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R-norm entropy for partitions of algebraic structures and dynamical systems

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Abstract

In this paper, we define R-norm entropy and conditional R-norm entropy of partitions of algebraic structures, and we establish some of their basic properties. We show that the Shannon entropy and the conditional Shannon entropy of partitions can be derived from the R-norm entropy and the conditional R-norm entropy of partitions, respectively, by letting R tend to 1. Finally, using the notion of entropy for partitions, we define the R-norm entropy of a dynamical system. We prove that the R-norm entropies of isomorphic dynamical systems are equal.

Keywords: Algebraic structure, R-norm entropy, partition, conditional R-norm entropy, dynamical system, isomorphism
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1 Introduction

The notion of entropy plays an important role in uncertain dynamical systems. It has been applied to information theory, physics, computer science, biology, statistics and many other fields. The classical approach in information theory is based on Shannon's entropy [25]. Shannon's entropy has been studied on different structures. We refer the reader to Ebrahimi and Mosapour [2], Eslami Giski and Ebrahimi [9], Khare [13], Markchova [18], and Ellerman [7]. Extensions of Shannon's original work have resulted in many alternative measures of entropy. To be used in the study of natural phenomena, some extensions of Shannon's entropy were developed. As an instance of such extensions, we can mention the Renyi entropy.

There are other notions of entropy, including logical entropy and R-norm entropy, that include more details of the aforementioned phenomena. Logical entropy has been studied on various algebraic structures, including fuzzy sigma-algebras [4, 13], quantum systems [5, 6], fuzzy dynamical systems [20], D-posets [22], effect algebras [8, 9], and MV-algebras [16]. The definition of R-norm entropy with Minkowski's inequality was presented in 1980 [1]. R-norm entropy has been discussed on fuzzy information [10], information measures [14], information measures of type a, fuzzy probability spaces [17], and generalized measures [11]. In this article, we introduce and study the notion of R-norm entropy.

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In this article, we introduce and study the notion of R-norm entropy. If $p = \{p_1, p_2, ..., p_n\}$ is a probability distribution and R is a positive real number other than 1, then the R-norm entropy is defined by

$$H_R(p) = \frac{R}{R-1}(1 - [\Sigma_{i=1}^n p_i^R]^{\frac{1}{R}}).$$

This entropy has found applications in coding theory, statistics and pattern recognition. In this paper, we prove some results similar to those of Markechova, Mosapour and Ebrahimzadeh obtained in [17] for the case of R-norm entropy on an appropriate algebraic structure. We define this entropy on algebraic structures and find its basic properties.

The paper is organized as follows. In the next section, we recall the required preliminaries and discuss some of the related work. Our main results are obtained in the next two sections. In fact, in Section 3 we define R-norm entropy and conditional R-norm entropy of finite partitions of an algebraic structure, and we present some of their basic properties. In Section 4, we use the proposed concept of R-norm entropy of finite partitions to define the notion of R-norm entropy for dynamical systems. The last section contains a brief conclusion outlining our achievements.

2 Preliminaries and related work

We begin with the definitions of basic terms and recalling some of the known results that will be used in this article.

Definition 2.1. A quadruple $(F, \oplus, \otimes, 1_F)$ is said to be an *algebraic structure* if F is a non-empty partially ordered set, $\oplus$ is a partial binary operation on F, $\otimes$ is a binary operation on F, 1_F is a fixed element of F, and there exist mappings $m : F \longrightarrow [0, 1]$ and $S : F \longrightarrow F$ for which the following conditions are satisfied.

($F1$) The operations $\oplus$ and $\otimes$ are m-commutative, that is, $m(f \otimes g) = m(g \otimes f)$ for any $f, g \in F$, and if $f \oplus g$ exists, then $g \oplus f$ exists, too, and $m(f \oplus g) = m(g \oplus f)$.
($F2$) The operations $\oplus$ and $\otimes$ are m-associative, that is, $m(f \otimes (g \otimes h)) = m((f \otimes g) \otimes h)$ for any $f, g, h \in F$, and if $(f \oplus g) \oplus h$ exists, then $f \oplus (g \oplus h)$ exists, too, and $m(f \oplus (g \oplus h)) = m((f \oplus g) \oplus h)$.
($F3$) The operations $\oplus$ and $\otimes$ satisfy the m-distributive law, that is, for any $f, g, h \in F$, if $(f \otimes h) \oplus (g \otimes h)$ exists, then $f \oplus g$ exists and $m(f \oplus g) \otimes h) = m((f \otimes h) \oplus (g \otimes h))$.
($F4$) For every $f, g \in F$, $f \otimes g \le f = 1_F \otimes f$.
($F5$) If $\oplus_{i=1}^n f_i$ exists, then $m(\oplus_{i=1}^n f_i) = \Sigma m(f_i)$.
($F6$) If $f, g \in F$ and $f \le g$, then $m(f) \le m(g)$.
($F7$) If $f \in F$ and $m(f) = m(1_F)$, then $m(f \otimes g) = m(g)$ for every $g \in F$.
($F8$) For any $f, g \in F$, if $f \oplus g$ exists, then $S(f) \oplus S(g)$ exists, too, and $m(S(f \oplus g)) = m(S(f) \oplus S(g))$.
($F9$) The mapping $S : F \longrightarrow F$ is an m-preserving transformation, that is, $m(S(f)) = m(f)$ for every $f \in F$.

Example 2.2. Consider a triple $(\Omega, p(\Omega), m)$, where $p(\Omega)$ is the power set of a finite set Ω, that is, the set of all subsets of Ω. Let the mapping $m : p(\Omega) \longrightarrow [0, 1]$ be defined by $m(A) = \frac{n(A)}{n(\Omega)}$, where $n(A)$ is the number of elements of the set A. Also, suppose that $S : p(\Omega) \longrightarrow p(\Omega)$ is defined by $S(A) = A$ for any $A \in p(\Omega)$, and $1_{p(\Omega)} = \Omega$. Then the binary operations $\oplus$ and $\otimes$, defined by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \otimes B = A \cap B$ for any $A, B \in p(\Omega)$, together with the mappings m and S satisfy the conditions $(F1) - (F9)$.

Remark 2.3. In this paper, the latter F means an algebric structure.

Definition 2.4. A partition of F is a finite collection $A = \{f_1, f_2, ..., f_n\} \subset F$ such that $\oplus_{i=1}^n f_i$ exists, and $m(1_F) = m(\oplus_{i=1}^n f_i) = \Sigma_{i=1}^n m(f_i)$.

If $A = \{f_1, f_2, ..., f_n\}$ and $B = \{g_1, g_2, ..., g_p\}$ are partitions of F, then

$$A \vee B = \{f_i \otimes g_j : i = 1, ..., n, \ j = 1, ..., p\}.$$

We say that B is a *refinement* of A, and we write $A < B$, if there exists a partition $I(1), I(2), ..., I(n)$ of the set $\{1, 2, ..., p\}$ such that $m(f_i) = \Sigma_{j \in I(i)} m(g_i)$, for every $i \in \{1, 2, ..., n\}$.

Fact [3]. If A and B are partitions of F, then $A \vee B$ is a partition of F, too. Fact [3]. If A and B are arbitrary partitions of F, then $A < A \vee B$.

Definition 2.5. Partitions $A = \{f_1, f_2, ..., f_n\}$ and $B = \{g_1, g_2, ..., g_p\}$ of F are said to be *statistically independent* if $m(f \otimes g) = m(f).m(g)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., p$.

Definition 2.6. Let $A = \{f_1, f_2, ..., f_n\}$ and $B = \{g_1, g_2, ..., g_p\}$ be partitions of F, and consider the mapping $m : F \longrightarrow [0, 1]$. Then, the *entropy* of A with respect to m is defined by Shannon's formula

$$H_m(A) = -\Sigma_{i=1}^{n} m(f_i). \log\ m(f_i).$$

The *conditional entropy* of A given B is defined by

$$H_m(A/B) = -\Sigma_{i=1}^{n} \Sigma_{j=1}^{p} m(f_i/g_j). \log \frac{m(f_i \otimes g_j)}{m(g_j)},$$

with the convention that $0. \log \frac{0}{x} = 0$ if $x \geq 0$.

In the proofs of our results, we will use the well-known *Jensen inequality.* It states that for a real, convex function ϕ, real numbers $x_1, x_2, ..., x_n$ in its domain, and non-negative real numbers $c_1, c_2, ..., c_n$ satisfying $\Sigma_{i=1}^{n} c_i = 1$,

$$\phi(\Sigma_{i=1}^{n} c_i x_i) \leq \Sigma_{i=1}^{n} c_i \phi(x_i).$$

Moreover, the inequality is reversed if ϕ is a real, concave function. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$ or ϕ is linear. In addition, we will use *Minkowski's inequality.* This says that for non-negative real numbers x_i and y_i, $i \in \{1, \ldots, n\}$,

$$[\Sigma_{i=1}^{n} x_i^R]^{\frac{1}{R}} + [\Sigma_{i=1}^{n} y_i^R]^{\frac{1}{R}} \geq [\Sigma_{i=1}^{n} (x_i + y_i)^R]^{\frac{1}{R}}, \text{ for R} > 1,$$

and

$$[\Sigma_{i=1}^{n} x_i^R]^{\frac{1}{R}} + [\Sigma_{i=1}^{n} y_i^R]^{\frac{1}{R}} \leq [\Sigma_{i=1}^{n} (x_i + y_i)^R]^{\frac{1}{R}}, \text{ for } 0 < \text{R} < 1.$$

Also, we will use L'Hôpital's rule, which can be stated as follows. Let f and g be functions that are differentiable on an open interval u, except possibly at a point $a \in u$. If $\lim f(x) = \lim g(x) = 0$ when $x \to a$, $g'(x) \neq 0$ for every x in u with $x \neq a$ and $\lim \frac{f'(x)}{g'(x)}$ exists, then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ when $x \to a$.

3 The R-norm entropy of a partition of F

In this section, we introduce the R-norm entropy of a partition of an algebraic structure, and we study its properties.

Definition 3.1. Let $A = \{f_1, f_2, ..., f_n\}$ be a partition of F, and R be a positive real number other than 1. The *R-norm entropy* of A with respect to m is defined by

$$H_R^m(A) = \frac{R}{R-1} \left(1 - \left[\Sigma_{i=1}^{n} \left(\frac{m(f_i)}{m(1_F)} \right)^R \right]^{\frac{1}{R}} \right).$$

Theorem 3.2. For an arbitrary partition A of F, the R-norm entropy $H_R^m(A)$ is non-negative.

Proof . Let $A = \{f_1, f_2, ..., f_n\}$, and $R > 0$. Then, $m(f_i)^R \leq m(f_i)$ for $i = 1, 2, ..., n$. Hence, $\Sigma_{i=1}^{n} m(f_i)^R \leq \Sigma_{i=1}^{n} m(f_i) = m(1_F)$. This implies that $[\Sigma_{i=1}^{n} \frac{m(f_i)^R}{m(1_F)^R}]^{\frac{1}{R}} \leq 1$. Since $\frac{R}{R-1} > 0$ for $R > 1$, it follows that $H_R^m(A) = \frac{R}{R-1}(1 - [\Sigma_{i=1}^{n} \frac{m(f_i)^R}{m(1_F)^R}]^{\frac{1}{R}}) \geq 0$. On the other hand, when $0 < R < 1$, $m(f_i)^R \geq m(f_i)$ for $i = 1, 2, ..., n$. Thus, $\Sigma_{i=1}^{n} m(f_i)^R \geq \Sigma_{i=1}^{n} m(f_i) = m(1_F)$. It follows that $[\Sigma_{i=1}^{n} \frac{m(f_i)^R}{m(1_F)^R}]^{\frac{1}{R}} \geq 1$. Since $\frac{R}{R-1} < 0$ for $0 < R < 1$, we obtain

$$H_R^m(A) = \frac{R}{R-1} \left(1 - \left[\Sigma_{i=1}^{n} \left(\frac{m(f_i)}{m(1_F)} \right)^R \right]^{\frac{1}{R}} \right) \geq 0.$$

□

Example 3.3. Consider the measurable space $([0,1],\beta)$, where β is the σ-algebra of all Borel subsets of the unit interval $[0,1]$. Let F be the family of all Borel measurable functions $f:[0,1]\longrightarrow[0,\frac{1}{2}]$. For every $t\in[0,\frac{1}{2}]$; we define $(t)_{[0,1]}:[0,1]\rightarrow[0,\frac{1}{2}]$ by $t(x)=t$. We define binary operations $\oplus$ and $\otimes$ by $f\oplus g=f+g$ if $f+g\leq(\frac{1}{2})_{[0,1]}$ and $f\otimes g=\max(f+g-(\frac{1}{2})_{[0,1]},0_{[0,1]})$. If we define the mappings $m:F\longrightarrow[0,1]$ and $S:F\longrightarrow F$ by $m(f)=\int_0^1 f(x)dx$ and $S(f)=f$ for any element f of F, then $(F,\oplus,\otimes,(\frac{1}{2})_{[0,1]})$ is an algebraic structure. The set $A=\{f_1,f_2\}$, where $f_1(x)=|x-\frac{1}{2}|$ and $f_2(x)=\frac{1}{2}-|x-\frac{1}{2}|$, is a partition of F. To calculate the R-norm entropy of A we write

$$H_R^m(A)=\frac{R}{R-1}\left(1-\left[\frac{(\int_0^1 f_1(x)dx)^R}{(\frac{1}{2})^R}+\frac{(\int_0^1 f_2(x)dx)^R}{(\frac{1}{2})^R}\right]^{\frac{1}{R}}\right)=\frac{R}{R-1}\left(1-2^{\frac{1-R}{R}}\right).$$

If we let $R=2$, then $H_R^m(A)=\sqrt{2}(\sqrt{2}-1)$.

Definition 3.4. Let $A=\{f_1,f_2,...,f_n\}$ and $B=\{g_1,g_2,...,g_k\}$ be partitions of F, and R be a positive real number other than 1. Then, the conditional R-norm entropy of A given B with respect to m is defined by

$$H_R^m(A/B)=\frac{R}{R-1}\left(\left[\Sigma_{j=1}^k(\frac{m(g_j)}{m(1_F)})^R\right]^{\frac{1}{R}}-\left[\Sigma_{j=1}^k\Sigma_{i=1}^n(\frac{m(f_i\otimes g_j)}{m(1_F)})^R\right]^{\frac{1}{R}}\right).$$

Remark 3.5. Let A be a partition of F. If $B=\{g\}$, where $g\in F$, then $H_R^m(A/B)=H_R^m(A)$.

Theorem 3.6. If $A=\{f_1,f_2,...,f_n\}$ and $B=\{g_1,g_2,...,g_k\}$ are partitions of F, then

$$\lim H_R^m(A/B)=C.H_m(A/B)$$

when $R\rightarrow1$, $C=\frac{1}{m(1_F).\log e}$ and $H_m(A/B)=-\Sigma_{i=1}^n\Sigma_{j=1}^k m(f_i\otimes g_j).\log\frac{m(f_i\otimes g_j)}{m(g_j)}$.

Proof . For every $R\in(0,1)\cup(1,\infty)$,

$$H_R^m(A/B)=\frac{1}{1-\frac{1}{R}}\left(\left[\Sigma_{j=1}^k\frac{m(g_j)^R}{m(1_F)^R}\right]^{\frac{1}{R}}-\left[\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)^R}{m(1_F)^R}\right]^{\frac{1}{R}}\right)=\frac{f(R)}{g(R)},$$

where the continuous functions f and g are defined by

$$f(R)=\left(\left[\Sigma_{j=1}^k\frac{m(g_j)^R}{m(1_F)^R}\right]^{\frac{1}{R}}-\left[\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)^R}{m(1_F)^R}\right]^{\frac{1}{R}}\right)$$

and $g(R)=1-\frac{1}{R}$. The functions f and g are differentiable and evidently, $\lim g(R)=g(1)=0$. It can be easily verified that

$$\lim f(R)=f(1)=\left[\Sigma_{j=1}^k\frac{m(g_j)}{m(1_F)}-\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)}{m(1_F)}\right]=1-1=0$$

when $R\rightarrow1$. Using L'Hôpital's rule, this implies

$$\lim H_R^m(A/B)=\frac{\lim f'(R)}{\lim g'(R)}$$

when $R\rightarrow1$, assuming that the right-hand side exists. To find the derivative of the function $f(R)$, we use the identity $a^x=e^{x\ln a}$. Now,

$$\begin{aligned}\frac{d}{dR}f(R) &= \left(\Sigma_{j=1}^k\frac{m(g_j)^R}{m(1_F)^R}\right)^{\frac{1}{R}}.\left(-\frac{1}{R^2}.\ln\Sigma_{j=1}^k\frac{m(g_j)}{m(1_F)}^R+\frac{1}{R}.\frac{1}{\Sigma_{j=1}^k\frac{m(g_j)^R}{m(1_F)^R}}\Sigma_{j=1}^k\frac{m(g_j)^R}{m(1_F)^R}.\ln\left(\frac{m(g_j)}{m(1_F)}\right)\right)\\ &- \left(\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)^R}{m(1_F)^R}\right)^{\frac{1}{R}}.(-\frac{1}{R^2}.\ln\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)^R}{m(1_F)^R}\\ &+ \frac{1}{R}.\frac{1}{\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)}{m(1_F)^R}}.\Sigma_{j=1}^k\Sigma_{i=1}^n\frac{m(f_i\otimes g_j)^R}{m(1_F)^R}.\ln\frac{m(f_i\otimes g_j)}{m(1_F)}).\end{aligned}$$

Since $\lim g'(R) = \lim \frac{1}{R^2} = 1$ when $R \to 1$,

$$\begin{aligned}
\lim H_R^m(A/B) &= \lim f'(R) \\
&= 1.\left(0 + 1.\Sigma_{j=1}^k \frac{m(g_j)}{m(1_F)}\right) \ln\left(\frac{m(g_j)}{m(1_F)}\right) - 1.\left(0 + 1.\Sigma_{j=1}^k \Sigma_{i=1}^n \frac{m(f_i \otimes g_j)}{m(1_F)}\right). \ln \frac{m(f_i \otimes g_j)}{m(1_F)} \\
&= \Sigma_{j=1}^k \frac{m(g_j)}{m(1_F)}. \ln \frac{m(g_j)}{m(1_F)} - \Sigma_{j=1}^k \Sigma_{i=1}^n \frac{m(f_i \otimes g_j)}{m(1_F)}. \ln \frac{m(f_i \otimes g_j)}{m(1_F)} \\
&= \Sigma_{j=1}^k \Sigma_{i=1}^n \frac{m(f_i \otimes g_j)}{m(1_F)}. \ln \frac{m(g_j)}{m(1_F)} - \Sigma_{j=1}^k \Sigma_{i=1}^n \frac{m(f_i \otimes g_j)}{m(1_F)}. \ln \frac{m(f_i \otimes g_j)}{m(1_F)} \\
&= -\Sigma_{j=1}^k \Sigma_{i=1}^n \frac{m(f_i \otimes g_j)}{m(1_F)}. \ln \frac{m(f_i \otimes g_j)}{m(g_j)} \\
&= \frac{-\ln e}{m(1_F)} \Sigma_{i=1}^n \Sigma_{j=1}^k m(f_i \otimes g_j). \log \frac{m(f_i \otimes g_j)}{m(g_j)} \\
&= C.H_m(A/B),
\end{aligned}$$

where $C = \frac{1}{m(1_F)}. \log e$. □

Corollary 3.7. Let $A = \{f_1, f_2, ..., f_n\}$ be a partition of F. Then, $\lim H_R^m(A) = C.H_m(A)$, where $H_m(A) = -\Sigma_{i=1}^n m(f_i) \log m(f_i$ and $C = \frac{1}{m(1_F). \log e}$.

Proof . By the previous theorem and Remark, it suffices to let $B = \{1_F\}$. Then, $\lim H_R^m(A) = C.H_m(A)$ when $R \to 1$. □

Theorem 3.8. If A, B and C are partitions of F, then

$$H_R^m(A \vee B/C) = H_R^m(A/C) + H_R^m(B/A \vee C).$$

Proof . Let $A = \{f_1, f_2, ..., f_n\}$, $B = \{g_1, g_2, ..., g_p\}$ and $C = \{h_1, h_2, ..., h_q\}$. Then,

$$\begin{aligned}
H_R^m(A \vee B/C) &= \frac{R}{R-1}\left(\left[\Sigma_{k=1}^q \frac{m(h_k)^R}{m(1_F)^R}\right]^{\frac{1}{R}} - \left[\Sigma_{i=1}^n \Sigma_{j=1}^p \Sigma_{k=1}^q \frac{m(f_i \otimes g_j \otimes h_k)^R}{m(1_F)^R}\right]^{\frac{1}{R}}\right) \\
&= \frac{R}{R-1}\left(\left[\Sigma_{k=1}^q \frac{m(h_k)^R}{m(1_F)^R}\right]^{\frac{1}{R}} - \left[\Sigma_{i=1}^n \Sigma_{k=1}^q \frac{m(f_i \otimes h_k)^R}{m(1_F)^R}\right]^{\frac{1}{R}}\right) \\
&+ \frac{R}{R-1}\left(\left[\Sigma_{i=1}^n \Sigma_{k=1}^q \frac{m(f_i \otimes h_k)^R}{m(1_F)^R}\right]^{\frac{1}{R}} - \left[\Sigma_{i=1}^n \Sigma_{j=1}^p \Sigma_{k=1}^q \frac{m(f_i \otimes g_j \otimes h_k)^p}{m(1_F)^R}\right]^{\frac{1}{R}}\right) \\
&= H_R^m(A/C) + H_R^m(B/A \vee C).
\end{aligned}$$

□

Corollary 3.9. For arbitrary partitions A and B of F,

$$H_R^m(A \vee B) = H_R^m(A) + H_R^m(B/A).$$

Proof . By the previous theorem and Remark, it suffices to let $C = \{1_F\}$. Then,

$$H_R^m(A \vee B) = H_R^m(A) + H_R^m(B/A).$$

□

Corollary 3.10. Let $A_1, A_2, ..., A_n$ and C be partitions of F. Then, for $n = 2, 3, ...,$

$(i) H_R^m(A_1 \vee A_2 \vee ... \vee A_n) = H_R^m(A_1) + \Sigma_{i=1}^n H_R^m(A_i / \vee_{k=1}^{i-1} A_k)$;

(ii) $H_R^m(\vee_{i-1}^n A_i/C) = H_R^m(A_1/C) + \Sigma_{i=2}^n H_R^m(A_i/(\vee_{k=1}^{i-1} A_k) \vee C)$.

Proof . The proof is straightforward and uses Theorem 3.8 and Theorem 3.9. □

Proposition 3.11. Let m and l be mappings from F to $[0,1]$. Then for every $\lambda \in [0,1]$, the mapping $\lambda m + (1-\lambda)l : F \longrightarrow [0,1]$ satisfies all conditions of an algebraic structure F i.e, the convex linear combination of these two maps can be used in definition as well as the m map.

Proof . It is easy to see that the mapping $\lambda m + (1-\lambda)l : F \longrightarrow [0,1]$ satisfies the conditions $(F1)-(F9)$ presented in Definition 3.11. □

Theorem 3.12. Let A be a partition of F, and m, l be partial binary operations on F. Then for every $\lambda \in [0,1]$,

$$\lambda H_R^m(A) + (1-\lambda)H_R^l(A) \leq H_R^{\lambda M+(1-\lambda)l}(A).$$

Proof . Let $A = \{f_1, f_2, ..., f_n\}$ and $\lambda \in [0,1]$. Letting $x_i = \lambda m(f_i)$ and $y_i = (1-\lambda)l(f_i)$ for $i = 1, 2, ..., n$, by applying Minkowski's inequality we find that for $R > 1$,

$$\lambda[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} + (1-\lambda)[\Sigma_{i=1}^n l(f_i)^R]^{\frac{1}{R}} \geq [\Sigma_{i=1}^n (\lambda m(f_i) + (1-\lambda)l(f_i))^R]^{\frac{1}{R}}.$$

Hence,

$$\lambda\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right) + (1-\lambda)\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n L(f_i)^R]^{\frac{1}{R}}\right) \leq 1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n (\lambda m(f_i) + (1-\lambda)l(f_i)^R]^{\frac{1}{R}}.$$

Thus, $\lambda\frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma m(f_i)^R]^{\frac{1}{R}}) + (1-\lambda)\frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n l(f_i)^R]^{\frac{1}{R}}) \leq \frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n (\lambda m(f_i) + (1-\lambda)l(f_i)^R]^{\frac{1}{R}}$, and for $0 < R < 1$,

$$\lambda[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} + (1-\lambda)[\Sigma_{i=1}^n l(f_i)^R]^{\frac{1}{R}} \leq [\Sigma_{i=1}^n (\lambda m(f_i)) + (1-\lambda)l(f_i)^R]^{\frac{1}{R}}.$$

Therefore, $\lambda(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}) + (1-\lambda)(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n l(f_i)^R]^{\frac{1}{R}}) \geq (1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n (\lambda m(f_i) + (1-\lambda)l(f_i)^R]^{\frac{1}{R}}$. Hence, $\lambda\frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}) + (1-\lambda)\frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n l(f_i)^R]^{\frac{1}{R}}) \leq \frac{R}{R-1}(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n (\lambda m(f_i) + (1-\lambda)l(f_i))^R]^{\frac{1}{R}}$. We find that for every $R \in (0,1) \cup (1,\infty)$, the function $\phi(m) = H_R^m(A)$ is concave on the family of all mappings $m : F \longrightarrow [0,1]$ in Definition 3.11. Thus, for every $\lambda \in [0,1]$,

$$\lambda H_R^m(A) + (1-\lambda)H_R^l(A) \leq H_R^{\lambda m+(1-\lambda)l}(A).$$

□

Theorem 3.13. Let A, B and C be partitions of F.

(i) If $A < B$, then $H_R^m(A) \leq H_R^m(B)$.

(ii) $H_R^m(A) \leq H_R^m(A \vee B)$.

(iii) $H_R^m(A \vee B) \geq \max(H_R^m(A), H_R^m(B))$.

(iv) If $A < B$, then $H_R^m(A/C) \leq H_R^m(B/C)$.

Proof . Let $A = \{f_1, f_2, ..., f_n\}$ and $B = \{g_1, g_2, ..., g_p\}$.

(i) If $A < B$, then there exists a partition $I(1), I(2), ..., I(n)$ of the set $\{1, 2, ..., p\}$ such that $m(f_i) = \Sigma_{j\in I(i)} m(g_j)$, for $i = 1, 2, ..., n$. Therefore, for any $R > 1$ and $i \in \{1, 2, ..., n\}$,

$$m(f_i)^R = (\Sigma_{j\in I(i)} m(g_j))^R \geq \Sigma_{j\in I(i)} m(g_j)^R.$$

Consequently,

$$\Sigma_{i=1}^n m(f_i)^R \geq \Sigma_{j=1}^P m(g_j)^R.$$

Hence,

$$[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} \geq [\Sigma_{j=1}^P m(g_j)^R]^{\frac{1}{R}}.$$

Since $\frac{R}{R-1} > 0$ for $R > 1$,

$$H_R^m(A) = \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right) \leq \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^p m(g_j)^R]^{\frac{1}{R}}\right) = H_R^m(B).$$

If $0 < R < 1$, then for $i = 1, 2, ..., n$,

$$m(f_i)^R = (\Sigma_{j\in I(i)} m(g_j))^R \leq \Sigma_{j\in I(i)} m(g_j)^R.$$

Consequently,

$$\Sigma_{i=1}^n m(f_i)^R \leq \Sigma_{j=1}^p m(g_j)^R.$$

Therefore,

$$[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} \leq [\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}}.$$

Since $\frac{R}{R-1} < 0$ for $0 < R < 1$,

$$H_R^m(A) = \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right) \leq \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}}\right) = H_R^m(B).$$

(ii) Since $A < A \vee B$, the desired result follows from (i).

(iii) This follows from (ii).

(iv) By the assumption $A < B$ we obtain $A \vee C < B \vee C$. Therefore,

$$H_R^m(A/C) = H_R^m(A \vee C) - H_R^m(C) = H_R^m(B/C).$$

□

Theorem 3.14. Let A and B be statistically independent partitions of F. Then,

$$H_R^m(A/B) = m(1_F).H_R^m(A) - \frac{R-1}{R} m(1_F) H_R^m(A).H_R^m(B) + (1 - m(1_F))\left(\frac{R}{R-1} - H_R^m(B)\right).$$

Proof . Let $A = \{f_1, f_2, ..., f_n\}$ and $B = \{g_1, g_2, ..., g_p\}$. By the assumption, $m(f_i \otimes g_j) = m(f_i).m(g_j)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., p$. Therefore,

$$\begin{aligned}
H_R^m(A/B) &= \frac{R}{R-1}\left(\frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}} - \frac{1}{m(1_F)}[\Sigma_{j=1}^P \Sigma_{i=1}^n m(f_i \otimes g_j)^R]^{\frac{1}{R}}\right) \\
&= \frac{R}{R-1}\left(\frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}} - \frac{1}{m(1_F)}[\Sigma_{j=1}^P m(g_j)^R]^{\frac{1}{R}}.[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right) \\
&= \frac{R}{R-1}(m(1_F) - [\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} - m(1_F) \\
&+ [\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}} + \frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}} - \frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}}.[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}) \\
&= \frac{R}{R-1} m(1_F)\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right) \\
&- \frac{R}{R-1} m(1_F)\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i)^R]^{\frac{1}{R}}\right).\frac{R-1}{R}\left(\frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}}\right)\right) \\
&+ \frac{1 - m(1_F)}{m(1_F)} \frac{R}{R-1}[\Sigma_{j=1}^p m(g_j)^R]^{\frac{1}{R}} \\
&= m(1_F) H_R^m(A) - \frac{R-1}{R} m(1_F) H_R^m(A).H_R^m(B) + (1 - m(1_F))\left(\frac{R}{R-1} - H_R^m(B)\right).
\end{aligned}$$

□

Theorem 3.15. Let A and B be statistically independent partitions of F. Then,

$$H_R^m(A \vee B) = m(1_F) H_R^m(A) + m(1_F) H_R^m(B) - \frac{R-1}{R} m(1_F) H_R^m(A).H_R^m(B) + \frac{R}{R-1}(1 - m(1_F)).$$

Proof . This follows from Theorem 3.9 and Theorem 3.14. □

4 R-norm entropy of a dynamical system

In this section, we define and study the R-norm entropy of a dynamical system (F, m, S).

Proposition 4.1. Let (F, m, S) be a dynamical system, and $A = \{f_1, f_2, ..., f_n\}$ be a partition of F. Then, $SA = \{S(f_1), S(f_2), ..., S(f_n)\}$ is a partition of F.

Proof . Since $\oplus_{i=1}^n f_i$ exists, $\oplus_{i=1}^n S(f_i)$ exists by $(F8)$. By $(F9)$,

$$m(\oplus_{i=1}^n S(f_i)) = m(S(\oplus_{i=1}^n f_i) = m(\oplus_{i=1}^n f_i) = m(1_F) = \Sigma_{i=1}^n m(f_i) = \Sigma_{i=1}^n m(S(f_i)).$$

□

Definition 4.2. Let $S^2 = S \circ S$ and $S^k = S \circ S^{k-1}$ for $k = 1, 2, ...$, where S^0 is an identical mapping on F.

Proposition 4.3. Let (F, m, S) be a dynamical system, and A, B be partitions of F. If $A < B$, then $S(A) < S(B)$.

Proof . The proof can be found in [3]. □

Theorem 4.4. Let (F, m, S) be a dynamical system, and A, B be partitions of F. Then, the following statements are true.

(i) For $k = 0, 1, ...$, $H_R^m(S^k A) = H_R^m(A)$.

(ii) If S is invertible, then $H_R^m(S^{-k}(A)) = H_R^m(A)$ for $k = 0, 1, ...$

(iii) For $k = 0, 1, ...$, $H_R^m(S^k A/S^k B) = H_R^m(A/B)$.

(iv) $H_R^m(\vee_{i=0}^{n-1} S^i(A)) = H_R^m(A) + \Sigma_{j=1}^{n-1} H_R^m(A/ \vee_{i=1}^j S^i A)$.

Proof . Let $A = \{f_1, f_2, ..., f_n\}$.

(i) Since $m(S(f)) = m(f)$, for every $f \in F$, $m(S^k(f_i)) = m(f_i)$ for $i = 1, 2, ..., n$ and $k = 0, 1, ...,$

$$H_R^m(S^k A) = \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(S^k(f_i))^R]^{\frac{1}{R}}\right) = \frac{R}{R-1}\left(1 - \frac{1}{m(1_F)}[\Sigma_{i=1}^n m(f_i))^R]^{\frac{1}{R}}\right) = H_R^m(A),$$

for $k = 0, 1, ...$

(ii), (iii) These can be proved similar to (i).

(iv) We use mathematical induction. The assertion is true for $n = 2$ according to Theorem 3.9 and the previous part of this theorem. Assume that the assertion is true for some $n \in \mathbb{N}$. Since by (i)

$$H_R^m(\vee_{i=1}^n S^i A) = H_R^m(S(\vee_{i=0}^{n-1} S^i A) = H_R^m(\vee_{i=0}^{n-1} S^i A),$$

by Theorem 3.9 and the induction hypothesis,

$$\begin{aligned} H_R^m(\vee_{i=0}^n S^i A) &= H_R^m((\vee_{i=1}^n S^i A) \vee A) \\ &= H_R^m(\vee_{i=1}^n S^i A) + H_R^m(A/ \vee_{i=1}^n S^i A) \\ &= H_R^m(\vee_{i=0}^{n-1} S^i A) + H_R^m(A/ \vee_{i=1}^n S^i A) \\ &= H_R^m(A) + \Sigma_{j=1}^{n-1} H_R^m(A/ \vee_{i=1}^j S^i A) + H_R^m(A/ \vee_{i=1}^n S^i A) \\ &= H_R^m(A) + \Sigma_{j=1}^n H_R^m(A/ \vee_{i=1}^j S^i A). \end{aligned}$$

□

Definition 4.5. Let (F, m, S) be a dynamical system, and A be a partition of F. The *R-norm entropy* of S with respect to A is defined by

$$H_R^m(S, A) = \limsup \frac{1}{n} H_R^m(\vee_{i=0}^{n-1} S^i A),$$

when $n \to \infty$. The *R-norm entropy* of a dynamical system (F, m, S) is defined by

$$H_R^m(S) = \sup\{H_R^m(S, A)\},$$

where A runs over all partitions of F.

Example 4.6. Consider a triplet (F, m, s) from example 3.3 and the partition $A = \{f_1, f_2\}$ of F. Then the $R-$norm entropy of S with respect to A is calculate

$$\begin{aligned}
H_R^m(S, A) &= \limsup \frac{1}{n} H_A^m\left(\bigvee_{i=0}^{n-1} S^i A\right) \\
&= \limsup \frac{1}{n} H_A^m\left(\bigvee_{i=0}^{n-1} A\right) \\
&= \limsup \frac{1}{n} H_A^m(A) \\
&= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1 - 2^{\frac{1-R}{R}}\right)\right] \\
&= 0
\end{aligned}$$

where $n \to \infty$. Since for every partition A of F, $H_A^m\left(\bigvee_{i=0}^{n-1} S^i A\right)$ is constant. Thus $H_R^m(S, A) = 0$. Hence

$$H_R^m(S) = \sup\{H_R^m(S, A)\} = 0$$

where A runs over all partitions of F.

Theorem 4.7. Let (F, m, S) be a dynamical system, and A, B be partitions of F. Then, the following statements are true.

(i) $H_R^m(S, A) \geq 0$.

(ii) If $A < B$, then $H_R^m(S, A) \leq H_R^m(S, B)$.

(iii) For $p = 1, 2, ...,$ $H_R^m(S, A) \leq H_R^m(S, \vee_{i=0}^{p} S^i A)$.

Proof . (i) If $C = \vee_{i=0}^{n-1} S^i A$, then C is a partition of F. By Theorem 3.2, $H_R^m(C) \geq 0$. Hence, $H_R^m(S, A) \geq 0$.

(ii) The assumption $A < B$ implies $\vee_{i=1}^{n-1} S^i A < \vee_{i=0}^{n-1} S^i B$ for $n = 1, 2, ...$ Thus, by Theorem 3.13 (i), $H_R^m(\vee_{i=0}^{n-1} S^i A) \leq H_R^m(\vee_{i=0}^{n-1} S^i B)$ for $n = 1, 2, ...$ Therefore, $H_R^m(S, A) \leq H_R^m(S, B)$.

(iii) By Definition 4.5,

$$\begin{aligned}
H_R^m(S, \vee_{i=0}^{p} S^i A) &= \limsup H_R^m(\vee_{j=0}^{n-1} S^j(\vee_{i=0}^{p} S^i A)) \\
&= \limsup \frac{p+n}{n} \cdot \frac{1}{p+n} H_R^m(\vee_{i=0}^{p+n-1} S^i A) \\
&= \limsup \frac{1}{p+n} H_R^m(\vee_{i=0}^{p+n-1} S^i A) \\
&= H_R^m(S, A),
\end{aligned}$$

when $n \to \infty$. $\square$

Definition 4.8. [3]. We say that dynamical systems (F_1, m_1, S_1) and (F_2, m_2, S_2) are *isomorphic* if there exists a bijective mapping $\psi : F_1 \longrightarrow F_2$ satisfying the following conditions.

(i) The diagram $S_1 : F_1 \longrightarrow F_1, \psi : F_1 \longrightarrow F_2, S_2 : F_2 \longrightarrow F_2$ is commutative, that is, $\psi(S_1(f)) = S_2(\psi(f))$ for every $f \in F_1$.

(ii) For every $f, g \in F_1$, $\psi(f \otimes g) = \psi(f) \otimes \psi(g)$.

(iii) For any $f, g \in F_1$, $f \oplus g$ exists if and only if $\psi(f) \oplus \psi(g)$ exists. In this case, $\psi(f \oplus g) = \psi(f) \oplus \psi(g)$.

(iv) $m_1(1_{F_1}) = m_2(1_{F_2})$.

(v) For every $f \in F_1$, $m_1(f) = m_2(\psi(f))$.

Lemma 4.9. Let (F_1, m_1, S_1) and (F_2, m_2, S_2) be isomorphic systems, and $\psi : F_1 \longrightarrow F_2$ be an isomorphism. Then, the following statements are true for the inverse $\psi^{-1} : F_2 \longrightarrow F_1$.

(i) For every $f, g \in F_2$, $\psi^{-1}(f \otimes g) = \psi^{-1}(f) \otimes \psi^{-1}(g)$.

(ii) For any $f, g \in F_2$, if $f \oplus g$ exists, then $\psi^{-1}(f) \oplus \psi^{-1}(g)$ exists, too, and $\psi^{-1}(f \otimes g) = \psi^{-1}(f) \otimes \psi^{-1}(g)$.

(iii) For every $f \in F_2$, $m_1(\psi^{-1}(f)) = m_2(f)$.

(iv) For every $f \in F_2$, $m_1((\psi^{-1} \circ S_2)(f)) = m_1((S_1 \circ \psi^{-1})(f))$.

Proof . The proof can be found in [3]. □

Theorem 4.10. If dynamical systems (F_1, m_1, S_1) and (F_2, m_2, S_2) are isomorphic, then

$$H_R^m(S_1) = H_R^m(S_2).$$

Proof . Let $\psi : F_1 \longrightarrow F_2$ be an isomorphism. If $A = \{f_1, f_2, ..., f_n\}$ is a partition of F_1, then it is easy to verify that $\psi(A) = \{\psi(f_1), ..., \psi(f_n)\}$ is a partition of F_2. Indeed, since $\oplus_{i=1}^n f_i$ exists and

$$m_2(\oplus_{i=1}^n \psi(f_i)) = m_2(\psi(\oplus_{i=1}^n f_i)) = m_1(\oplus_{i=1}^n f_i) = m_1(1_{F_1}) = m_2(1_{F_2})$$

by Definition 4.8, $\oplus_{i=1}^n \psi(f_i)$ exists. On the other hand,

$$m_2(\oplus_{i=1}^n \psi(f_i)) = m_1(\oplus_{i=1}^n f_i) = \Sigma_{i=1}^n m_1(f_i) = \Sigma_{i=1}^n m_2(\psi F_i).$$

So,

$$\begin{aligned}
H_R^m(\psi(A)) &= \frac{R}{R-1}\left(1 - \left[\Sigma_{i=1}^n \left(\frac{m_2(\psi(f_i))}{m_2(1_{F_2})}\right)^R\right]^{\frac{1}{R}}\right) \\
&= \frac{R}{R-1}\left(1 - \left[\Sigma_{i=1}^n \left(\frac{m_1(f_i)}{m_1(1_{F_1})}\right)^R\right]^{\frac{1}{R}}\right) \\
&= H_R^m(A).
\end{aligned}$$

Therefore, using conditions (i) and (ii) of Definition 4.8 we obtain

$$\begin{aligned}
H_R^m(S_2, \psi(A)) &= \limsup \frac{1}{n} H_R^m(\vee_{i=0}^{n-1} S_2^i \psi(A)) \\
&= \limsup \frac{1}{n} H_R^m(\vee_{i=0}^{n-1} \psi(S_1^i A)) \\
&= \limsup \frac{1}{n} H_R^m(\psi(\vee_{i=0}^{n-1} S_1^i A)) \\
&= \limsup \frac{1}{n} H_R^m(\vee_{i=0}^{n-1} S_1^i A) \\
&= H_R^m(S_1, A),
\end{aligned}$$

when $n \to \infty$. Hence, $\{H_R^m(S_1, A) : A \text{ is a partition of } F_1\}$ is a subset of $\{H_R^m(S_2, B) : B \text{ is a partition of } F_2\}$. Thus,

$$H_R^m(S_1) = \sup\{H_R^m(S_1, A)\} \le \sup\{H_R^m(S_2, B)\} = H_R^m(S_2).$$

The reverse inequality can be proved in a similar way by the previous lemma concerning the isomorphism $\psi^{-1} : F_2 \longrightarrow F_1$. Therefore, $H_R^m(S_2) \le H_R^m(S_1)$. This completes the proof. □

Example 4.11. Consider the measurable space $([0, 1], B)$ where B is the $\sigma-$algebra of all Borel subsets of the unit interval $[0, 1]$. Put $F_1 = \{\chi_E : E \in B\}$, where $\chi_E : [0, 1] \to [0, 1]$ is the characteristic function of the set $E \in B$. We define binary operations $\oplus_1$ and $\otimes_1$ by $f \oplus_1 g = f + g$ and $f + g \le (1)_{[0,1]}$ and $f \otimes_1 g = \max\big(f + g - (1)_{[0,1]}, 0_{[0,1]}\big)$.

If we define the mapping $m_1 : F_1 \to [0, 1]$ and $s_1 : F_1 \to F_1$ by $m_1(f) = \int_0^1 f(x)dx$ and $d_1(f) = f$ for any element f of F. Then $\big(F_1, \oplus_1, \otimes_1, (1)_{[0,1]}\big)$ is an algebric structure. Let us consider triplet $([0, 1], B, \mu)$ where μ is lebesque measure. Put $F_2 = B$. We define binary operation $\oplus_2$ and $\otimes_2$ by $A \oplus_2 B = A \cup B$ if $A \cap B = \emptyset$ and $A \otimes_2 B = A \cap B$ where $A, B \in B$ and $1_{F_2} = [0, 1]$. If we define the mapping $m_2 : F_2 \to [0, 1]$ and $F_2 \to F_2$ by $m_2(A) = \mu(A)$ and $S_2(A) = A$ for any element A of F_2. Then $(F_2, \oplus_2, \otimes_1, 1_{F_2})$ is an algebric structure. The dynamical system (F_1, m_1, s_1) and (F_2, m_2, s_2) because bijective mapping and $\psi(s_1(\chi_E)) = \psi(\chi_E) = s_2(\psi(\chi_E))$ therefore the diagram

$$\begin{tikzcd} F_1 \arrow[r, "s_1"] \arrow[d, "\psi"'] & F_1 \arrow[d, "\psi"] \\ F_2 \arrow[r] & F_2 \end{tikzcd}$$

is commutative.

ii) For every $f, g \in F_1$,

$$\begin{aligned} \psi(f \otimes g) &= \psi(\chi_{E_1} \otimes_1 \chi_{E_2}) \\ &= \psi\left(\max\left[\chi_{E_1 \cup E_2} - (1)_{[0,1]}, 0_{[0,1]}\right]\right) \\ &= E_1 \cap E_2 \\ &= \psi(\chi_{E_1}) \otimes_2 \psi(\chi_{E_2}) \\ &= \psi(f) \otimes_2 \psi(g). \end{aligned}$$

iii) For any $f, g \in F_1$,

$$\begin{aligned} \psi(f \oplus_1 g) &= \psi(\chi_{E_1} \oplus_1 \chi_{E_2}) \\ &= \psi(\chi_{E_1} + \chi_{E_2}) \\ &= \psi(\chi_{E_1 \cup E_2}) \\ &= E_1 \cup E_2, \end{aligned}$$

if $\chi_{E_1} + \chi_{E_2} \leq (1)_{[0,1]}$ hence $E_1 \cap E_2 = \emptyset$. On the other hand

$$\psi(f) - \psi(g) = \psi(\chi_{E_1}) \oplus_2 \psi(\chi_{E_2}) = E_1 \oplus_2 E_2 = E_1 \cup E_2.$$

Hence

$$\psi(f \oplus_1 g) = \psi(f) \oplus_2 \psi(g).$$

(iv) $m_1(1_F) = \int_0^1 1dx = 1$ and $m_2(1_{F_2}) = \mu([0,1]) = 0 \Rightarrow m_1(1_F) = m_2(1_F)$.

(v) For every $f \in F_1$, $m_1(f) = m_1(\chi_E) = \mu(E)$ and $m_2(\psi(f)) = m_2(\psi(\chi_E)) = m_2(E) = \mu(E)$. Then $m_1(f) = m_2(\psi(f))$. If $A = \{f_1, f_2, \ldots, f_n\}$ be a partition of F_1 where $f_i = \chi_{E_i}$, for $i = 1, 2, \ldots, n$, then

$$\begin{aligned} H_R^{m_1}(s_1, A) &= \limsup \frac{1}{n} H_R^{m_1}\left(\bigvee_{i=0}^{n-1} s_i^i A\right) \\ &= \limsup \frac{1}{n} H_R^{m_1}(A) \\ &= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1 - \left[\sum_{i=1}^{n}\left(\frac{m_1(f_i)}{m(1_{F_1})}\right)^R\right]^{\frac{1}{R}}\right)\right] \\ &= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1 - \left[\sum_{i=1}^{n}\left(\frac{m_1(\chi_{E_i})}{1}\right)^R\right]^{\frac{1}{R}}\right)\right] \\ &= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1 - \left[\sum_{i=1}^{n}(\mu(E_i))^R\right]^{\frac{1}{R}}\right)\right] \qquad (4.1) \end{aligned}$$

when $n \to \infty$.

If $A = \{E_1, E_2, \ldots, E_n\}$ be a partition of F_2, then

$$\begin{aligned} H_R^{m_2}(S_2, A) &= \limsup \frac{1}{n} H_R^{m_2}\left(\bigvee_{i=0}^{n-1} S_2^i A\right) \\ &= \limsup \frac{1}{n} H_R^{m_2}(A) \\ &= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1-\left[\sum_{i=1}^{n}\left(\frac{m_2(E_i)}{m_2(1_{F_2})}\right)^R\right]^{\frac{1}{R}}\right)\right] \\ &= \limsup \frac{1}{n}\left[\frac{R}{R-1}\left(1-\left[\sum_{i=1}^{n}(\mu(E_i))^R\right]^{\frac{1}{R}}\right)\right] \qquad (4.2) \end{aligned}$$

when $n \to \infty$.

By (4.1) and (4.2), we get

$$H_R^{m_1}(S_1, A) = H_R^{m_2}(S_2, A).$$

Hence

$$H_R^{m_1}(S_1) = H_R^{m_2}(S_2).$$

If $A = \{f_1, f_2, \ldots, f_n\}$ be a partition of F_1 where $f_i = \chi_{E_i}$ for $i = 1, 2, \ldots, n$.

Conclusion

In the second section of this paper, we introduced and studied the notions of R-norm entropy and conditional R-norm entropy of finite partitions of algebraic structures. In Section 3, we observed that the R-norm entropy was non-negative and did not satisfy the property of additivity. In addition, it was shown that the conditional Shannon entropy of finite partitions in an algebraic structure could be derived from the conditional R-norm entropy by letting R tend to 1. In Section 4, using the proposed concept of R-norm entropy of partitions, we defined the R-norm entropy of a dynamical system and proved that such an entropy was invariant under isomorphisms. In the proofs, the Jensen inequality, L'Hôpital's rule and Minkowski's inequality were used.

References

[1] D. E. Boekke and J.C.A. Van der lubbe, *The R-norm information measure*. In. Control **45** (1980), 136–155.

[2] M. Ebrahimi and B. Mosapour, *The concept of entropy on D-posets*, Cankaya Univ. J. Sci. Eng. **10** (2013), 137–151.

[3] A. Ebrahimzadeh, Z. Eslami Giski, and D. Markechova, *Logical entropy of dynamical systems-A general model*, Mathematics **5** (2017), no. 1, 4.

[4] A. Ebrahimzadeh and J. Jamalzadeh, *Conditional logical entropy of fuzzy σ-algebras*, J. Intell. Fuzzy Syst. **33** (2017), 1019–1027.

[5] A. Ebrahimzadeh, *Logical entropy of quantum systems*, Open Phys. **14** (2016), 1–5.

[6] A. Ebrahimzadeh, *Quantum conditional logical entropy of dynamical systems*, Ital. J. Pure Apple. Math. **36** (2016), 879–886.

[7] D. Ellerman, *An introduction to logical entropy and its relation to Shannon entropy*, Int. J. Semant. Comput. **7** (2013), 121–145.

[8] Z. Eslami Giski and M. Ebrahimi, *Entropy of countable partitions on effect algebras with the Riesz decomposition property and weak sequential effect algebras*, Cankaya Univ. J. Sci. Eng. **12** (2015), 20–39.

[9] Z. Eslami Giski and A. Ebrahimzadeh, *An introduction of logical entropy on sequential effect algebra*, Indag. Math. **28** (2017), 928–937.

[10] D.S. Hooda and R.K. Bajij, *On generalized R-norm information measures of fuzzy information*, J. Appl. Math. Stat. Inf. **4** (2008), 199–212.

[11] D. S. Hooda and A. Ram, *Charactrizatiion of a generalized measure of R-norm entropy*, Caribb. J. Math. Comput. Sci. **8** (2002), 18–31.

[12] D.S. Hooda and D.K. Sharma, *Generalized R-norm information measures*, J. Appl. Math. Stat. Inf. **4** (2008), 153–168.

[13] M. Khare, *Fuzzy σ-algebras and conditional entropy*, Fuzzy Sets Syst. **102** (1999), 287–292.

[14] S. Kumar and A. Choudhary, *Generalized parametric R-norm information measure*. Trends Appl. Sci. Res. **7** (2012), 350–369.

[15] S. Kumar, A. Choudhary, and R. Kumar, *Some more results on a generalized parametric R-norm information measure of type α*, J. Appl. Sci. Eng. **7** (2014), 447–453.

[16] D. Markechova, B. Mosapour, and A. Ebrahimzadeh, *Logical divergence, logical entropy, and logical mutual information in product MV-algebras*, Entropy **20** (2018), 129.

[17] D. Markechova, B. Mosapour, and A. Ebrahimzadeh, *R-norm entropy and R-norm divergence in fuzzy probability spaces*, Entropy **20** (2018), 272.

[18] D. Markechova, *Kullback-Leibler divergence and mutual information of experiment in the fuzzy case*, Axioms **6** (2017), no. 1, 5.

[19] D. Markechova and B. Riecan, *Kullback-Leibler divergence and mutual information of partitions in product MV-algebras*, Entropy **19** (2017), 267.

[20] D. Markechova and B. Riecan, *Logical entropy of fuzzy dynamical systems*, Entropy **18** (2016), 157.

[21] D. Markechova and B. Riecan, *Renyi entropy and Renyi divergence in product MV-algebras*, Entropy **20** (2018), 587.

[22] U. Mohammadi, *The concept of logical entropy on D-posets*, J. Algebra Struct. Appl. **1** (2016), 53–61.

[23] B. Riecan, *Kolmogorov-sinaj entropy on MV-algebras*, Int. J. Theor. Phys. **44** (2005), 1041–1052.

[24] B. Riecan and D. Mundici, *Probability on MV-algebras*, Handbook of measure theory, Elsevier. Amsterdam, The Netherlands, 2002, pp. 869–910.

[25] C.E. Shannon, *Mathematical Theory of communication*, Bell Syst. Tech. J. **27** (1948), 379–423.