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Multiple solutions for a class of nonlinear elliptic equations on Carnot groups

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Abstract

In this paper, using variational methods and critical point theory we establish the existence of multiple solutions for a class of elliptic equations on Carnot groups depending on one real positive parameter and involving a subcritical nonlinearity. Some recent results are extended and improved.

Keywords: Variational methods, Heisenberg group, Folland-Stein space
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1 Introduction

In this paper, we study the existence of two and infinitely many weak solutions for the following problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda f(\xi, u), & \text{in } D \\ u|_{\partial D} = 0, \end{cases} \tag{1.1}$$

where D is a smooth bounded domain of the Carnot group $\mathbb{G}$, $\Delta_{\mathbb{G}}$ is the subelliptic Laplacian on $\mathbb{G}$, and λ is a positive real parameter. Study on Carnot-Carathèodory (briefly CC) spaces can be considered as an field which presently experiencing vast improvement. In these abstract structures, the interactions between analytical and geometric tools have been performed with prosperous results and they are a particular category of metric spaces. In this situation, Carnot groups play a primary role, as it is famous, they are finite dimensional, simply connected Lie groups $\mathbb{G}$ whose Lie algebra g of left invariant vector fields is stratified (see Section 2). Approximately Carnot groups can be recognized as local models of CC spaces. In fact, they are the natural digression spaces to CC spaces, exactly as Euclidean spaces are digression to manifolds. It is distinguished that many authors have focused on the study of subelliptic equations on Carnot groups and especially, on the Heisenberg group $\mathbb{H}^n$, see the papers [2, 1, 5, 9, 8, 10, 15, 12, 14, 13] and references therein. For example, in [13] Molica Bisci and Ferrara, by using variational methods and a direct consequence of the celebrated Pucci-Serrin theorem, have established the existence of at least two weak solutions for the problem (1.1), and especially Ferrara et.al [8] have proved the existence of at least one nontrivial solution by using variational methods for the problem (1.1).

In the present paper, we use a variational method to prove the existence of solutions for the problem (1.1) under suitable conditions imposed on f (see, the conditions (f_0), (f_1) and (f_2) in Theorem 3.3 and the condition (f_0) in

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Theorem 3.4). In Theorem 3.3 we establish the existence of at least two weak solutions for the problem (1.1), while in Theorem 3.4 we discuss the existence of infinitely many weak solutions for the problem (1.1). The key used theorems here are completely different from the theorems used in [13]. In addition, we also prove the existence of infinitely many weak solution for the problem (1.1).

The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main results of the paper. Then, we give two examples to illustrate our results.

2 Preliminaries and Basic Notation

We now introduce a few basic notations and definitions about Carnot groups. A Carnot group G of step $r \geq 1$ is a simply connected nilpotent Lie group whose Lie algebra g is stratified. This means that g admits a decomposition as a vector space sum

$$g = \bigoplus_{k=1}^{r} g_k$$

such that

$$[g_1, g_i] = g_{i+1}, \text{ for } 1 < i \leq r-1,$$

and

$$[g_1, g_i] = \{0\} \text{ for } i > r.$$

Note that g is generated as a Lie algebra by g_1. The exponential map is a diffeomorphism from g onto $\mathbb{G}$. Using these exponential coordinates, consider $(\mathbb{R}^n, \circ)$, where $n = \dim g_1 + ... + \dim g_r$ and the operation $\circ$ is given by the Baker-Campbell-Hausdorff formula. We suppose $\mathbb{R}^n$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\mu\}_{\mu>0}$ (called dilations) of the form

$$\delta_\mu(\xi^{(1)}, \xi^{(2)}, ..., \xi^{(r)}) := (\mu^1\xi^{(1)}, \mu^2\xi^{(2)}, ..., \mu^r\xi^{(r)}),$$

where $\xi^k \in \mathbb{R}^{n_k}$ for every $k \in \{1, ..., r\}$ and $\sum_{k=1}^{r} n_k = n$.

The structure $\mathbb{G} := (\mathbb{R}^n, \circ, \delta_\mu)$ is called a homogeneous group with homogeneous dimension

$$\dim_h \mathbb{G} = \sum_{k=1}^{r} kn_k. \tag{2.1}$$

In this paper, we let $\dim_h \mathbb{G} \geq 3$. A Carnot group is a homogeneous group $\mathbb{G}$ such that the Lie algebra g associated to $\mathbb{G}$ is stratified. Moreover, the subelliptic Laplacian operator on $\mathbb{G}$ is the second order differential operator, given by

$$\Delta_{\mathbb{G}} := \sum_{k=1}^{n_1} X_{n_k}^2$$

where $\{X_1, ..., X_{n_1}\}$ is a basis of g_1 and

$$\Delta_{\mathbb{G}} := (X_1, ..., X_{n_1})$$

the related horizontal gradient. The following Sobolev-type inequality plays a crucial role in the functional analysis on Carnot groups:

$$\int_D |u(\xi)|^{2^*} d\xi \leq C \int_D |\nabla_{\mathbb{G}} u(\xi)|^2 d\xi, \quad \forall u \in C_0^\infty, \tag{2.2}$$

(see [6]). In the above expression, C is a positive constant (independent of u) and

$$2^* = \frac{2 \dim_h \mathbb{G}}{\dim_h \mathbb{G} - 2}$$

is the critical Sobolev exponent. Inequality (2.2) ensures that if D is a bounded open (smooth) subset of $\mathbb{G}$, then the function

$$u \to \|u\|_{S_0^1} := (\int_D |\nabla_{\mathbb{G}} u(\xi)|^2 d\xi)^{\frac{1}{2}} \tag{2.3}$$

is a norm in $C_0^\infty(D)$.

We will mean by $S_0^1(D)$ the Folland-Stein space defined as the completion of $C_0^\infty(D)$ with respect to the norm $\|.\|_{S_0^1(D)}$. The exponent 2^* is critical for $\Delta_{\mathbb{G}}$ since, as in the classical Laplacian setting, the embedding $S_0^1(D) \hookrightarrow L^q(D)$ is compact when $1 \leq q < 2^*$, while it is only continuous if $q = 2^*$ (see Folland and Stein [7]).

Definition 2.1. We say $u : D \to \mathbb{R}$ where $u \in X$ is a weak solution of (1.1), if

$$\int_D < \nabla_{\mathbb{G}} u(\xi), \nabla_{\mathbb{G}} u(\xi) > d\xi = \lambda \int_D f(\xi, u(\xi))v(\xi)d\xi, \quad \forall v \in S_0^1(D) \tag{2.4}$$

Weak solutions to the problem (1.1) mean the critical points of the associated energy functional J_λ acting on the space $S_0^1(D)$. We consider the functional $J_\lambda : S_0^1(D) \to \mathbb{R}$ denoted by

$$J_\lambda = \frac{1}{2}\|u\|^2_{S_0^1(D)} - \int_D F(x, u)dx \quad \forall u \in S_0^1(D) \tag{2.5}$$

where $\lambda \in \mathbb{R}$, and $F(\xi, t) = \int_0^t f(\xi, \tau)d\tau$. Now, under our growth condition on f the functional $J_\lambda \in C^1(S_0^1(D))$ and its derivative at $u \in S_0^1(D)$ is defined by

$$< J'_\lambda(u), v >= \int_D < \nabla_{\mathbb{G}} u(\xi), \nabla_{\mathbb{G}} v(\xi) > d\xi - \lambda \int_D f(\xi, u(\xi))v(\xi)d\xi, \quad \forall v \in S_0^1(D) \tag{2.6}$$

for all $v \in S_0^1(D)$. Therefore, the weak solutions of the problem (1.1) are the critical points of the energy functional J_λ.

Definition 2.2. Consider E to be a real reflexive Banach space. If any sequence $\{u_k\} \subset E$ for which $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence. Then we say J satisfies Palais-Smale condition (denoted by PS condition in short).

The proofs of our theorems are based on Theorems 2.3 and 2.4 below.

Theorem 2.3. [16, Theorem 4.10] Assume $J \in C^1(X, \mathbb{R})$, and J satisfies the PS condition. Let that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and

$$\inf_{u \in \partial\Omega} J(u) > \max\{J(u_0), J(u_1)\},$$

then there exists a critical point u of J, i.e. $J'(u) = 0$ with $J(u) > \max\{J(u_0), J(u_1)\}$.

Theorem 2.4. [17, Theorem 9.12] Let E be an infinite dimensional real Banach space. Let $J \in C^1(E, \mathbb{R})$ be an even functional which satisfies the (PS)-condition, and $J(0) = 0$. Consider that $E = V \bigoplus X$, where V is finite dimensional, and J satisfies that

- (i_1) There exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$;
- (i_2) For any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $J(u) \leq 0$ on $W \setminus B_R$.

Then J possesses an unbounded sequence of critical values.

Theorem 2.5. [18, Theorem 38] For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset, \min_{u \in M} F(u) = \alpha$ has a solution in case the following conditions hold:

- (i_3) X is a real reflexive Banach space, (i$_4$) M is bounded and weak sequentially closed,

(i$_5$) F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leq \lim_{n\to\infty} \inf F(u_n)$ holds.

We refer to the papers [3, 19] in which Theorems 2.3 and 2.4 have been applied to obtain the existence of multiple solutions for some boundary value problems. Moreover, in the paper [20], Theorem 2.4 has been successfully applied to obtain the existence of infinitely many solutions for a boundary value problem.

3 Main results

We take the following assumptions on the function f:

(f_0) there exist constants $\nu > 2$ and $T > 0$ such that $0 < \nu F(x,t) \leq tf(x,t)$, $|t| > T$.

(f_1) $f : \overline{D} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathèodory condition and there exists $c > 0$ such that

$$|f(x,t)| \leq c(1 + |t|^{q-1}) \quad for \quad t \in \mathbb{R},$$

where $q \in (2, 2^*)$ and $x \in \overline{D}$.

(f_2) $f(x,t) = o(|t|)$, $t \longrightarrow 0$, for $x \in \overline{D}$ uniformly.

We need the following lemmas to prove our main results.

Lemma 3.1. If (f_0) holds. Then $J_\lambda(u)$ satisfies the (PS)-condition.

Proof . To prove the lemma, we use [4, Lemma 2.4]. Let that $\{u_n\}$ be a sequence in X such that $\{J_\lambda(u_n)\}$ is bounded and $J'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|J_\lambda(u_n)| \leq c_0$ and $|J'_\lambda(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Therefore, by the assumptions (f_0) and definition of J'_λ, we have

$$\begin{aligned}
c_0 + \|u_n\|_{S_0^1(D)} &\geq J_\lambda(u_n) - \frac{1}{\nu} J'_\lambda(u_n)(u_n) \\
&\geq (\frac{1}{2\lambda} - \frac{1}{\lambda\nu})\|u\|^2_{S_0^1(D)} + \int_\Omega \left(\frac{1}{\nu} f(x,u_n)u_n - F(x,u_n) \right) dx \\
&\geq (\frac{1}{2\lambda} - \frac{1}{\lambda\nu})\|u\|^2_{S_0^1(D)}
\end{aligned}$$

Since $\nu > 2$, this implies that $\{u_n\}$ is bounded. By using the same argument given in [4, Lemma 2.4], it can easily be proved that $\{u_n\}$ converges strongly to u in X. Overall, this implies J_λ satisfies the (PS)-condition. □

Lemma 3.2. ([21, Lemma 2.2]) If condition (f_0) holds, then for every $x \in D$, the following inequalities hold:

$$\begin{aligned}
F(x,t) &\leq F(x, \frac{t}{|t|})|t|^\nu, if \ \ 0 < |t| \leq 1; \\
F(x,t) &\geq F(x, \frac{t}{|t|})|t|^\nu, if \ \ |t| \geq 1.
\end{aligned}$$

In view Lemma 3.2, (f_0) implies that for every $x \in D$,

$$F(x,t) \leq a_3|t|^\nu, if \ \ |t| \leq 1$$

and

$$F(x,t) \geq a_1|t|^\nu, if \ \ |t| \geq 1 \tag{3.1}$$

where $a_3 = \max_{x\in D,|t|=1} F(x,t)$ and $a_1 = \min_{x\in D,|t|=1} F(x,t)$. Assumption (f_0) implies $a_1, a_3 > 0$. In addition, since $F(x,t) - a_1|t|^\nu$ is continuous on $D \times [0,T]$, there exists a constant $a_2 > 0$ such that

$$F(x,t) \geq a_1|t|^\nu - a_2 \ \ for \ \ all \ \ (x,t) \in D \times [0,T]. \tag{3.2}$$

Then, it follows from (3.1) and (3.2) that

$$F(x,t) \geq a_1|t|^\nu - a_2 \ \ for \ \ all \ \ (x,t) \in D \times \mathbb{R}. \tag{3.3}$$

Now, we state our main results as follows.

Theorem 3.3. Suppose that (f_0), (f_1) and (f_2) hold. Then, if $f(x,t) \geq 0$ for all $(x,t) \in \overline{D} \times \mathbb{R}$, the problem (1.1) has at least two weak solutions.

Proof . By Lemma 3.1 we know that J satisfies the (PS)-condition, also from the definition of J_λ implies that $J_\lambda(0) = 0$.

Step 1. We shall prove that there exists $M > 0$ such that the functional J_λ has a local minimum $u_0 \in B_M = \{u \in X; \|u\|_{S_0^1(D)} < M\}$. To show this, we will apply Mazur's lemma (see, e.g., [11]) which states that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit. Let $\{u_n\} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to \infty$, then there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \quad \sum_{j=1}^{n} a_{n_j} = 1, \qquad a_{n_j} \geq 0, \ j \in N$$

such that $v_n \to u$ in X. Since $\overline{B}_M$ is a closed convex set, we have $\{v_n\} \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Since J_λ is weak sequentially lower semi-continuous on $\overline{B}_M$, and X is a reflexive Banach space, then by Theorem 2.5 we can imply that J_λ has a local minimum $u_0 \in \overline{B}_M$.

Now, we assume that $J_\lambda(u_0) = \min_{u\in\overline{B}_M} J(u)$, and show that

$$J_\lambda(u_0) < \inf_{u\in\partial B_M} J(u).$$

We have the embedding $X \hookrightarrow L^2(D)$ and $X \hookrightarrow L^q(D)$ which means that there exists $c_2, c_q > 0$ such that $|u|_2 \leq c_2\|u\|_{s_0^1(D)}$ and $|u|_q \leq c_q\|u\|_{s_0^1(D)}$, $\forall u \in X$. Let $\varepsilon > 0$ be small enough such that $\varepsilon c_2 < \frac{1}{4\lambda}$, by the assumptions (f_1) and (f_2), we have

$$F(x,t) \leq \varepsilon|t|^2 + c|t|^q \ \text{ for } \ (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.4}$$

Then, from (3.4) it reads

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{2\lambda}\|u\|^2_{s_0^1(D)} - \varepsilon\int_D |u|^2 dx - c\int_D |u|^q dx \\
&\geq \frac{1}{2\lambda}\|u\|^2_{s_0^1(D)} - \varepsilon c_2\|u\|^2_{S_0^1(D)} - cc_q\|u\|^q_{S_0^1(D)} \\
&\geq \frac{1}{4\lambda}\|u\|^2_{s_0^1(D)} - cc_q\|u\|^q_{S_0^1(D)}, \ \text{ when } \|u\|_{S_0^1(D)} < 1
\end{aligned}$$

Since $q > 2$, therefore, there exist $r, \delta > 0$ such that $J_\lambda(u) \geq \delta > 0$ for every $\|u\|_{S_0^1(D)} = r < 1$. If we let $M = r$, then $J_\lambda(u) > 0 = J_\lambda(0) \geq J_\lambda(u_0)$ for $u \in \partial B_M$. Hence $u_0 \in B_M$ and $J'_\lambda(u_0) = 0$.

Step 2. Since u_0 is a minimum point of J_λ on X, we can assume $M > 0$ be sufficiently large such that $J_\lambda(u_0) \leq 0 < \inf_{u\in\partial B_M} J_\lambda(u)$, where $B_M = \{u \in X; \|u\|_{S_0^1(D)} < M\}$.
Now we will show that there exists $u_1 \in X$ with $\|u_1\|_{S_0^1(D)} > M$ such that $J_\lambda(u_1) < \inf_{\partial B_M} J_\lambda(u)$. To prove this claim, consider $e_1(x) \in X$ and $u_1 = \gamma e_1, \gamma > 0$ and $\|e_1\|_{S_0^1(D)} = 1$. By (f_0) and (3.4) there exist constants $a_1, a_2 > 0$ such that $F(x,t) \geq a_1|t|^\nu - a_2$ for all $x \in \overline{D}$, $|t| \geq T$. Thus

$$J_\lambda(u_1) = \frac{1}{2\lambda}\|\gamma e_1\|^2_{S_0^1(D)} - \int_\Omega F(x,\gamma e_1)dx \leq \frac{1}{2\lambda}\|\gamma e_1\|^2_{S_0^1(D)} - a_1\gamma^\nu\int_\Omega |e_1|^\nu dx + a_2.$$

Since $\nu > 2$, there exists sufficiently large γ such that $\gamma > M > 0$ which infers $J_\lambda(\gamma e_1) < 0$. Hence, $\inf_{\partial B_M} J_\lambda(u) > \max\{J_\lambda(u_0), J_\lambda(u_1)\}$. Then, Theorem 2.3 assures the existence of the second critical point u^*. Therefore, u_0, u^* are two critical points of J_λ, which are two nontrivial solutions of the problem (1.1). □

Theorem 3.4. Suppose that (f_0) holds. Then, if $f(x,t)$ is odd in t, the problem (1.1) has infinitely many weak solutions.

Proof . By definition J_λ we infer that J_λ is even and $J_\lambda(0) = 0$. The rest of the proof is split into two steps:

Step 1. Since its proof is straightforward, we only depict briefly how J_λ satisfies condition (i_1) in Theorem 2.4. Since, J_λ is coercive and also satisfies (PS)-condition, by the minimization theorem [16, Theorem 4.4], the functional J_λ has a minimum critical point $u \in X$ with $J_\lambda(u) \geq \alpha > 0$ and $\|u\|_{S_0^1(D)} = \rho$ for $\rho > 0$ small enough.

Step 2. Now, we will show that J satisfies condition (i_2) in Theorem 2.4. Let $W \subset X$ be a finite dimensional subspace. Any non-zero vector $u \in W$ has a unique representation $u = \theta e_2$, where $\theta = \|u\|_{S_0^1(D)}$ and $\|e_2\|_{S_0^1(D)} = 1$. Then, similar to Step 2 in the proof of Theorem 3.3, it follows

$$J_\lambda(\theta e_2) = \frac{1}{2\lambda}\|\theta e_2\|^2_{S_0^1} - \int_\Omega F(x, \theta e_2)dx \leq \frac{1}{2\lambda}\|\theta e_2\|^2_{S_0^1} - a_1\theta^\nu \int_\Omega |e_2|^\nu dx + a_2.$$

The above inequality implies that there exists θ_0 such that $\|\theta e_2\| > \rho$ and $J(\theta e_2) < 0$ for every $\theta \geq \theta_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that for all $u \in W \setminus B_R$, that is, when $\|u\|_{S_0^1(D)} \geq R$, we have $J_\lambda(u) \leq 0$. According to Theorem 2.4, the functional $J_\lambda(u)$ possesses infinitely many critical points, i.e., the problem (1.1) has infinitely many weak solutions. □

Remark 3.5. According to the condition used in Theorem 3.4 , the solutions obtained in this theorem are different from the two solutions obtained in [13].

Finally, we give two examples to illustrate the applicability of our results.

Example 3.6. Consider $\lambda = 2$ and $T = 1$, and let D be a smooth and bounded domain of a Carnot group $\mathbb{G}$ with $\dim_h \mathbb{G} = 3$ so $2^* = 6$. Let $f(x,t) = t^4 + \sin^2 t$ for all $(x,t) \in \overline{D} \times \mathbb{R}$. We have $F(x,t) = \frac{t^5}{5} + \frac{1}{2}t - \frac{1}{4}\sin 2t$ for all $(x,t) \in \overline{D} \times \mathbb{R}$. We have $f(x,t) = o(|t|), t \to 0$, and by choosing $q = 5$ and $c = 3$, we observe $|f(x,t)| < c(1 + |t|^{q-1})$ for all $t \in \mathbb{R}$. Since $\lim_{t\to\infty} \frac{tf(x,t)}{F(x,t)} = \lim_{t\to\infty} \frac{t^5 + t\sin^2 t}{\frac{t^5}{5} + \frac{1}{2}t - \frac{1}{4}\sin 2t} = 5$, by choosing $\nu = 5$, that $\nu > 2$ then $5F(x,t) \leq tf(x,t)$, so we see that the conditions (f_0), (f_1), and (f_2) are satisfied, and $f(x,t) \geq 0$. Hence, using Theorem 3.3 the problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = 2(u^4 + \sin^2 u), & \text{in } D \\ u|_{\partial D} = 0, \end{cases} \tag{3.5}$$

has at least two nontrivial weak solutions.

Example 3.7. Choose $\lambda = T = 1$, and let D be a smooth and bounded domain of a Carnot group $\mathbb{G}$ with $\dim_h \mathbb{G} = 4$ so $2^* = 4$. Let $f(x,t) = t^3 + \sin t$ for all $(x,t) \in \overline{D} \times \mathbb{R}$. By the expression of f, we have $F(x,t) = \frac{t^4}{4} - \cos t$ for all $(x,t) \in \overline{D} \times \mathbb{R}$. Since $\lim_{t\to\infty} \frac{tf(x,t)}{F(x,t)} = \lim_{t\to\infty} \frac{t^4 + t\sin t}{\frac{t^4}{4} - \cos t} = 4$, by choosing $\nu = 4$, that $\nu > 2$ we have $4F(x,t) \leq tf(x,t)$, so we see that the condition (f_0) is satisfied, and $f(x,t)$ is odd in t, therefore, applying Theorem 3.4 the problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = u^3 + \sin u, & \text{in } D \\ u|_{\partial D} = 0, \end{cases} \tag{3.6}$$

has infinitely many weak solutions.

4 Conclusion

In this article, using variational methods and critical point theory, we have proved the existence of multiple solutions for the problem (1.1) under suitable conditions imposed on the nonlinear term f. We have illustrated the results by giving convenience examples.

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