5}$$

Subtracting (5.5) from (1.1) gives

$$r\left(x,y,z\right) = \widetilde{w}\left(x,y,z\right) - w\left(x,y,z\right) - \int_0^x\int_0^y\int_0^z k\left(x,y,z,p,q,r\right)\left[\psi\left(p,q,r,\widetilde{w}(p,q,r)\right) - \psi\left(p,q,r,w(p,q,r)\right)\right]drdqdp.$$

Table 1: Absolute errors for Example 6.1.

(x, y, z)	Exact solution	$N = 1, M = 2$
$(0.1, 0.1, 0.1)$	0.3	$1.1e - 16$
$(0.3, 0.3, 0.3)$	0.9	$4.4e - 16$
$(0.5, 0.5, 0.5)$	1.5	0
$(0.7, 0.7, 0.7)$	2.1	$4.4e - 16$
$(0.9, 0.9, 0.9)$	2.7	$4.4e - 16$

Therefore

$$|r(x,y,z)| \leqslant |\widetilde{w}(x,y,z) - w(x,y,z)| + \int_0^1\int_0^1\int_0^1 \left|k(x,y,z,p,q,r)\right|\left|\psi(p,q,r,\widetilde{w}(p,q,r)) - \psi(p,q,r,w(p,q,r))\right| drdqdp,$$

from (5.3), (5.4) and using L^2 norm, we get

$$\|r\|_2 \leqslant (1+GS)\,\|w-\widetilde{w}\|_2 \leqslant \frac{(1+GS)\xi}{2^{2M-1}N^M M!}.$$

6 Numerical illustration

Here, four examples have been used to investigate the proficiency and the usefulness of our method. In each examples, We have supposed that $(x,y,z) \in \Lambda$. Let $e_{N,M}(x,,y,z)$ be the error involved in the approximation, so

$$e_{N,M}(x,,y,z) = \left|w(x,y,z) - w_{N,M}(x,y,z)\right|.$$

Matlab programs and Maple have been used to obtain the numerical solutions.

Example 6.1. Suppose that

$$w(x,y,z) = v(x,y,z) - \int_0^x\int_0^y\int_0^z w(p,q,r)drdqdp,$$

where

$$v(x,y,z) = x+y+z+\frac{x^2yz+xy^2z+xyz^2}{2},$$

and its exact solution is $w(x,y,z) = x+y+z$. When $N=1$ and $M=2$, the proposed method by substituting collocation points (4.3) gives

$$\begin{aligned}
&(1+x_iy_jz_l)w_{101010} + \left[(2z_l-1)+x_iy_j(z_l^2-z_l)\right]w_{101011} + \left[(2y_j-1)+x_iz_l(y_j^2-y_j)\right]w_{101110}\\
&\quad+\left[(2y_j-1)(2z_l-1)+x_i(y_j^2-y_j)(z_l^2-z_l)\right]w_{101111} + \left[(2x_i-1)+(x_i^2-x_i)y_jz_l\right]w_{111010}\\
&\quad+\left[(2x_i-1)(2z_l-1)+(x_i^2-x_i)y_j(z_l^2-z_l)\right]w_{111011} + \left[(2x_i-1)(2y_j-1)+(x_i^2-x_i)(y_j^2-y_j)z_l\right]w_{111110}\\
&\quad+\left[(2x_i-1)(2y_j-1)(2z_l-1)+(x_i^2-x_i)(y_j^2-y_j)(z_l^2-z_l)\right]w_{111111}\\
&=(x_i+y_j+z_l)\left(1+\frac{x_iy_jz_l}{2}\right),
\end{aligned}$$

where $i,j,l = 1,2$. Computational results are displayed in Table 1.

Example 6.2. Suppose that

$$w(x,y,z) = v(x,y,z) - 24\int_0^x\int_0^y\int_0^z x^2yw(p,q,r)drdqdp,$$

where $v(x,y,z) = 4x^5y^3z+4x^3y^3z^3+3x^4y^3z^2+x^2y+yz^2+xyz$. Its exact solution is specified by $w(x,y,z) = x^2y+yz^2+xyz$. In Table 2, we have compared absolute errors of the proposed method with those in [9].From Table 2, can be understood that the errors calculated by our technique are better than those in [9].

Example 6.3. We consider the following Volterra-Hammerstein integral equation:

$$w(x,y,z) = v(x,y,z) + \int_0^x\int_0^y\int_0^z w^2(p,q,r)drdqdp,$$

where

$$v(x,y,z) = xyz - \frac{(xyz)^3}{27}.$$

Its exact solution is $w(x,y,z) = xyz$. When $N=1$ and $M=3$, the computational results are shown in Table 3. The graph of $e_{1,3}(x,y,z)$ for $z = 0.1, 0.7, 0.9$ is displayed in Figure 1. Our results are approximately consistent with those in [9].

Table 2: Absolute errors for Example 6.2.

$(x,y,z)=(\frac{1}{2^t},\frac{1}{2^t},\frac{1}{2^t})$	$N=1,M=3$	Method of [6] with $m=8$
$t=1$	0	$3.63109e-12$
$t=2$	0	$3.65291e-10$
$t=3$	0	$4.05249e-10$
$t=4$	$3.6970e-17$	$5.03878e-10$
$t=5$	$4.4070e-17$	$5.63516e-10$

Table 3: Absolute errors for Example 6.3.

(x,y,z)	Exact solution	$N=1,M=3$
$(0.1,0.1,0.1)$	0.001	$5.1e-11$
$(0.3,0.3,0.3)$	0.027	$6.4e-12$
$(0.5,0.5,0.5)$	0.125	0
$(0.7,0.7,0.7)$	0.343	$6.40e-12$
$(0.9,0.9,0.9)$	0.729	$5.17e-11$

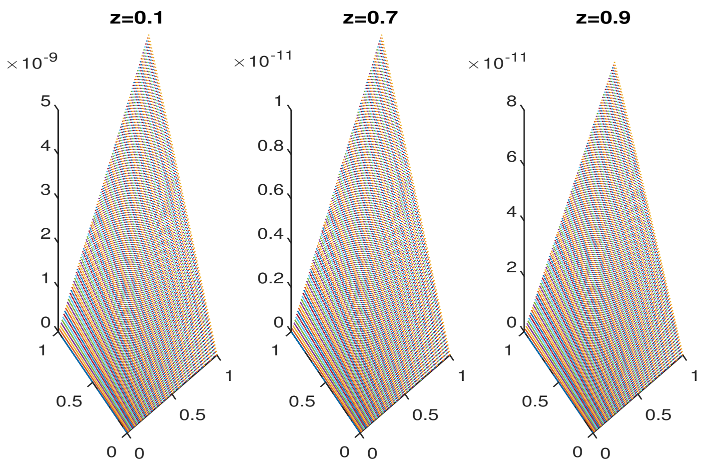

Figure 1: Plot of the error function for Example 6.3.

Table 4: Absolute errors for Example 6.4.

(x,y,z)	Exact solution	$N=1,M=2$
$(0.1,0.1,0.1)$	0.000998	0.000174
$(0.3,0.3,0.3)$	0.026597	0.002957
$(0.5,0.5,0.5)$	0.119856	0.003729
$(0.7,0.7,0.7)$	0.315667	0.002957
$(0.9,0.9,0.9)$	0.634495	0.023105

Example 6.4. We consider the following Volterra-Hammerstein integral equation:

$$w(x,y,z)=v(x,y,z)+\int_0^x\int_0^y\int_0^z w^3(p,q,r)drdqdp,$$

where

$$v\left(x,y,z\right)=yz\sin z-\frac{1}{24}y^4z^4+\frac{1}{48}y^4z^4\sin^2 x\cos x+\frac{1}{48}y^4z^4\cos x.$$

Its analytical solution is $w(x,y,z)=yz\sin x$. When $N=1$ and $M=2$, computational results are given in Table 4.

7 Conclusions

We used hybrid of block-pulse functions and Legendre polynomials to find the solution of three-dimensional Volterra-Hammerstein integral equations. With the aim of finding approximate solution, we collocated integral equation at N^3M^3 collocation points and obtained a system of nonlinear equations. Using the suggested method for some numerical examples provides a very good approximation, although we used small values of M and N. The advantages of our method are simple calculations and high accuracy.

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