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The existence of a solution to more general proportional forms of fractional integrals via a measure of noncompactness

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Abstract

A fixed point theorem is proved using a newly constructed contraction operator in this article, and the solvability of a more general type of fractional integrals based here on the proportional derivative is analyzed. We also use suitable examples to illustrate our findings.

Keywords: Measure of noncompactness(MNC), integral equation, Fixed point
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1 Introduction

Fractional integral equations play a decisive role in real-world problems. The importance of fractional order integral equations has gained much research interest. The concept of an MNC is important in fixed point theory. Kuratowski [23] pioneered the idea of an MNC. Using the idea of an MNC, Darbo [12] established a result proving the presence of a fixed point for the so-called condensing operators in 1955. Fixed point theory and the MNC have numerous applications in analyzing various integral equations found in a wide range of real-world problems (see [3, 18, 14, 15, 17, 19, 20, 25, 13]). This theorem was highly valuable in establishing the solvability of several kinds of differential and integral equations ([6, 7, 8, 14, 16, 30], for example).

This article aims to generalize the fixed-point theorem of Darbo and apply this theorem in the control of the solvability of a fractional integral equation.

Let $(\mathfrak{Z}, \|\,.\,\|)$ be a real Banach space and $B(\theta, r) = \{z \in \mathfrak{Z} : \| z - \theta \| \leq r\}$. If $\mathfrak{E}(\neq \emptyset) \subseteq \mathfrak{Z}$. Also, $\bar{\mathfrak{E}}$ and Conv$\mathfrak{E}$ represent the closure and convex closure of $\mathfrak{E}$. Furthermore, let

- $\mathfrak{M}_{\mathfrak{Z}}$ = The collection of all non-empty and bounded subsets of $\mathfrak{Z}$,
- $\mathfrak{N}_{\mathfrak{Z}}$ = The collection of all relatively compact sets,
- $\mathbb{R} = (-\infty, \infty)$,
  and

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- $\mathbb{R}_+ = [0, \infty)$.

The definition of an MNC is as follows: [9].

Definition 1.1. A function $\Omega : \mathfrak{M}_{\mathfrak{Z}} \to [0, \infty)$ is said to be an MNC in $\mathfrak{Z}$ if it fulfills axioms:

(i) for all $\mathfrak{E} \in \mathfrak{M}_{\mathfrak{Z}}$, $\Omega(\mathfrak{E}) = 0$ gives $\mathfrak{E}$ is relatively compact.
(ii) ker $\Omega = \{\mathfrak{E} \in \mathfrak{M}_{\mathfrak{Z}} : \Omega(\mathfrak{E}) = 0\} \neq \phi$ and ker $\Omega \subset \mathfrak{N}_{\mathfrak{Z}}$.
(iii) $\mathfrak{E} \subseteq \mathfrak{E}_1 \implies \Omega(\mathfrak{E}) \leq \Omega(\mathfrak{E}_1)$.
(iv) $\Omega(\bar{\mathfrak{E}}) = \Omega(\mathfrak{E})$.
(v) $\Omega(Conv\mathfrak{E}) = \Omega(\mathfrak{E})$.
(vi) $\Omega(\chi\mathfrak{E} + (1-\chi)\mathfrak{E}_1) \leq \chi\Omega(\mathfrak{E}) + (1-\chi)\Omega(\mathfrak{E}_1)$ for $\chi \in [0,1]$.
(vii) if $\mathfrak{E}_c \in \mathfrak{M}_{\mathfrak{Z}}$, $\mathfrak{E}_c = \bar{\mathfrak{E}}_c$, $\mathfrak{E}_{c+1} \subset \mathfrak{E}_c$ for $c = 1, 2, 3, ...$ and $\lim_{c\to\infty} \Omega(\mathfrak{E}_c) = 0$ then $\bigcap_{c=1}^{\infty} \mathfrak{E}_c \neq \emptyset$.

The family $ker\Omega$ is said to be the *kernel of measure* Ω. Since $\Omega(\mathfrak{E}_\infty) \leq \Omega(\mathfrak{E}_c)$, $\Omega(\mathfrak{E}_\infty) = 0$. So, $\mathfrak{E}_\infty = \bigcap_{c=1}^{\infty} \mathfrak{E}_c \in ker\Omega$.

Some important theorems and definitions

The following are some fundamental theorems to recall:

Theorem 1.2. (Shauder [1]) Let $\mathfrak{U}$ be a non-empty, closed and convex subset of a Banach Space $\mathfrak{Z}$. Then every compactt continuous map $\mathfrak{G} : \mathfrak{U} \to \mathfrak{U}$ has at least one fixed point.

Theorem 1.3. (Darbo[12]) Let $\mathfrak{U}$ be a non-empty, bounded, closed and convex (NBCC) subset of a Banach Space $\mathfrak{Z}$. Let $\mathfrak{G} : \mathfrak{U} \to \mathfrak{U}$ be a continuous mapping and there is a constant $\chi \in [0, 1)$ such that

$$\Omega(\mathfrak{G}\mathfrak{B}) \leq \chi\Omega(\mathfrak{B}), \ \mathfrak{B} \subseteq \mathfrak{U}.$$

Then $\mathfrak{G}$ has a fixed point.

The following related concepts are needed to establish an extension of Darbo's fixed point theorem:

Definition 1.4. ([26]) Let $\Lambda_1, \Lambda_2 : [0, \infty) \to \mathbb{R}$ be the two functions. Then the pair of maps (Λ_1, Λ_2) is called a pair of shifting distance functions, if it satisfies following conditions:

1. For $x, y \in [0, \infty)$ if $\Lambda_1(x) \leq \Lambda_2(y)$ then $x \leq y$.
2. For $x_n, y_n \in [0, \infty)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z$, if $\Lambda_1(x_n) \leq \Lambda_2(y_n) \ \forall \ n$ then $z = 0$.

We denote by Λ a pair (Λ_1, Λ_2) of shifting distance functions.

As examples, we put $\Lambda_1(x) = x$, $\Lambda_2(x) = \epsilon x$, $x \geq 0$ and $\epsilon \in [0, 1)$. They are obviously a pair of shifting distance functions.

Definition 1.5. [2] A continuous function $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a function of $\mathcal{C}$- class if subsequent axioms hold true:
(1) $g(m, n) \leq m$,
(2) $g(m, n) = m$ implies that either $m = 0$ or $n = 0$. Also $g(0, 0) = 0$. A $\mathcal{C}$- class function is symbolized by $\mathcal{C}$.

For example,

(1) $g(m, n) = m - n$,

(2) $g(m, n) = am, \ 0 < a < 1$.

Definition 1.6. [22] A function $\xi : [0, \infty) \to [0, \infty)$ is an alternating distance function if:
(1) $\xi(x) = 0$ if and only if $x = 0$.
(2) ξ is continuous and increasing.

We use Ξ to denote this class of functions. For example, $\xi(x) = (1-b)x,\ 0 \le b < 1$.

Definition 1.7. [2] A continuous function $\phi : [0,\infty) \to [0,\infty)$ is an ultra altering distance function if $\phi(0) \ge 0$ and $\phi(t) > 0,\ t > 0$.

We use Φ to denote this class of functions.

Definition 1.8. A continuous function $h : [0,\infty) \to [0,\infty)$ is a function of $\mathcal{A}$ class if $h(x) > x,\ x \in (0,\infty)$. Also $h(0) = 0$.

For example, $h(x) = \bar{m}x,\ \bar{m} > 1$.

Definition 1.9. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non-decreasing mapping of $\mathcal{B}$ class if $\gamma(t) = t,\ t \ge 0$.

2 Main Results

Theorem 2.1. Let $\mathbb{U}$ be a NBCC subset of a Banach space $\mathfrak{Z}$. Also, let $\mathcal{T} : \mathbb{U} \to \mathbb{U}$ be continuous mapping with

$$\Lambda_1[h\left[\xi\left\{\mu\left(\mathcal{T}\Omega\right) + \gamma\left(\mu\left(\mathcal{T}\Omega\right)\right)\right\}\right]] \le \Lambda_2[g\left[\xi\left\{\mu\left(\Omega\right) + \gamma\left(\mu\left(\Omega\right)\right)\right\}, \phi\left\{\mu\left(\Omega\right) + \gamma\left(\mu\left(\Omega\right)\right)\right\}\right]] \tag{2.1}$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC and $(\Lambda_1, \Lambda_2) \in \Lambda,\ \phi \in \Phi,\ \xi \in \Xi,\ g \in \mathcal{C},\ h \in \mathcal{A}$ and $\gamma \in \mathcal{B}$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{U}$.

Proof . Let us create a sequence $\{\mathbb{U}_p\}_{p=1}^{\infty}$ with $\mathbb{U}_1 = \mathbb{U}$ and $\mathbb{U}_{p+1} = Conv(\mathcal{T}\mathbb{U}_p)$ for $p \in \mathbb{N}$. Also $\mathcal{T}\mathbb{U}_1 = \mathcal{T}\mathbb{U} \subseteq \mathbb{U} = \mathbb{U}_1,\ \mathbb{U}_2 = Conv(\mathcal{T}\mathbb{U}_1) \subseteq \mathbb{U} = \mathbb{U}_1$. Continuing in the similar manner gives $\mathbb{U}_1 \supseteq \mathbb{U}_2 \supseteq \mathbb{U}_3 \supseteq \ldots \supseteq \mathbb{U}_p \supseteq \mathbb{U}_{p+1} \supseteq \ldots$.

If there exists $p_0 \in \mathbb{N}$ satisfying $\mu(\mathbb{U}_{p_0}) = 0$ then $\mathbb{U}_{p_0}$ is a compact set. In this case Schauder's theorem implies $\mathcal{T}$ has a FP in $\mathbb{U}$. Let $\mu(\mathbb{C}_p) > 0,\ p \in \mathbb{N}$. Now, for $p \in \mathbb{N}$, we have

$$\begin{aligned}\Lambda_1[h[\xi\left\{\mu\left(\mathbb{U}_{p+1}\right) + \gamma\left(\mu\left(\mathbb{U}_{p+1}\right)\right)\right\}]] &= \Lambda_1[h\left[\xi\left\{\mu\left(Conv\mathcal{T}\mathbb{U}_p\right) + \gamma\left(\mu\left(Conv\mathcal{T}\mathbb{U}_p\right)\right)\right\}\right]] \\ &= \Lambda_1[h\left[\xi\left\{\mu\left(\mathcal{T}\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathcal{T}\mathbb{U}_p\right)\right)\right\}\right]] \\ &\le \Lambda_2[g\left[\xi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}, \phi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}\right]].\end{aligned}$$

Using the condition (1) of definition 1.4, we get

$$\begin{aligned}h[\xi\left\{\mu\left(\mathbb{U}_{p+1}\right) + \gamma\left(\mu\left(\mathbb{U}_{p+1}\right)\right)\right\}] &\le g\left[\xi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}, \phi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}\right] \\ &\le \xi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}.\end{aligned}$$

Clearly $\{\xi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}\}_{p=1}^{\infty}$ is a non-negative and non-increasing sequence hence there exists $a \ge 0$ such that

$$\lim_{p\to\infty} \xi\left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\} = a.$$

If possible let $a > 0$. As $p \to \infty$, we get

$$h(a) \le a$$

which is a contradiction hence $a = 0$, i.e.,

$$\xi[\lim_{p\to\infty} \left\{\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right\}] = 0$$

i.e.,

$$\lim_{p\to\infty} \left[\mu\left(\mathbb{U}_p\right) + \gamma\left(\mu\left(\mathbb{U}_p\right)\right)\right] = 0.$$

Using the definition 1.9 ,we get

$$\lim_{p\to\infty} \mu\left(\mathbb{U}_p\right) = 0.$$

Since $\mathbb{U}_p \supseteq \mathbb{U}_{p+1}$, by definition 1.1, we get $\mathbb{U}_\infty = \bigcap_{p=1}^{\infty} \mathbb{U}_p$ is a nonempty, closed and convex subset of $\mathbb{U}$ and $\mathbb{U}_\infty$ is $\mathcal{T}$ invariant. Thus theorem 1.2 implies that $\mathcal{T}$ has a fixed point in $\mathbb{U}$. This completes the proof. □

Theorem 2.2. Let $\mathbb{U}$ be a NBCC subset of a Banach space $\mathfrak{Z}$. Also $\mathcal{T}:\mathbb{U}\to\mathbb{U}$ is a continuous mapping with

$$h\left[\xi\left\{\mu\left(\mathcal{T}\Omega\right)+\gamma\left(\mu\left(\mathcal{T}\Omega\right)\right)\right\}\right]\leq kg\left[\xi\left\{\mu\left(\Omega\right)+\gamma\left(\mu\left(\Omega\right)\right)\right\},\phi\left\{\mu\left(\Omega\right)+\gamma\left(\mu\left(\Omega\right)\right)\right\}\right] \tag{2.2}$$

where $\Omega\subset\mathbb{U}$ and μ is an arbitrary MNC and $\phi\in\Phi,\xi\in\Xi,\ g\in\mathcal{C},\ h\in\mathcal{A}$ and $\gamma\in\mathcal{B}$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{U}$.

Proof . The result follows by taking $\Lambda_1(x)=x$ and $\Lambda_2(x)=kx$ in Theorem 2.1. □

Theorem 2.3. Let $\mathbb{U}$ be a NBCC subset of a Banach space $\mathfrak{Z}$. Also, let $\mathcal{T}:\mathbb{U}\to\mathbb{U}$ be a continuous mapping with

$$h\left[\xi\left\{2\mu\left(\mathcal{T}\Omega\right)\right\}\right]\leq kg\left[\xi\left\{2\mu\left(\Omega\right)\right\},\phi\left\{2\mu\left(\Omega\right)\right\}\right] \tag{2.3}$$

where $\Omega\subset\mathbb{U}$ and μ is an arbitrary MNC and $\phi\in\Phi,\xi\in\Xi,\ g\in\mathcal{C}\ and\ h\in\mathcal{A}$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{U}$.

Proof . The result follows by taking $\gamma(x)=x$ in Theorem 2.2. □

Theorem 2.4. Let $\mathbb{U}$ be a NBCC subset of a Banach space $\mathfrak{Z}$. Also, let $\mathcal{T}:\mathbb{U}\to\mathbb{U}$ be a continuous mapping with

$$h\left[\xi\left\{2\mu\left(\mathcal{T}\Omega\right)\right\}\right]\leq k\xi\left\{2\mu\left(\Omega\right)\right\} \tag{2.4}$$

where $\Omega\subset\mathbb{U}$ and μ is an arbitrary MNC and $\xi\in\Xi\ and\ h\in\mathcal{A}$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{U}$.

Proof . Use $g(m,n)\leq m$ in Theorem 2.3. □

Corollary 2.5. Let $\mathbb{U}$ be a NBCC subset of a Banach space $\mathfrak{Z}$. Also, let $\mathcal{T}:\mathbb{U}\to\mathbb{U}$ be a continuous mapping with

$$\mu\left(\mathcal{T}\Omega\right)\leq\lambda\mu\left(\Omega\right),\ \lambda=\frac{k}{\bar{k}}\in(0,1). \tag{2.5}$$

where $\Omega\subset\mathbb{U}$ and μ is an arbitrary MNC. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{U}$.

Proof . Using $\xi(x)=x$ and $h(x)=\bar{k}x$ where $0<k<1,\ \bar{k}>1$ in Theorem 2.4. we get DPFT. □

3 Measure of noncompactness on $C([0,I])$

Consider the space $\mathfrak{Z}=C(U)$ which is the set of real continuous functions on U, where $U=[0,I]$. Then $\mathfrak{Z}$ is a Banach space with the norm

$$\parallel\mathit{\Lambda}\parallel=\sup\left\{|\mathit{\Lambda}(t)|:t\in U\right\},\ \mathit{\Lambda}\in\mathfrak{Z}.$$

Let $T(\neq\emptyset)\subseteq\mathfrak{Z}$ be bounded. For $\mathit{\Lambda}\in T$ and $\varepsilon>0$, denote by $\mu(\mathit{\Lambda},\varepsilon)$ the modulus of the continuity of $\mathit{\Lambda}$, i.e.,

$$\mu(\mathit{\Lambda},\varepsilon)=\sup\left\{|\mathit{\Lambda}(t_1)-\mathit{\Lambda}(t_2)|:t_1,t_2\in U,|t_1-t_2|\leq\varepsilon\right\}.$$

Moreover, we set

$$\mu(T,\varepsilon)=\sup\left\{\mu(\mathit{\Lambda},\varepsilon):\mathit{\Lambda}\in T\right\};\ \mu_0(T)=\lim_{\varepsilon\to0}\mu(T,\varepsilon).$$

It is well-known that the function μ_0 is a MNC in $\mathfrak{Z}$ such that the Hausdorff MNC Γ is given by $\Gamma(T)=\frac{1}{2}\mu_0(T)$ (see [9]).

4 Solvability of a fractional integral equation

For $h\in(0,1]$ and $\omega\in\mathbb{C},\ Re(\omega)>0$, we define the left fractional integral of w by [21]

$$\left({}_aU^{\omega,h,\sigma}w\right)(\varphi)=\frac{1}{h^{\omega}\Gamma(\omega)}\int_a^{\varphi}e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}w(\vartheta)\sigma'(\vartheta)d\vartheta.$$

In this section, we will study the fractional integral equation shown below

$$\mathcal{H}(\varphi)=\Psi(\varphi,\mathcal{J}\left(\varphi,\mathcal{H}(\varphi)\right),\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)), \tag{4.1}$$

where $\omega>0,\ h\in(0,1],\ \varphi\in U=[0,I]$. Let

$$D_{e_0}=\{\mathcal{H}\in\mathfrak{Z}:\|\mathcal{H}\|\le e_0\}.$$

Assume that

(A) $\Psi:U\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \mathcal{J}:U\times\mathbb{R}\to\mathbb{R}$ be continuous and there exists constants $\beta_1,\ \beta_2,\ \beta_3\ge 0$ satisfying

$$\left|\Psi(\varphi,\mathcal{J},U_1)-\Psi(\varphi,\bar{\mathcal{J}},\bar{U}_1)\right|\le\beta_1\left|\mathcal{J}-\bar{\mathcal{J}}\right|+\beta_2\left|U_1-\bar{U}_1\right|,\ \varphi\in U;\ \mathcal{J},U_1,\bar{\mathcal{J}},\bar{U}_1\in\mathbb{R}$$

and

$$|\mathcal{J}(\varphi,L_1)-\mathcal{J}(\varphi,L_2)|\le\beta_3|L_1-L_2|,L_1,L_2\in\mathbb{R}.$$

(B) There exists $e_0>0$ satisfying

$$\bar{\Psi}=\sup\left\{|\Psi(\varphi,\mathcal{J},U_1)|:\varphi\in U,\mathcal{J}\in[-\hat{\mathcal{J}},\hat{\mathcal{J}}],U_1\in[-\hat{\mathcal{U}},\hat{\mathcal{U}}]\right\}\le e_0,$$

and

$$\beta_1\beta_3<1,$$

where

$$\hat{\mathcal{J}}=\sup\left\{|\mathcal{J}\left(\varphi,\mathcal{H}(\varphi)\right)|:\varphi\in U,\mathcal{H}(\varphi)\in[-e_0,e_0]\right\}$$

and

$$\hat{U}=\sup\left\{\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)\right|:\varphi\in U,\mathcal{H}(\varphi)\in[-e_0,e_0]\right\}.$$

(C) Let $\sigma:\mathbb{R}\to\mathbb{R}$ be a strictly increasing continuous function.

(D) $|\Psi(\varphi,0,0)|=0,\ \mathcal{J}(\varphi,0)=0$.

(E) There exists a positive solution e_0 of the inequality

$$\beta_1\beta_3e_0+\frac{\beta_2e_0I^{\omega-1}}{h^{\omega-1}(h-1)\Gamma(\omega)}.e^{\frac{(h-1)I}{h}}\le e_0.$$

Theorem 4.1. If conditions (A)-(E) hold, then the Eq.(4.1) has a solution in $\mathfrak{Z}=C(U)$.

Proof . Set the operator $\mathcal{S}:\mathfrak{Z}\to\mathfrak{Z}$ as follows:

$$(\mathcal{S}\mathcal{H})(\varphi)=\Psi(\varphi,\mathcal{J}\left(\varphi,\mathcal{H}(\varphi)\right),\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)).$$

Step 1: We show that the function $\mathcal{S}$ maps D_{e_0} into D_{e_0}. Let $\mathcal{H}\in D_{e_0}$. We have

$$\begin{aligned}|(\mathcal{S}\mathcal{H})(\varphi)|&\le\left|\Psi(\varphi,\mathcal{J}\left(\varphi,\mathcal{H}(\varphi)\right),\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi))-\Psi(\varphi,0,0)\right|+|\Psi(\varphi,0,0)|\\&\le\beta_1|\mathcal{J}(\varphi,\mathcal{H}(\varphi))-0|+\beta_2\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)-0\right|\\&\le\beta_1\beta_3|\mathcal{H}(\varphi)|+\beta_2\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)\right|.\end{aligned}$$

Also,

$$\begin{aligned}\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)\right|&=\left|\frac{1}{h^\omega\Gamma(\omega)}\int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}\mathcal{H}(\vartheta)\sigma'(\vartheta)d\vartheta\right|\\&\le\frac{1}{h^\omega\Gamma(\omega)}\int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}\sigma'(\vartheta)|\mathcal{H}(\vartheta)|d\vartheta\\&\le\frac{e_0}{h^\omega\Gamma(\omega)}\int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}\sigma'(\vartheta)d\vartheta\\&\le\frac{e_0I^{\omega-1}e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)}.\end{aligned}$$

Hence, $\| \mathcal{S} \| < e_0$ gives

$$\| \mathcal{S} \| \leq \beta_1\beta_3 e_0 + \frac{\beta_2 e_0 I^{\omega-1}}{h^{\omega-1}(h-1)\Gamma(\omega)}.e^{\frac{(h-1)I}{h}} \leq e_0.$$

Due to the assumption (E), $\mathcal{S}$ maps D_{e_0} into D_{e_0}.

Step 2: We show that $\mathcal{S}$ is continuous on D_{e_0}. Let $\varepsilon > 0$ and $\mathcal{H}, \bar{\mathcal{H}} \in D_{e_0}$ such that $\| \mathcal{H} - \bar{\mathcal{H}} \| < \varepsilon$. We now have

$$\begin{aligned}\left|(\mathcal{S}\mathcal{H})(\varphi) - \left(\mathcal{S}\bar{\mathcal{H}}\right)(\varphi)\right| &\leq \left|\Psi\left(\varphi, \mathcal{J}(\varphi, \mathcal{H}(\varphi)), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi)\right) - \Psi\left(\varphi, \mathcal{J}(\varphi, \bar{\mathcal{H}}(\varphi)), \left({}_0U^{\omega,h,\sigma}\bar{\mathcal{H}}\right)(\varphi)\right)\right| \\ &\leq \beta_1 \left|\mathcal{J}(\varphi, \mathcal{H}(\varphi)) - \mathcal{J}(\varphi, \bar{\mathcal{H}}(\varphi))\right| + \beta_2 \left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi) - \left({}_0U^{\omega,h,\sigma}\bar{\mathcal{H}}\right)(\varphi)\right|.\end{aligned}$$

Also,

$$\begin{aligned}\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi) - \left({}_0U^{\omega,h,\sigma}\bar{\mathcal{H}}\right)(\varphi)\right| &= \left|\frac{1}{h^\omega\Gamma(\omega)}\int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}\sigma'(\vartheta)\left\{\mathcal{H}(\vartheta) - \bar{\mathcal{H}}(\vartheta)\right\}d\vartheta\right| \\ &\leq \frac{1}{h^\omega\Gamma(\omega)}\int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi)-\sigma(\vartheta))}{h}}(\sigma(\varphi)-\sigma(\vartheta))^{\omega-1}\sigma'(\vartheta)\left|\mathcal{H}(\vartheta) - \bar{\mathcal{H}}(\vartheta)\right|d\vartheta \\ &< \frac{\varepsilon I^{\omega-1}e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)}.\end{aligned}$$

Hence, $\| \mathcal{H} - \bar{\mathcal{H}} \| < \varepsilon$ gives

$$\left|(\mathcal{S}\mathcal{H})(\varphi) - \left(\mathcal{S}\bar{\mathcal{H}}\right)(\varphi)\right| < \beta_1\beta_3\varepsilon + \frac{\varepsilon\beta_2 I^{\omega-1}e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)}.$$

As $\varepsilon \to 0$, we get $\left|(\mathcal{S}\mathcal{H})(\varphi) - \left(\mathcal{S}\bar{\mathcal{H}}\right)(\varphi)\right| \to 0$. This shows that $\mathcal{S}$ is continuous on D_{e_0}.

Step 3: An estimate of $\mathcal{S}$ with respect to μ_0: Assume that $\Delta(\neq \emptyset) \subseteq D_{e_0}$. Let $\varepsilon > 0$ be arbitrary and choose $\mathcal{H} \in \Delta$ and $\varphi_1, \varphi_2 \in U$ such that $|\varphi_2 - \varphi_1| \leq \varepsilon$ and $\varphi_2 \geq \varphi_1$.

Now,

$$\begin{aligned}\left|(\mathcal{S}\mathcal{H})(\varphi_2) - (\mathcal{S}\mathcal{H})(\varphi_1)\right| &= \left|\Psi(\varphi_2, \mathcal{J}\left(\varphi_2, \mathcal{H}(\varphi_2)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2)) - \Psi(\varphi_1, \mathcal{J}\left(\varphi_1, \mathcal{H}(\varphi_1)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1))\right| \\ &\leq \left|\Psi(\varphi_2, \mathcal{J}\left(\varphi_2, \mathcal{H}(\varphi_2)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2)) - \Psi(\varphi_2, \mathcal{J}\left(\varphi_2, \mathcal{H}(\varphi_2)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1))\right| \\ &+ \left|\Psi(\varphi_2, \mathcal{J}\left(\varphi_2, \mathcal{H}(\varphi_2)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)) - \Psi(\varphi_2, \mathcal{J}\left(\varphi_1, \mathcal{H}(\varphi_1)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1))\right| \\ &+ \left|\Psi(\varphi_2, \mathcal{J}\left(\varphi_1, \mathcal{H}(\varphi_1)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)) - \Psi(\varphi_1, \mathcal{J}\left(\varphi_1, \mathcal{H}(\varphi_1)\right), \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1))\right| \\ &\leq \beta_2\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| + \beta_1\left|\mathcal{J}\left(\varphi_2, \mathcal{H}(\varphi_2)\right) - \mathcal{J}\left(\varphi_1, \mathcal{H}(\varphi_1)\right)\right| + \mu_\Psi(U, \varepsilon) \\ &\leq \beta_2\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| + \beta_1\beta_3\left|\mathcal{H}(\varphi_2) - \mathcal{H}(\varphi_1)\right| + \mu_\Psi(U, \varepsilon),\end{aligned}$$

where

$$\mu_\Psi(U, \varepsilon) = \sup\left\{\begin{array}{c}|\Psi(\varphi_2, \mathcal{J}, U_1) - \Psi(\varphi_1, \mathcal{J}, U_1)| : |\varphi_2 - \varphi_1| \leq \varepsilon; \varphi_1, \varphi_2 \in U; \\ \mathcal{J} \in [-\hat{\mathcal{J}}, \hat{\mathcal{J}}]; U_1 \in [-\hat{\mathcal{U}}, \hat{\mathcal{U}}]\end{array}\right\}.$$

Also,

$$\begin{aligned}\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| &= \left|\frac{1}{h^\omega\Gamma(\omega)}\int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}}(\sigma(\varphi_2)-\sigma(\vartheta))^{\omega-1}\mathcal{H}(\vartheta)\sigma'(\vartheta)d\vartheta\right. \\ &\quad \left. - \frac{1}{h^\omega\Gamma(\omega)}\int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}}(\sigma(\varphi_1)-\sigma(\vartheta))^{\omega-1}\mathcal{H}(\vartheta)\sigma'(\vartheta)d\vartheta\right| \\ &\leq \frac{1}{h^\omega\Gamma(\omega)}\left|\int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}}(\sigma(\varphi_2)-\sigma(\vartheta))^{\omega-1}\mathcal{H}(\vartheta)\sigma'(\vartheta)d\vartheta\right.\end{aligned}$$

$$
\begin{aligned}
& - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta)\sigma'(\vartheta) d\vartheta \Bigg| \\
& \leq \frac{1}{h^\omega \Gamma(\omega)} \Bigg| \int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta)\sigma'(\vartheta) d\vartheta \\
& - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta)\sigma'(\vartheta) d\vartheta \Bigg| \\
& + \frac{1}{h^\omega \Gamma(\omega)} \Bigg| \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta)\sigma'(\vartheta) d\vartheta \\
& - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta)\sigma'(\vartheta) d\vartheta \Bigg| \\
& \leq \frac{1}{h^\omega \Gamma(\omega)} \int_{\varphi_1}^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} |\mathcal{H}(\vartheta)| \sigma'(\vartheta) d\vartheta \\
& + \frac{1}{h^\omega \Gamma(\omega)} \int_0^{\varphi_1} \Bigg| \left(e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} - e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \right) \mathcal{H}(\vartheta)\sigma'(\vartheta) \Bigg| d\vartheta \\
& \leq \frac{-e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)} \parallel \mathcal{H} \parallel (\varphi_2 - \varphi_1)^{\omega-1} \\
& + \frac{\parallel \mathcal{H} \parallel}{h^\omega \Gamma(\omega)} \int_0^{\varphi_1} \Bigg| \left(e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} - e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \right) \sigma'(\vartheta) \Bigg| d\vartheta.
\end{aligned}
$$

As $\varepsilon \to 0$, then $\varphi_2 \to \varphi_1$ and so, $\left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| \to 0$. Hence,

$$|(\mathcal{SH})(\varphi_2) - (\mathcal{SH})(\varphi_1)| \leq \beta_2 \left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| + \beta_1\beta_3\mu(\mathcal{H},\varepsilon) + \mu_\Psi(U,\varepsilon),$$

gives

$$\mu(\mathcal{SH},\varepsilon) \leq \beta_2 \left|\left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_2) - \left({}_0U^{\omega,h,\sigma}\mathcal{H}\right)(\varphi_1)\right| + \beta_1\beta_3\mu(\mathcal{H},\varepsilon) + \mu_\Psi(U,\varepsilon).$$

By the uniform continuity of Ψ on $U \times [-\hat{\mathcal{J}}, \hat{\mathcal{J}}] \times [-\hat{\mathcal{U}}, \hat{\mathcal{U}}]$ we have $\mu_\Psi(U,\varepsilon) \to 0$, as $\varepsilon \to 0$.

Taking $\sup_{\mathcal{H}\in\Delta}$ and $\varepsilon \to 0$ we get,

$$\mu_0(\mathcal{S}\Delta) \leq \beta_1\beta_3\mu_0(\Delta).$$

Thus by Corollary 2.5, $\mathcal{S}$ has a fixed point in $\Delta \subseteq D_{e_0}$ i.e. equation (4.1) has a solution in $\mathfrak{Z}$. □

Example 4.2. Consider the equation below

$$\mathcal{H}(\varphi) = \frac{\mathcal{H}(\varphi)}{9+\varphi^4} + \frac{\left({}_0U^{1,\frac{1}{3},\varphi}\mathcal{H}\right)(\varphi)}{20} \tag{4.2}$$

for $\varphi \in [0,3] = U$.

We have

$$\sigma(\varphi) = \varphi;$$

$$\left({}_0U^{1,\frac{1}{3},\varphi}\mathcal{H}\right)(\varphi) = \frac{3}{\Gamma(1)} \int_0^\varphi e^{-2(\varphi-\vartheta)} \mathcal{H}(\vartheta) d\vartheta.$$

Also, $\Psi(\varphi, \mathcal{J}, U_1) = \mathcal{J} + \frac{U_1}{20}$ and $\mathcal{J}(\varphi, \mathcal{H}) = \frac{\mathcal{H}}{9+\varphi^4}$. It is trivial that both Ψ, $\mathcal{J}$ are continuous satisfying

$$|\mathcal{J}(\varphi, L_1) - \mathcal{J}(\varphi, L_2)| \leq \frac{|L_1 - L_2|}{9},$$

and

$$\left|\Psi(\varphi, \mathcal{J}, U_1) - \Psi(\varphi, \bar{\mathcal{J}}, \bar{U}_1)\right| \leq \left|\mathcal{J} - \bar{\mathcal{J}}\right| + \frac{1}{20}\left|U_1 - \bar{U}_1\right|.$$

Therefore, $\beta_1 = 1,\ \beta_2 = \frac{1}{20},\ \beta_3 = \frac{1}{9}$ and $\beta_1\beta_3 = \frac{1}{9} < 1$. If $\| \mathcal{H} \| \leq e_0$ then

$$\hat{\mathcal{J}} = \frac{e_0}{9}$$

and

$$\hat{U} = \frac{3e_0}{2}\Big(1 - \frac{1}{e^6}\Big).$$

Further,

$$|\Psi(\varphi, \mathcal{J}, U_1)| \leq \frac{e_0}{9} + \frac{3e_0}{40}\Big(1 - \frac{1}{e^6}\Big) \leq e_0.$$

If we choose $e_0 = 3$ then

$$\hat{\mathcal{J}} = \frac{1}{3},\ \hat{U} = \frac{9}{2}\Big(1 - \frac{1}{e^6}\Big),$$

which gives

$$\bar{\Psi} \leq 3.$$

For $e_0 = 3$, however, assumption (E) is also satisfied. We can see that all of Theorem 4.1's assumptions are achieved, from (A) to (E). Equation (4.2), according to Theorem 4.1, has a solution in $\mathfrak{Z} = C(U)$.

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