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Two step size algorithms for strong convergence for a monotone operator in Banach spaces

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Abstract

For $p \geq 2$, let E be a 2 uniformly smooth and p uniformly convex real Banach spaces and let a mapping $\Phi : E \to E^*$ be Lipschitz, and strongly monotone such that $\Phi^{-1}(0) \neq \emptyset$. For an arbitrary $(\{\xi_1\}, \{\psi_1\}) \in E$, we define the sequences $\{\xi_n\}$ and $\{\psi_n\}$ by

$$\begin{cases} \psi_{n+1} = J^{-1}(J\xi_n - \theta_n \Phi\xi_n), & n \geq 0 \\ \xi_{n+1} = J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}), & n \geq 0 \end{cases}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0, \Lambda_p)$ where $\Lambda_p > 0$, then ξ_n and ψ_n converges strongly to ξ^*, a unique solution of the equation $\Phi\xi = 0$.

Keywords: Lipschitz, Equations, generalized monotone, Bounded Operator, Accretive operator, Uniformly convex
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1 Introduction

Many physical problems in applications can be modeled in the following form: find $x \in H$ such that

$$0 \in \Phi\xi \tag{1.1}$$

where Φ is a monotone operator on a real Hilbert space H. Typical examples where monotone operators occur and satisfy the inclusion $0 \in \Phi\xi$ include the equilibrium state of evolution equations and critical points of some functionals and convex optimization, linear programing, monotone inclusions and elliptic differential equations defined on Hilbert spaces (see e.g., Browder [5], Mustafa [21], Stephen [27], Khorasani and Adibi [15], Mendy et la, [18] and Chidume [9]). For precisely, the classical convex optimization problem: let $h : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The sub-differential of h at $x \in H$; is defined by $\partial h : H \to 2^H$

$$\partial(\xi) = \{\xi^* \in h : h(\psi) - h(\xi) \geq \langle \psi - \xi, \xi^* \rangle \forall \psi \in h\}. \tag{1.2}$$

Clearly, $\partial h : H \to 2^H$is monotone operator on H, and $0 \in \partial(\xi_0)$ if and only if ξ_0 is a minimizer of h. In the case of setting $\partial(\xi) \equiv \Phi$; solving the inclusion $0 \in \Phi\xi$ is solving for a minimizer of h.

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Many authors have done a great works in find zero point of Φ in Hilbert spaces (see e.g., Takahashi and Ueda [30], Song and Chen [26], and Cho et al. [11]). The proximal point algorithm (PPA) is recognized as a powerful and successful algorithm in finding a numerical solution of monotone operators equation $0 \in Ax$ which was introduced by Martinet [17] and studied further by Rockafellar [25], Mendy et la [4] and a host of other authors. That is, given $x_k \in H$;

$$\xi_{n+1} = J_{\lambda_n}\xi_n \tag{1.3}$$

where $J_{\lambda_n} = (I + \lambda_n\Phi)^{-1}$ is the resolvent of operator Φ. Since Rockafellar [25] only obtained the weak convergence of the algorithm 1.3 as $\lambda_n \to \infty$. He asked the following two questions for obtaining the strong convergence of the proximal point algorithm

1. Does the proximal point algorithm always converge weakly?
2. Can the proximal point algorithm be modified to guarantee strong convergence?

So many authors modify the the proximal point algorithm (PPA) to converges strongly under different setting, see Takahashi [29], Reich [22], Lehdili and Moudafi [16], Chidume et al. [7], and the references therein.

Let E be a real normed space, E^* its topological dual space. The map $J : E \to 2^{E^*}$ defined by

$$J_\xi : \Big\{\xi^* \in E^* : \quad \langle \xi, \xi^*\rangle = \|\xi\|.\|\xi^*\| = \|\xi\|^2 = \|\xi^*\|^2\Big\}$$

is called the normalized duality map on E. where $\langle , \rangle$ denotes the generalized duality pairing between E and E^*.

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, monotonicity and accretivity coincide. For an accretive-type operator Φ,

The solutions of the equation $\Phi\xi = 0$, in many cases, represent the equilibrium state of some dynamical system (see, for example, [11, page 116]). To approximate a solution of $\Phi\xi = 0$, assuming existence, where $\Phi : E \to E$ is of accretive type, Browder [5] defined an operator $T : E \to E$ by $T := I - \Phi$, where I is the identity map on E. He called such an operator pseudo-contractive. It is trivial to observe that zeros of Φ correspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume [6] proved the following theorem.

Theorem 1.1. (Chidume, [6]. Let $E = Lp, 2 \le p < 8$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $\xi_0 \in K$, let a sequence $\{\xi_n\}$ be defined iteratively by $\xi_{n+1} = (1-\lambda_n)\xi_n + \lambda_n T\xi_n, n \ge 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: $(i)\displaystyle\sum_{n=1}^{\infty}\lambda_n = \infty$, $(ii)\sum_{n=1}^{\infty}\lambda_n^2 \le \infty$. Then $\{\xi_n\}$ converges strongly to the unique fixed point of T.

By setting $T := I - \Phi$ in Theorem 1.1, the following theorem for approximating a solution of $\Phi\xi = 0$ where A is a strongly accretive and bounded operator can be proved.

Unfortunately, the success achieved in using geometric properties developed from the mid-1980s to early 1990s in approximating zeros of accretive-type mappings has not carried over to approximating zeros of monotone-type operators in general Banach spaces. Part of the problem is that since Φ maps E to E^*, for $\xi_n \in E, \Phi\xi_n$ is in E^*. Consequently, a recursion formula containing ξ_n and $\Phi\xi_n$ may not be well defined. Attempts have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.Examples Chidume [6],[10], Moudafi[20], Reich [24], Takahashi [28], Zegeye [32], Djitte[18], Mendy [[19],[13]] Chidume et al. [8] ,Djitte et la [13].

Following this great work, in 2019, Tan [31] constructed the following two-step proximal algorithm for the zero point of monotone mapping and proof a strong convergency of the sequences $\{\xi_n\}$ and $\{\psi_n\}$ to a unique point $\xi^* \in \Phi^{-1}(0)$.

$$\begin{cases} \psi_{n+1} = J^{-1}(J\xi_n - \lambda_n\Phi\xi_n), & n \ge 0 \\ \xi_{n+1} = J^{-1}(J\psi_{n+1} - \lambda_{n+1}\Phi\psi_{n+1}), & n \ge 0 \end{cases} \tag{1.4}$$

In this paper, we study the two step size Krasnoselskii-type algorithm introduced by Chidume et al.[6] and prove a strong convergence theorem to approximate the unique zero of a Lipschitz strongly monotone mapping 2−uniformly smooth and p−uniformly convex real Banach space for $p \ge 2$. This class of Banach spaces contains all Lp-spaces, $2 \le p < \infty$. Then we apply our results to the convex minimization problem. Finally, our method of proof is of independent interest

2 Preliminaries

Let E be a normed linear space. E is said to be smooth if

$$\lim_{t\to 0}\frac{\|\xi+t\psi\|-\|\xi\|}{t} \tag{2.1}$$

exist for each $\xi,\psi\in S_E$ (Here $S_E:=\{\xi\in E:||\xi||=1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $\xi,\psi\in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $\psi\in S_E$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau):=\sup\left\{\frac{\|\xi+\psi\|+\|\xi-\psi\|}{2}-1:\|x\|=1,\|\psi\|=\tau\right\};\quad \tau>0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

If there exist a constant $c>0$ and a real number $q>1$ such that $\rho_E(\tau)\leq c\tau^q$, then E is said to be *q-uniformly smooth.*

A normed linear space E is said to be strictly convex if:

$$\|\xi\|=\|\psi\|=1,\ x\neq\psi\ \Rightarrow\ \left\|\frac{\xi+\psi}{2}\right\|<1.$$

The modulus of convexity of E is the function $\delta_E:(0,2]\to[0,1]$ defined by:

$$\delta_E(\epsilon):=\inf\left\{1-\frac{1}{2}\|\xi+\psi\|\ :\|\xi\|=\|\psi\|=1,\ \|\xi-\psi\|\geq\epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon)>0$ for every $\epsilon\in(0,2]$. For $p>1$, E is said to be p-uniformly convex if there exists a constant $c>0$ such that $\delta_E(\epsilon)\geq c\epsilon^p$ for all $\epsilon\in(0,2]$. Observe that every p-uniformly convex space is uniformly convex.

Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1<p<\infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m\text{ is }\begin{cases}2-\text{uniformly smooth and }p-\text{uniformly convex} & \text{if}\quad 2\leq p<\infty;\\ 2-\text{uniformly convex and }p-\text{uniformly smooth} & \text{if}\quad 1<p<2.\end{cases}$$

Remark 2.1. Note also that duality mapping exists in each Banach space.We recall from [12] some of the examples of this mapping in $\ell_p,L_p,W^{m,p}-$spaces, $1<p<\infty$

- $\ell_p:J\xi=\|\xi\|_{\ell_p}^{2-p}\psi\in\ell_q,\xi=(\xi_1,\xi_2,...,\xi_n,...),\psi=(\xi_1|\xi_1|^{p-2},\xi_2|\xi_2|^{p-2},...,\xi_n|\xi_n|^{p-2},...)$
- $L_p:Ju=\|u\|_{L_p}^{2-p}|u|^{p-2}u\in L_q$
- $W^{m,p}:Ju=\|u\|_{W^{m,p}}^{2-p}\sum_{|\alpha\leq m|}(-1)^{|\alpha|}D^\alpha(|D^\alpha u|^{p-2}D^\alpha u)\in W^{-m,p}$

Definition 2.2.

- A map $\Phi:E\to E^*$ is called monotone if for each $\xi,\psi\in E$, the following inequality holds:

$$\langle\Phi\xi-\Phi\psi,\xi-\psi\rangle\geq 0.$$

- Φ is called strongly monotone if there exists $k\in(0,1)$ such that for each $\xi,\psi\in E$, the following inequality holds:

$$\langle\Phi\xi-\Phi\psi,\xi-\psi\rangle\geq k\|\xi-\psi\|^2.$$

- A map $\Phi : E \to E$ is called accretive if for each $\xi, \psi \in E$, there exists $j(\xi - \psi) \in J(\xi - \psi)$ such that
$$\langle \Phi\xi - \Phi\psi, j(\xi - \psi)\rangle \geq 0.$$
- Φ is called strongly accretive if there exists $k \in (0, 1)$ such that for each $\xi, \psi \in E$, there exists $j(\xi-\psi) \in J(\xi-\psi)$ such that
$$\langle \Phi\xi - \Phi\psi, j(\xi - \psi)\rangle \geq k\|\xi - \psi\|^2.$$

It is well known that

- E is smooth if and only if J is single-valued.
- If E is uniformly smooth then J is uniformly continuous on bounded subsets of E.
- If E is reflexive and strictly convex dual then J^{-1} is single-valued, one-to-one, surjective, uniformly continuous on bounded subsets and it is the duality mapping from E^* into E and $J^{-1}J = I_E$ and $JJ^{-1} = I_E$.
- J^{-1} is uniformly continuous if and only if it has a modulus of continuity.

Let E be a smooth real Banach space with dual space E^*. The function $\phi : E \times E \to \mathbb{R}$, defined by
$$\phi(\xi, \psi) = \|\xi\|^2 - 2\langle \xi, J\psi\rangle + \|\psi\|^2,\ \xi, \psi \in E, \tag{2.2}$$
where J is the normalized duality mapping from E into E^*, introduced by Alber has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi[14], Reich[23] and a host of other authors. This functional ϕ will play a central role in what follows. If $E = H$, a real Hilbert space, then relation (2.2) reduces to $\phi(\xi, \psi) = \|\xi - \psi\|^2$ for $\xi, \psi \in H$. It is obvious from the definition of the function ϕ that
$$(\|\xi\| - \|\psi\|)^2 \leq \phi(\xi, \psi) \leq (\|\xi\| + \|\psi\|)^2\ \forall\, \xi, \psi \in E. \tag{2.3}$$

Let $V : E \times E^* \to \mathbb{R}$ be the functional defined by:
$$V(\xi, \xi^*) = \|\xi\|^2 - 2\langle \xi, \xi^*\rangle + \|\xi^*\|^2,\ \forall\, \xi \in E, \xi^* \in E^*. \tag{2.4}$$

Then, one can observe that
$$V(\xi, \xi^*) = \phi(\xi, J^{-1}\xi^*)\ \forall\, \xi \in E,\ \xi^* \in E^*. \tag{2.5}$$

Lemma 2.3 (Alber, [1]). Let X be a reflexive strictly convex and smooth real Banach space with X^* as its dual. Then,
$$V(\xi, \xi^*) + 2\langle J^{-1}\xi^* - \xi, \psi^*\rangle \leq V(\xi, \xi^* + \psi^*) \tag{2.6}$$
for all $\xi \in X$ and $\xi^*, \psi^* \in X^*$.

From the definition of ϕ and inequality (2.3), we can observe that for all $\xi, \psi \in E$, $\phi(\psi, \xi) \geq 0$ and
$$2\langle \xi - \psi, J\xi - J\psi\rangle - \phi(\xi, \psi) = \phi(\psi, \xi).$$

This leads to the following.

Lemma 2.4. Let E be a smooth real Banach space. Then, for all $x, y \in E$, the following holds
$$\phi(\xi, \psi) \leq 2\langle J\psi - J\xi, \psi - \xi\rangle.$$

Similarly, if E is a reflexive smooth and strictly convex real Banach space, we introduce the functional $\phi_* : E^* \times E^* \to \mathbb{R}$, defined by:

$$\phi_*(\xi^*, \psi^*) = \|\xi^*\|^2 - 2\langle \xi^*, J^{-1}\psi^*\rangle + \|\psi^*\|^2,\ \xi^*, \psi^* \in E^*, \tag{2.7}$$

and the functional $V_* : E^* \times E \to \mathbb{R}$ defined from $E^* \times E$ to $\mathbb{R}$ by:

$$V_*(\xi^*, \xi) = \|\xi^*\|^2 - 2\langle \xi^*, \xi\rangle + \|\xi\|^2,\ \xi \in E, \xi^* \in E^*. \tag{2.8}$$

It is easy to see that

$$V_*(\xi^*, \xi) = \phi_*(\xi^*, J\xi)\ \ \forall \xi \in E,\ \xi^* \in E^*. \tag{2.9}$$

In what follows, the product space $E \times E^*$ is equiped with the following norm:

$$\|w_1 - w_2\| = \Big(\|\xi - \psi\|^2 + \|\xi^* - \psi^*\|^2\Big)^{\frac{1}{2}}\ \ \forall w_1 = (\xi, \xi^*) \in E \times E^*,\ w_2 = (\psi, \psi^*) \in E \times E^*.$$

Finally, we introduce the functional $\psi : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ defined by:

$$\psi(w_1, w_2) := \phi(\xi, \psi) + \phi_*(\xi^*, \psi^*)\ \ \forall\ w_1 = (\xi, \xi^*) \in E \times E^*,\ w_2 = (\psi, \psi^*) \in E \times E^*. \tag{2.10}$$

The following results will be useful.

Lemma 2.5 (Kamimura and Takahashi, [14]). Let E be a smooth and uniformly convex real Banach space, and let $\{\xi_n\}$ and $\{\psi_n\}$ be two sequences of E. If either $\{\xi_n\}$ or $\{\psi_n\}$ is bounded and $\phi(\xi_n, \psi_n) \to 0$ as $n \to \infty$, then $\|\xi_n - \psi_n\| \to 0$ as $n \to \infty$.

Lemma 2.6 (Tan and Xu, [31]). Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:

$$a_{n+1} \le a_n + \sigma_n\ \ \forall n \ge 0.$$

Assume that $\sum_{n=0}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

3 Main Result

We now prove the following result

Theorem 3.1. Let $p \ge 2$, let E be a 2 uniformly smooth and p uniformly convex real Banach space and let a mapping $\Phi : E \to E^*$ be lipschitz, and strongly monotone such that $\Phi^{-1}(0) \neq \emptyset$. For an arbitrary $(\{\xi_1\}, \{\psi_1\}) \in E$, we define the sequences $\{\xi_n\}$ and $\{\psi_n\}$ by

$$\begin{cases} \psi_{n+1} = J^{-1}(J\xi_n - \theta_n \Phi\xi_n), & n \ge 0 \\ \xi_{n+1} = J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}), & n \ge 0 \end{cases} \tag{3.1}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0, \Lambda_p)$ where $\Lambda_p > 0$, then ξ_n and ψ_n converges strongly to ξ^*, a unique solution of the equation $\Phi\xi = 0$

i) $\lambda_n + \theta_n = \dfrac{1}{2}$.

ii) $\lim_{n\to\infty} \lambda_n = 0, \quad \lim_{n\to\infty} \theta_n = \dfrac{1}{2}$.

iii) $\sum_{n=0}^{\infty} \lambda_n < \infty$.

iv) $\lim_{n\to\infty} (\lambda_n k \theta_n \delta) = 0$.

Remark 3.1. Real sequences that satisfy conditions *(i)-(iv)* are $\lambda_n = (2(n+1))^{-1}$ *and* $\theta_n = n(2(n+1))^{-1}$.

Proof . Letting L and k the Lipschitz and strongly monotone constants of A. The proof is in two steps.

Step 1 . We prove that $\{\xi_n\}$ and $\{\psi_n\}$ is bounded. The proof is by induction. Let $\xi^* \in \Phi^{-1}(0)$. Then there exists $r > 0$ such that $\phi(\xi^*, \xi_1) = r$. Suppose that $\phi(\xi^*, \xi_n) = r$ for some $n = 1$. We prove that $\phi(\xi^*, \xi_{n+1}) = r$. By construction, $\phi(\xi^*, \xi_1) = r$. From the Lipschitz property on bounded sets of J^{-1} (Lemma 2.3) and the boundedness of A, there exists a positive constant M_1 and M_2 such that

$$\|J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}) - J^{-1}(J\psi_{n+1})\| \leq \lambda_n M_1 \|\Phi\psi_{n+1}\|, \quad \forall \lambda_n \in (0,1), \xi \in E : \phi(\xi*, \xi) \leq r \tag{3.2}$$

$$\|J^{-1}(J\xi_n - \theta_n \Phi\xi_n) - J^{-1}(J\xi_n)\| \leq \theta_n M_2 \|\Phi\xi_n\|, \quad \forall \theta_n \in (0,1), \xi \in E : \phi(\xi*, \xi) \leq r \tag{3.3}$$

$$\Lambda_p = \min\Big\{1, \frac{k}{2M_1 L^2}, \frac{k}{2M_2 L^2}\Big\}.$$

Now with lemma 2.3 and 3.1, we compute the following

$$\begin{aligned}
\phi(\xi^*, \xi_{n+1}) &= \phi(\xi^*, J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1})) \\
&= V(\xi^*, J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}) \\
&\leq V(\xi^*, J\psi_{n+1}) - 2\lambda_n \langle J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}) - \xi^*, \Phi\psi_{n+1} - \Phi\xi^* \rangle \\
&= V(\xi^*, J\psi_{n+1}) - 2\lambda_n \langle \psi_{n+1} - \xi^*, \Phi\psi_{n+1} - \Phi\xi^* \rangle \\
&\quad +2\lambda_n \langle \psi_{n+1} - \xi^*, \Phi\psi_{n+1} - \Phi\xi^* \rangle - 2\lambda_n \langle J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}) - J^{-1}(J\psi_{n+1}), \Phi\psi_{n+1} - \Phi\xi^* \rangle \\
&\leq \phi(\xi^*, \psi_{n+1}) - 2\lambda_n k \|\psi_{n+1} - \xi^*\| \\
&\quad +2\lambda_n \|J^{-1}(J\psi_{n+1} - \lambda_n \Phi\psi_{n+1}) - J^{-1}(J\psi_{n+1})\| \|\Phi\psi_{n+1} - \Phi\xi^*\|.
\end{aligned}$$

Using the strong monotonicity, Lipschitz property of Φ, inequality 3.2 and definition of Λ_p

$$\begin{aligned}
\phi(\xi^*, \xi_{n+1}) &\leq \phi(\xi^*, \psi_{n+1}) - 2\lambda_n k \|\psi_{n+1} - \xi^*\|^2 + 2\lambda_n^2 M_1 L^2 \|\psi_{n+1} - \xi^*\|^2 \\
&\leq \phi(\xi^*, \psi_{n+1}) - \lambda_n k \|\psi_{n+1} - \xi^*\|^2.
\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}
\phi(\xi^*, \psi_{n+1}) &= \phi(\xi^*, J^{-1}(J\xi_n - \theta_n \Phi\xi_n)) \\
&= V(\xi^*, J\xi_n - \theta_n \Phi\xi_n) \\
&\leq V(\xi^*, J\xi_n) - 2\theta_n \langle J^{-1}(J\xi_n - \theta_n \Phi\xi_n) - \xi^*, \Phi\xi_n - \Phi\xi^* \rangle \\
&= V(\xi^*, J\xi_n) - 2\theta_n \langle \xi_n - \xi^*, \Phi\xi_n - \Phi\xi^* \rangle \\
&\quad +2\theta_n \langle \xi_n - \xi^*, \Phi\xi_n - \Phi\xi^* \rangle - 2\theta_n \langle J^{-1}(J\xi_n) - \theta_n \Phi\xi_n - J^{-1}(J\xi_n), \Phi\xi_n - \Phi\xi^* \rangle \\
&\leq \phi(\xi^*, \xi_n) - 2\theta_n k \|\xi_n - \xi^*\| \\
&\quad +2\theta_n \|J^{-1}(J\xi_n - \theta_n \Phi\xi_{n+1}) - J^{-1}(J\xi_n)\| \|\Phi\xi_n - \Phi\xi^*\|.
\end{aligned}$$

Using the strong monotonicity, Lipschitz property of Φ, inequality 3.3 and definition of Λ_p

$$\begin{aligned}
\phi(\xi^*, \psi_{n+1}) &\leq \phi(\xi^*, \xi_n) - 2\theta_n k \|\xi_n - \xi^*\|^2 + 2\theta_n^2 M_2 L^2 \|\xi_n - \xi^*\|^2 \\
&\leq \phi(\xi^*, \xi_n) - \theta_n k \|\xi_n - \xi^*\|^2.
\end{aligned} \tag{3.5}$$

Now substituting 3.5 in 3.4, we have

$$\phi(\xi^*, \xi_{n+1}) \leq \phi(\xi^*, \xi_n) - \theta_n k \|\xi_n - \xi^*\|^2 - \lambda_n k \|\psi_{n+1} - \xi^*\|^2.$$

Using the fact that

$$\begin{aligned}
\|\psi_{n+1} - \xi^*\|^2 &= \|J^{-1}(J\xi_n - \theta_n \Phi\xi_n) - \xi^*\|^2 \\
&\leq \|J^{-1}(J\xi_n) - J^{-1}(J\xi^*)\|^2 + \|J^{-1}(J\xi^*) - J^{-1}(J\xi_n)\|^2 \\
&\quad +\theta_n \|J^{-1}\Phi\xi_n\|^2 + \|J^{-1}(J\xi_n) - \xi^*\|^2
\end{aligned} \tag{3.6}$$

Since $J^{-1}J = I_E$ and letting $\delta = \sup\left\{\|J^{-1}\Phi\xi_n\|^2\right\}$, we have the following estimate

$$\|\psi_{n+1} - \xi^*\|^2 \quad \leq \quad \theta_n\delta + \|\xi_n - \xi^*\|^2.$$

Putting 3.7 into 3.6, gives

$$\begin{aligned}
\phi(\xi^*, \xi_{n+1}) &\leq \phi(\xi^*, \xi_n) - \theta_n k\|\xi_n - \xi^*\|^2 \\
&- \lambda_n k\theta_n\delta - \lambda_n k\|\xi_n - \xi^*\|^2 \\
&\leq \phi(\xi^*, \xi_n) - (\theta_n k + \lambda_n k)\|\xi_n - \xi^*\|^2 - \lambda_n k\theta_n\delta \\
&\leq \phi(\xi^*, \xi_n) - \frac{1}{2}k\|\xi_n - \xi^*\|^2 - \lambda_n k\theta_n\delta \\
&\leq \phi(\xi^*, \xi_n) - \frac{1}{2}k\|\xi_n - \xi^*\|^2 < r.
\end{aligned}$$

Hence, by induction, $\{\xi_n\}$ and $\{\psi_n\}$ are bounded.

Step 2 We now prove that $\{\xi_n\}$ and $\{\psi_n\}$ converges strongly to $\xi^* \in \Phi^{-1}(0)$. With the same computation as above, we have that the following

$$\phi(\xi^*, \xi_{n+1}) \leq \phi(\xi^*, \xi_n) - \frac{1}{2}k\|\xi_n - \xi^*\|^2 - \lambda_n k\theta_n\delta$$

which implies that $\lim \phi(\xi^*, \xi_n)$ exists. Therefore,

$$0 \leq \lim_{n\to\infty}\left(\frac{1}{2}k\|\xi_n - \xi^*\|^2\right) \leq \lim_{n\to\infty}\phi(\xi^*, \xi_n) - \lim_{n\to\infty}\phi(\xi^*, \xi_{n+1}) - \lim_{n\to\infty}\left(\lambda_n k\theta_n\delta\right) = 0$$

Therefore $\{\xi_n\} \to \xi^*$ and $\{\psi_n\} \to \xi^*$ as $n \to \infty$.

□

Corollary 3.2. For $E = L_p, 2 \leq p < \infty$, and $\Phi : E \to E^*$ be a Lipschitz, and strongly monotone mapping such that $\Phi^{-1}(0) \neq \emptyset$. For arbitrary $(\xi_1, \psi_1) \in E$, define the sequence $\{\xi_n\}$ and $\{\psi_n\}$ iteratively by

$$\begin{cases} \psi_{n+1} = J^{-1}(J\xi_n - \theta_n\Phi\xi_n), & n \geq 0 \\ \xi_{n+1} = J^{-1}(J\psi_{n+1} - \lambda_n\Phi\psi_{n+1}), & n \geq 0 \end{cases} \tag{3.7}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0, \delta_p)$ where $\delta_p > 0$, then ξ_n and ψ_n converges strongly to ξ^*, a unique solution of the equation $\Phi\xi = 0$.

Proof . Since $E = L_p$ spaces, $2 \leq p < \infty$, are $2-$uniformly smooth and $p-$uniformly convex real Banach spaces, then the proof follows from Theorem 3.1. □

4 Convex minimization problem

Now, we present a convex minimization problem for a convex function $\nabla : E \to \mathbb{R}$. The following results are well known.

Remark 4.1. Let $\nabla : E \to \mathbb{R}$ be a differentiable convex function and $\eta^* \in E$, then the point η^* is a minimizer of ∇ on E if and only if $d\nabla(\eta^*) = 0$.

Definition 4.2. A function $\nabla : E \to \mathbb{R}$ is said to be strongly convex if there exists $\gamma > 0$ such that the following condition holds:

$$\nabla(\beta\xi + (1-\beta)\psi) \leq \beta\nabla\xi + (1-\beta)\nabla\psi - \gamma\|\xi - \psi\|^2 \tag{4.1}$$

for all $\xi, \psi \in E$ with $\xi \neq \psi$ and $\beta \in (0, 1)$,

Lemma 4.3. Let E be normed linear space and $\nabla : E \to \mathbb{R}$ a convex differentiable function. Suppose that ∇ is strongly convex. Then the differential map $d\nabla : E \to E^*$ is strongly monotone, i.e., there exists $k > 0$ such that

$$\langle d\nabla\xi - d\nabla\psi, \xi - \psi\rangle \geq k\|\xi - \psi\|^2 \;\forall\, \xi, \psi \in E. \tag{4.2}$$

Now we present the following result.

Theorem 4.4. Let $d\nabla : E^* \to E$ be a L-Lipschitz continuous and monotone mapping such that $d\nabla^{-1}(0) \neq \emptyset$. For given $\xi_1, \psi_1 \in E$, define the sequence $\{\xi_n\}$ and $\{\psi_n\}$ as follows:

$$\begin{cases} \psi_{n+1} = J^{-1}(J\xi_n - \theta_n d\nabla\xi_n), & n \geq 0 \\ \xi_{n+1} = J^{-1}(J\psi_{n+1} - \lambda_n d\nabla\psi_{n+1}), & n \geq 0. \end{cases} \tag{4.3}$$

where J is the normalized duality mapping from E into E^* and the sequences $\{\lambda_n\}$ *and* $\{\theta_n\}$, are in the interval $[0, 1]$ satisfying assumptions (i) to (iv). Then ∇ has a unique minimizer $\xi^* \in E$ and there exists a positive real number δ_p such that if $\Big(\lambda_n, \theta_n\Big) \in (0, \delta_p)$, the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to ξ^*.

Proof . From Remark 4.1 it follows that ∇ has a unique minimizer ξ^* and is obtained by $d\nabla(\xi^*) = 0$. From Lemma 4.3 and using the fact that the differential mapping $d\nabla : E \to E^*$ is Lipschitz, considering the result of Theorem 3.1 we can complete the proof. □

Conclusion

In this paper, we proposed and analyzed the strong convergence theorem of two step size Krasnoselskii-type algorithm introduced by Chidume et al.[6] and prove a strong convergence theorem to approximate the unique zero of a Lipschitz strongly monotone mapping 2−uniformly smooth and p−uniformly convex real Banach space for $p \geq 2$. This class of Banach spaces contains all Lp-spaces, $2 \leq p < \infty$. Then we apply our results to the convex minimization problem. We also complemented and generalized previous worked been done under this setting.

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