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Existence results for some weakly singular integral equations via measures of non-compactness

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Abstract

In this paper, the existence of the solutions of a class of weakly singular integral equations in Banach algebra is investigated. The basic tool used in investigations is the technique of the measure of non-compactness and Petryshyn's fixed point theorem. Also, for the applicability of the obtained results, some examples are given.

Keywords: Weakly singular integral equations, Fixed point theorem, Measure of non-compactness (MNC), Existence of the solution
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1 Introduction

Kuratowski [13] introduced the concept of a measure of noncompactness (MNC). Then it is used by Darbo [3], Furi and Vignoli [8], Nussbaum [20], Petryshyn [22], and others. For details on measures of noncompactness see [2, 14]. Recently, there have been several successful attempts to apply the concept of (MNC) in the study of the existence of solutions of nonlinear integral equations and integro-differential equations [21, 28, 4, 11, 10, 9, 18, 5, 7, 24, 25, 26, 12, 27]. Ordinary integral equations with weakly singular kernels arise in many problems of science and engineering. This study deals with the existence of solution of the following weakly singular integral equations of the form

$$u(s) = q\left(s, g(s, u(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right), \tag{1.1}$$

for all $s \in I_a = [0, a]$. For the existence of solutions of the integral equation (1.1), we use a fixed point theorem due to Petryshyn [22] that has been analyzed as a generalization of Darbo fixed point theorem [2]. Many authors have successful efforts to solve many functional integral equations by powerful tools of Darbo condition [23, 19, 15, 1, 16, 17, 6]. The advantage of Petryshyn's fixed point theorem among the others (Darbo and Schauder fixed point theorems) lie in that in applying the theorem, one does not need to verify the involved operator maps a closed convex subset onto itself. The article is organized into 4 sections including the introduction. In Section 2, we recall some preliminaries and specify the concept of MNC. Section 3 is applied to state and prove an existence theorem for Eq.

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(1.1) including densifying operators by Petryshyn's fixed point theorem. In Section 4, we provide some examples that verify the applications of these kinds of weakly singular integral equations in nonlinear analysis. Finally, conclusions of the work are given in Section 5.

2 Preliminaries

In this article, we have some notations:
- E : Real Banach space;
- $B_\rho(z)$: Open ball with center z and radius ρ.

Definition 2.1. [2] Let $S \in E$ and

$$\alpha(S) = \inf \left\{ \varepsilon > 0 : S = \bigcup_{i=1}^{n} S_i \text{ with diam } S_i \leq \varepsilon,\ i = 1, 2, ..., n \right\},$$

is called the Kuratowski MNC.

Definition 2.2. [2] The Hausdroff MNC

$$\psi(S) = \inf \left\{ \varepsilon > 0 : \exists \text{ a finite } \varepsilon\text{- net for S in } E \ \right\}, \tag{2.1}$$

where, by a finite ε net for S in E it involves, as a set $\{z_1, z_2, ..., z_n\} \subset S$ such that the ball $B_\varepsilon(E, z_1), B_\varepsilon(E, z_2), ..., B_\varepsilon(E, z_n)$ over S. Those MNC are commonly related that is $\psi(S) \leq \alpha(S) \leq 2\psi(S)$, for any bounded set $S \subset E$.

Theorem 2.3. Let $S, \bar{S} \in E$ and $\lambda \in \mathbb{R}$. Then

(i) $\psi(S) = 0$ if and only if S is pre-compact;

(ii) $S \subseteq \bar{S} \implies \psi(S) \leq \psi(\bar{S})$;

(iii) $\psi(\text{Conv}S) = \psi(S)$;

(iv) $\psi(S \cup \bar{S}) = \max\{\psi(S), \psi(\bar{S})\}$;

(v) $\psi(\lambda S) = |\lambda|\psi(S)$, where $\lambda S = \{\lambda z : z \in S\}$;

(vi) $\psi(S + \bar{S}) \leq \psi(S) + \psi(\bar{S})$.

Let, $C[0, a]$ be the space of all real valued continuous function defined on $I_a = [0, a]$ with the usual norm $\|z\| = \sup\{|z(s)| : s \in [0, a]\}$. The space $C[0, a]$ is also the structure of Banach algebra. The modulus of continuity of $z \in C[0, a]$ is defined as

$$\omega(z, \varepsilon) = \sup\{|z(s) - z(\bar{s})| : s, \bar{s} \in [0, a], |s - \bar{s}| \leq \varepsilon\}.$$

and,

$$\omega(S, \varepsilon) = \sup\{\omega(z, \varepsilon) : z \in S\}, \quad \omega_0(S) = \lim_{\varepsilon \to 0} \omega(S, \varepsilon).$$

Theorem 2.4. [11] On the space $C[0, a]$, the Hausdorff MNC is equivalent to

$$\psi(S) = \lim_{\varepsilon \to 0} \sup\ \omega(z, \varepsilon) \tag{2.2}$$

for all bounded sets $S \subset C[0, a]$.

Definition 2.5. [20] Assume $T : E \to E$ be a continuous mapping of E. T is called a k-set contraction if for all $S \subset E$ with S bounded, $T(S)$ is bounded and

$$\alpha(TS) \leq k\alpha(S), \text{ for } k \in (0, 1).$$

Moreover, if $\alpha(TS) < \alpha(S)$, for all$\alpha(S) > 0$, then T is called densifying or condensing map.

Theorem 2.6. [22] Let $T : B_\rho \to E$ be a condensing mapping which satisfying the boundary condition, if $T(z) = kz$, for some $z \in \partial B_\rho$ implies $k \leq 1$, then the set of fixed points in B_ρ is non-empty.

This is called Petryshyn's fixed point theorem.

3 Main results

Now, we investigate the existence of the Eq.(1.1) under the following assumptions;

(1) $q \in C(I_a \times \mathbb{R}^2, \mathbb{R})$, $g \in C(I_a \times \mathbb{R}, \mathbb{R})$ $k \in C(I_a^2 \times \mathbb{R}, \mathbb{R})$, and $\beta, \gamma : I_a \to I_a$, are continuous.

(2) there are non-negative constants c_1, k_1, k_2, with $k_1 c_1 < 1$ such that

$$|q(s, u_1, u_2) - q(s, \bar{u_1}, \bar{u_2}| \leq k_1|u_1 - \bar{u_1}| + k_2|u_2 - \bar{u_2}|; \quad \text{and} \quad |g(s, u) - g(s, \bar{u})| \leq c_1|u - \bar{u}|.$$

(3) there exists $\rho > 0$ such that q fulfill the inequality

$$\sup\{|q(s, u_1, u_2)| : s \in I_a, u_1 \in [-\rho, \rho], u_2 \in [-H|a(\ln a - 1)|, H|a(\ln a - 1)|]\} \leq \rho,$$

where $H = \sup\{|k(s, \nu, u)| : \forall s, \nu \in I_a \quad and \quad u \in [-\rho, \rho]\}$.

Theorem 3.1. Under the assumptions (1) – (3) the Eq. (1.1) has at least one solution in $E = C(I_a)$.

Proof . Define the operator $T : B_\rho \to E$, where $B_\rho = \{u \in C(I_a) : \|u\| \leq \rho\}$ in the following form

$$(Tu)(s) = q\left(s, g(s, u(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right).$$

Now, we show that T is continuous on B_ρ. Choose $\varepsilon > 0$ and any $u, x \in B_\rho$ such that $\|u - x\| < \varepsilon$. Then

$$\begin{aligned}
|(Tu)(s) - (Tx)(s)| &= \Bigg| q\left(s, g(s, u(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right) \\
&\quad - q\left(s, g(s, x(\gamma(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, x(\gamma(\nu))\big) d\nu\right)\Bigg| \\
&\leq \Bigg| q\left(s, g(s, u(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right) \\
&\quad - q\left(s, g(s, x(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right)\Bigg| \\
&\quad + \Bigg| q\left(s, g(s, x(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, u(\gamma(\nu))\big) d\nu\right) \\
&\quad - q\left(s, g(s, x(\beta(s))), \int_0^s \ln|\nu - s| k\big(s, \nu, x(\gamma(\nu))\big) d\nu\right)\Bigg| \\
&\leq k_1|g(s, u(\beta(s))) - g(s, x(\beta(s)))| \\
&\quad + k_2 \int_0^s \ln|\nu - s| |k\big(s, \nu, u(\gamma(\nu))\big) - k\big(s, \nu, x(\gamma(\nu))\big)| d\nu \\
&\leq k_1 c_1 |u(\beta(s)) - x(\beta(s))| + k_2 \omega(k, \varepsilon) \int_0^s \ln|\nu - s| d\nu, \\
&\leq k_1 c_1 \|u - x\| + k_2 \omega(k, \varepsilon) |s \ln s - s|, \\
&\leq k_1 c_1 \|u - x\| + k_2 \omega(k, \varepsilon),
\end{aligned}$$

where

$$\omega(k, \varepsilon) = \sup\{|k(s, \nu, u) - k(s, \nu, x)| : s, \nu \in I_a, u, x \in [-\rho, \rho], |u - x| \leq \varepsilon\}.$$

From the uniform continuity of $k(s, \nu, u)$ on the subset $I_a \times I_a \times [-\rho, \rho]$, we judge that $\omega(k, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the above fact prove that the operator T is continuous on B_ρ. Next, we prove that T fulfils the condensing map. For arbitrary $\varepsilon > 0$ and $u \in S$, where S is bounded subset of E, $s_1, s_2 \in I_a$ with $|s_2 - s_1| \le \varepsilon$, we have

$$\begin{aligned}
|(Tu)(s_2) - (Tu)(s_1)| &= \bigg| q\bigg(s_2, g(s_2, u(\beta(s_2))), \int_0^{s_2} \big(\ln|\nu - s_2| k\big(s_2, \nu, u(\gamma(\nu))\big) d\nu\bigg) \\
&- q\bigg(s_1, g(s_1, u(\beta(s_1))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg)\bigg| \\
&\le \bigg| q\bigg(s_2, g(s, u(\beta(s_2))), \int_0^{s_2} \ln|\nu - s_2| k\big(s_2, \nu, u(\gamma(\nu))\big) d\nu\bigg) \\
&- q\bigg(s_2, g(s_2, u(\beta(s_2))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg)\bigg| \\
&+ \bigg| q\bigg(s_2, g(s_2, u(\beta(s_2))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg) \\
&- q\bigg(s_2, g(s_1, u(\beta(s_1))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg)\bigg| \\
&+ \bigg| q\bigg(s_2, g(s_1, u(\beta(s_1))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg) \\
&- q\bigg(s_1, g(s_1, u(\beta(s_1))), \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg)\bigg| \\
&\le k_1|g(s_2, u(\beta(s_2)) - g(s_2, u(\beta(s_1))| + k_1|g(s_2, u(\beta(s_1)) - g(s_1, u(\beta(s_1))| + \omega_q(I_a, \varepsilon), \\
&+ k_2\bigg| \int_0^{s_2} \ln|\nu - s_2| k\big(s_2, \nu, u(\gamma(\nu))\big) d\nu - \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg| \\
&\le k_1 c_1 |u(\beta(s_2) - u(\beta(s_1)| + k_1\omega_g(I_a, \varepsilon) + \omega_q(I_a, \varepsilon), \\
&+ k_2\bigg[\bigg| \int_0^{s_2} \ln|\nu - s_2| k\big(s_2, \nu, u(\gamma(\nu))\big) d\nu - \int_0^{s_2} \ln|\nu - s_2| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg| \\
&+ \bigg| \int_0^{s_2} \ln|\nu - s_2| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu - \int_0^{s_1} \ln|\nu - s_1| k\big(s_1, \nu, u(\gamma(\nu))\big) d\nu\bigg|\bigg] \\
&\le k_1 c_1 |u(\beta(s_2) - u(\beta(s_1)| + k_1\omega_g(I_a, \varepsilon) \\
&+ k_2\big[\int_0^{s_2} \ln|\nu - s_2| |k\big(s_2, \nu, u(\gamma(\nu)) - k\big(s_1, \nu, u(\gamma(\nu))\big)| d\nu \\
&+ |k\big(s_1, \nu, u(\gamma(\nu))\big)| \big[\int_0^{s_2} \ln|\nu - s_2| d\nu - \int_0^{s_1} \ln|\nu - s_1| d\nu\big] + \omega_q(I_a, \varepsilon),
\end{aligned}$$

where

$$\omega_g(I_a, \varepsilon) = \sup\{|g(s, u) - g(\bar{s}, u)| : |s - \bar{s}| \le \varepsilon, s, \bar{s} \in I_a, u \in [-\rho, \rho]\},$$

$$\begin{aligned}
\omega_q(I_a, \varepsilon) \quad = \quad & \sup\{|q(s, u_1, u_2) q(\bar{s}, u_1, u_2)| : |s - \bar{s}| \le \varepsilon, s, \bar{s} \in I_a, u_1 \in [-\rho, \rho], u_2 \in [-H|a(\ln a - 1)|, \\
& H|a(\ln a - 1)|]\},
\end{aligned}$$

$$\omega_k(I_a, \varepsilon) = \sup\{|k(s, \nu, u) - k(\bar{s}, \nu, u)| : |s - \bar{s}| \le \varepsilon, \quad s, \bar{s}, \nu \in I_a, u \in [-\rho, \rho]\}.$$

From above relations, we have

$$\begin{aligned}
|(Tu)(s_2) - (Tu)(s_1)| \quad \le \quad & k_1\omega_g(I_a, \varepsilon) + k_1 c_1 |u(\beta(s_2) - u(\beta(s_1)| + k_2\omega_k(I_a, \varepsilon)|a(\ln a - 1)| \\
& + k_2 H|s_2 - s_1| + \omega_q(I_a, \varepsilon).
\end{aligned}$$

So,

$$\omega(Tu, \varepsilon) \quad \le \quad k_1\omega_g(I_a, \varepsilon) + k_1 c_1 \omega(u, \omega(\beta, \varepsilon)) + k_2\omega_k(I_a, \varepsilon) + k_2 H\varepsilon + \omega_q(I_a, \varepsilon).$$

This yields the following estimate:

$$\omega(Tu,\varepsilon) \le (k_1c_1)\omega(u,\varepsilon).$$

Taking limit as $\varepsilon \to 0$, we get

$$\psi(TS) \le (k_1c_1)\psi(S).$$

Hence T is a condensing map. Now, let $u \in \partial B_\rho$ and if $Tu = ku$ then $\|Tu\| = k\|u\| = k\rho$ and by assumptions (3), then

$$|Tu(s)| = \left|q\left(s, g(s,u(\beta(s))), \int_0^s \ln|\nu - s| k\big(s,\nu,u(\gamma(\nu))\big)d\nu\right)\right| \le \rho$$

for all $s \in I_a$, hence $\|Tu\| \le \rho$ i.e $k \le 1$. This completes the proof. □

Corollary 3.2. Let

(1) $f, F \in C(I_a \times \mathbb{R}, \mathbb{R})$, $k \in C(I_a^2 \times \mathbb{R}, \mathbb{R})$, and $\alpha, \gamma : I_a \to I_a$, are continuous.

(2) there are non-negative constants c_1, k_1 with $c_1 < 1$ such that

$$|F(s,u_2) - F(s,\bar{u_2}| \le k_1|u_2 - \bar{u_2}|$$

$$|f(s,u_1) - f(s,\bar{u_1})| \le c_1|u_1 - \bar{u_1}|.$$

(3) there exists $\rho > 0$ such that the following bounded condition is satisfied

$$\sup\{|D + F|\} \le \rho,$$

where,

$$\sup D = \sup\{|f(s,u)| : \text{for all } s \in I_a \;\; and \;\; u \in [-\rho,\rho]\},$$
$$\sup F = \sup\{|F(s,u)| : \text{for all } s \in I_a \;\; u \in [-Ha(\ln a - 1), Ha(\ln a - 1)]\},$$
$$H = \sup\{|k(s,\nu,u)| : \text{for all } s,\nu \in I_a \;\; and \;\; u \in [-\rho,\rho]\}.$$

Then

$$u(s) = f(s,u(\alpha(s))) + F\left(s, \int_0^s \ln|\nu - s| k\big(s,\nu,u(\gamma(\nu))\big)d\nu\right) \tag{3.1}$$

has at least one solution in $C(I_a)$.

Proof . The proof is relevant to Theorem 3.1 and we can leave the proof parts. □

4 Examples

In this section, we present some examples of functional integral equations to illustrate the usefulness of our results.

Example 4.1. Consider the following weakly singular integral equation

$$\begin{aligned} u(s) \;\; = \;\; & \frac{1}{2+\sqrt{s}}e^{s-1} + \frac{u(s^3)s^2}{3(1+s^2)} \\ & + \frac{\cos(s)}{4(e^{s^2} + 3\sin(\sqrt{s}))}\int_0^s \ln|\nu - s| \frac{1 + \cos\sqrt{\nu} + |u(\sqrt{\nu})|}{1 + \nu s^2 + \ln(s)} d\nu, \;\; s \in [0,1]. \end{aligned} \tag{4.1}$$

Eq. (4.1) is particular form of Eq. (1.1) with

$$\gamma(s) = \sqrt{s}, \beta(s) = s^3, \forall\; s \in [0,1],$$

and

$$q(s,u_1,u_2) = q_1(s,u_1) + q_2(s,u_2),$$

where

$$q_1(s,u_1)=\frac{1}{2+\sqrt{s}}e^{s-1}+\frac{1}{3}u_1,\quad u_1=\frac{u(s^3)s^2}{(1+s^2)},$$

$$q_2(s,u_2)=\frac{\cos(s)}{4(e^{s^2}+3\sin(\sqrt{s}))}u_2,$$

$$u_2=\int_0^s \ln|\nu-s|\frac{1+\cos\sqrt{\nu}+|u(\sqrt{\nu})|}{1+\nu s^2+\ln(s)}d\nu,\quad k(s,\nu,u)=\frac{1+\cos\sqrt{\nu}+|u(\sqrt{\nu})|}{1+\nu s^2+\ln(s)}.$$

It is obvious that assumptions (1) and (2) of Theorem 3.1 are satisfied. We need to check that assumption (3) holds true. Suppose that $\|u\|\le\rho,\rho>0$, then

$$\begin{aligned}|u(s)|=&|\frac{1}{2+\sqrt{s}}e^{s-1}+\frac{u(s^3)s^2}{3(1+s^2)}\\&+\frac{\cos(s)}{4(e^{s^2}+3\sin(\sqrt{s}))}\int_0^s \ln|\nu-s|\frac{1+\cos\sqrt{\nu}+|u(\sqrt{\nu})|}{1+\nu s^2+\ln(s)}d\nu|\le\rho,\end{aligned}$$

for all $s\in I_a$. Hence (3) holds if,

$$\frac{1}{2}+\frac{1}{3}\rho+\frac{1}{4}(2+\rho)\le\rho.$$

It can be check that $\rho=2.4$ satisfies in the last inequality. Hence, all conditions of Theorem 3.1 are fulfill, then Eq. (4.1) has at least one solution in $C[0,1]$.

Example 4.2. Consider the following weakly singular integral equation

$$\begin{aligned}u(s)\;=\;&\frac{s^4u(\sqrt{s})}{3(1+s^4)}\\&+\frac{e^{-2s^2}}{5(1+s)}\int_0^s \ln|\nu-s|(e^s+\nu\sin(\sqrt{s})+\ln(1+u(s)))d\nu,\ s\in[0,1].\end{aligned}\tag{4.2}$$

Now, we can see that assumptions (1) and (2) of Theorem 3.1 are satisfied. We check that (3) also holds. Suppose that $\|u\|\le\rho,\rho>0$, then

$$|u(s)|=|\frac{s^4}{3(1+s^4}u(\sqrt{s})+\frac{e^{-2s^2}}{5(1+s)}\int_0^s \ln|\nu-s|(e^s+\nu\sin(\sqrt{s})+\ln(1+u(s)))d\nu|\le\rho,$$

for all $s\in I_a$. Hence (3) holds if,

$$\frac{1}{3}\rho+\frac{1}{5}(e+1+\rho)\le\rho.$$

It can be check that $\rho\ge\frac{3}{7}(e+1)$ satisfies in the last inequality. Hence, Eq. (4.2) has at least one solution in $C[0,1]$.

5 Conclusion

In the current study, we examined the existence of solutions for non-linear weakly singular integral equations by Petryshyn's fixed point theorem and the concept of a measure of noncompactness. We gave some examples to confirm the efficiency of our results.

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