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Generalized hybrid contraction in weak partial metric spaces

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Abstract

In this paper, a common fixed point theorem for a generalized hybrid contraction map in weak partial metric space is proved. We also give illustrated examples in support of our result. Moreover, we provide a homotopy result as an application of our result.

Keywords: Generalized Hybrid contraction mapping, Weak Partial metric space, Partial Hausdorff metric, Coincidence Point, Common fixed point
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1 Introduction

The theory of non-linear analysis has emerged as a fascinating field. Many authors have generalized and extended Banach contraction principle. In 1969, the generalization of the famous Banach contraction principle for multi-valued mappings using Hausdorff metric is done by Nadler [10]. A rapid progress has been observed using weak and generalized contraction mappings afterwards. Multi-valued contraction mapping has many applications in differential equations, economics and control theory. Let (X, d) be a metric space and $CB(X)$, the class of all nonempty closed and bounded subsets of X. The Hausdorff metric [2] induced by d on $CB(X)$ is

$$H(A,B) = max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for every $A, B \in CB(X)$, where $d(a, B) = inf\{d(a, b) : b \in B\}$ is the distance from a to $B \subseteq X$. Let $f : X \to X$ be a single-valued mapping and $U : X \to CB(X)$ be a multi-valued mapping.

(i) A point $w \in X$ is said to be a fixed point of f (resp. U) if $fw = w$ (resp. $w \in X$). The set of all fixed points of f (resp. U) is denoted by $F(f)$(resp. $F(U)$).

(ii) A point $w \in X$ is said to be a coincidence point of f and U if $fw \in Uw$. The set of all coincidence points of f and U is denoted by $C(f, U)$.

(iii) A point $w \in X$ is a common fixed point of f and U if $w = fw \in Uw$. The set of all common fixed points of f and U is denoted by $F(f, U)$.

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In 1969, Nadler proved the following theorem-

Theorem 1.1. [10] Let (X, d) be a complete metric space and $U : X \to CB(X)$ be a multi-valued mapping satisfying

$$H(Ux, Uy) \leq kd(x, y), \;\; \forall x, y \in X$$

where $k \in [0, 1)$ then there exists $x \in X$ such that $x \in Ux$.

The concept of (IT)- commutativity for a hybrid pair of single-valued and multivalued mappings is introduced by Singh and Mishra [14]. Further In 2004, Kamran [8] introduced a weaker condition than (IT)- commutativity for a hybrid pair of single-valued and multivalued maps which is the notion of $T-$ weak commutativity. The definitions of (IT)- commutativity and T- weak commutativity are as follows:

Definition 1.2. [14] A mapping $f : X \longrightarrow X$ and $U : X \longrightarrow CB(X)$ is known as (IT)- commuting at $w \in X$ if $fUw \subseteq Ufw$.

Definition 1.3. [8] Let $f : X \longrightarrow X$ and $U : X \longrightarrow CB(X)$, the map f is known $T-$ weakly commuting at $w \in X$ if $ffw \in Ufw$.

On the other hand, many authors introduced and generalized the distance notion in the metric fixed point theory in several different ways. In 1992, Mathews [9] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow networks. He presented a modified version of Banach contraction principle. Several authors have done work in this direction [1, 3, 6, 7].

2 Preliminaries

Mathews gave the following definition of partial metric space:

Definition 2.1. [9] Let X be a non empty set. Then a mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x, y, z \in X$,

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$;

(P4) $p(x, y) \leq p(x, z) + p(z, x) - p(z, z)$.

The pair (X, p) is called a partial metric space.

Recently, a weaker form of partial metric space is introduced by Ismat Beg and H. K. Pathak [5] known as weak partial metric space and defined as:

Definition 2.2. [5] Let X be a non empty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$, the following conditions hold :

(WP1) $q(x, x) = q(x, y) \Leftrightarrow x = y$;

(WP2) $q(x, x) \leq q(x, y)$;

(WP3) $q(x, y) = q(y, x)$;

(WP4) $q(x, y) \leq q(x, z) + q(z, x)$.

The pair (X, q) is a weak partial metric space. Further, many authors have worked on weak partial metric space [4, 11, 12].

Example 2.3. (i) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = \max\{x, y\} + e^{|x-y|} \;\forall\; x, y \in \mathbb{R}^+.$$

(ii) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = \frac{1}{6} \max\{x, y\} \;\; \forall x, y \in \mathbb{R}^+.$$

- ⋄ If $q(x, y) = 0$, then (WP1) and (WP2) $\Rightarrow x = y$. But the converse need not be true.
- ⋄ (P1) $\Rightarrow$(WP1), but the converse need not be true.
- ⋄ (P4) $\Rightarrow$ (WP4), but the converse need not be true.

Each weak partial metric q on X generates a T_0 topology τ_q on X. Topology τ_q has as a base the family of open q-balls $\{B_q(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x, \epsilon) = \{y \in X : q(x, y) < q(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.
If q is weak partial metric on X, then the function $q^s : X \times X \to \mathbb{R}^+$ given by

$$q^s(x, y) = q(x, y) - \frac{1}{2}[q(x, x) + q(y, y)]$$

defines a metric on X.

Definition 2.4. [5] Let (X, q) be a weak partial metric space. Then

(i) P is said to be a bounded subset in (X, q) if there exists $x \in X$ and $L \geq 0$ such that for all $p \in P$, we have $p \in B_q(x_0, L)$ that is

$$q(x_0, p) < q(p, p) + L.$$

(ii) A sequence $\{x_n\}$ in (X, q) converges to a point $x \in X$, w.r.t. τ_q iff $q(x, x) = \lim_{n\to\infty} q(x, x_n)$. Moreover, a sequence $\{x_n\}$ converges in (X, q^s) to a point $x \in X$ iff

$$\lim_{n\to\infty m\to\infty} q(x_n, x_m) = \lim_{n\to\infty} q(x_n, x) = q(x, x)$$

(iii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} q(x_n, x_m)$ exists and is finite.

(iv) (X, q) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ with respect to topology τ_q.

Lemma 2.5. [5] Let (X, q) be a weak partial metric space. Then

(a) A sequence $\{x_n\}$ in X is Cauchy sequence in (X, q) if and only if it is a Cauchy sequence in the metric space (X, q^s).

(b) (X, q) is complete iff the metric space (X, q^s) is complete.

For $L, M \in CB^q(X)$ and $x \in X$ define $q(x, L) = \inf\{q(x, l) : l \in L\}, \delta_q(L, M) = \sup\{q(l, M) : l \in L\}$ and $\delta_q(M, L) = \sup\{q(m, L) : m \in M\}$. Clearly, $q(x, L) = 0$ implies that $q^s(x, L) = 0$ where $q^s(x, L) = \inf\{q^s(x, l) : l \in L\}$.

Remark 2.6. [1] Let (X, q) be a weak partial metric space and L be any non empty set in (X, q), then

$$l \in \bar{L} \Leftrightarrow q(l, L) = q(l, l)$$

where $\bar{L}$ denotes the closure of L with respect to weak partial metric q. Observe that L is closed in (X, q) iff $L = \bar{L}$.

Now, we study the following properties of the mapping $\delta_q : CB^q(X) \times CB^q(X) \to [0, \infty)$.

Proposition 2.7. [5] Let (X, q) be a weak partial metric space. For all $L, M, N \in CB^q(X)$, we have the following :

(a) $\delta_q(L, L) = sup\{q(l, l) : l \in L\}$;

(b) $\delta_q(L, L) \leq \delta_q(L, M)$;

(c) $\delta_q(L, M) = 0 \Rightarrow L \subseteq M$;

(c) $\delta_q(L, M) \leq \delta_q(L, N) + \delta_q(N, M)$.

Proposition 2.8. [5] Let (X, q) be a weak partial metric space. For all $L, M, N \in CB^q(X)$, we have

(wh1) $H_q^+(L, L) \leq H_q^+(L, M)$;

(wh2) $H_q^+(L, M) = H_q^+(M, L)$;

(wh3) (wh3) $H_q^+(L, M) \leq H_q^+(L, N) + H_q^+(N, M)$.

Definition 2.9. [5] Let (X, q) be a weak partial metric space. For $L, M \in CB^q(X)$, define

$$H_q^+(L, M) = \frac{1}{2}\{\delta_q(L, M) + \delta_q(M, L)\}$$

The mapping $H_q^+ : CB^q(X) \times CB^q(X) \to [0, +\infty)$ is called H_q^+- type Hausdorff metric induced by q.

Definition 2.10. [5] Let (X, q) be a weak partial metric space. A multi-valued map $U : X \to CB^q(X)$ is called H_q^+-contraction if

(1) There exists $\alpha \in (0, 1)$ such that

$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq q(x, y) \quad \text{for every } x, y \in X$$

(2) For every x in X, y in $U(x)$ and $\epsilon > 0$, there exists z in $U(y)$ such that

$$q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon$$

Remark 2.11. Since, $\max\{a, b\} \geq \frac{1}{2}(a + b)$, for all $a, b \geq 0$, which follows that H_q contraction always implies H_q^+-contraction but the converse need not be true.

A variant of Nadler's fixed point theorem is given by Beg and Pathak [5] which is stated as:

Theorem 2.12. [5] Every H_q^+- type multi-valued contraction map $U : X \to CB^q(X)$ on a complete weak partial metric space has a fixed point.

Definition 2.13. [13] Let (X, q) be a weak partial metric space. A mapping $f : X \to X$ be a single valued mapping and $U : X \to CB^q(X)$ be a multi-valued mapping. U is said to be a H_q^+ - hybrid contraction if

(1) There exists $\alpha \in (0, 1)$ such that

$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq \alpha q(fx, fy) \quad \text{for every } x, y \in X.$$

(2) For every x in X, y in $U(x)$ and $\epsilon > 0$, there exists z in $U(y)$ such that

$$q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

Recently, Saxena and Gairola [13] prove a fixed point theorem for hybrid contraction map in weak partial metric space.

Theorem 2.14. Let (X, q) be a weak partial metric space, $f : X \longrightarrow X$ be a single-valued mapping and $U : X \longrightarrow CB^q(X)$ be a H_q^+- type hybrid contraction mapping. Suppose fX is a complete subspace of X and $Ux \subset fX$. Then f and U have a coincidence point. Furthermore, if f is $T-$ weakly commuting at coincidence points of f and U, then f and U have a common fixed point.

3 Main Result

We define H_q^+-type generalized hybrid contraction mapping as follows:

Definition 3.1. Let (X, q) be a weak partial metric space. A mapping $f : X \to X$ be a single valued mapping and $U : X \to CB^q(X)$ be a multi-valued mapping. U is said to be a H_q^+ - generalized hybrid contraction if

(1) There exists $\alpha > 0,\ \ \beta > 0,\ \ \gamma > 0$ such that

$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq \alpha M(x,y) + \beta.max\{q(fx,Ux), q(fy,Uy)\} + \gamma[q(fx,Uy) + q(fy,Ux)]$$

where

$$M(x,y) = max\{q(fx,fy), q(fx,Ux), q(fy,Uy), \frac{1}{2}[q(fx,Uy) + q(fy,Ux)]\}$$

and $2\alpha + \beta + 3\gamma \leq k < 1$.

(2) For every x in X, y in $U(x)$ and $\epsilon > 0$, there exists z in $U(y)$ such that

$$q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon$$

Example 3.2. Let (X, q) be a weak partial metric space w.r.t. weak partial metric $q : X \times X \to [0, \infty)$ where $X = [0, 1]$ and q is defined by $q = \frac{1}{4}\max\{x, y\}$, for all $x, y \in X$, define the maps $U : X \to CB^q(X)$ such that

$$U(x) = \left[0, \frac{3}{4}\right] \qquad \forall x \in X,$$

and $f : X \to X$ such that

$$f(x) = \begin{cases} 1 \ \ if \ \ x \in \left[0, \frac{1}{2}\right) \\ \frac{3x}{4} \ \ if \ \ x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Clearly we can see that q is weak partial metric on X and (X, q) is a weak partial metric space w.r.t. q. Now, for all $x, y \in X$, we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

Case(i) $x \in \left[0, \frac{1}{2}\right], y \in \left[0, \frac{1}{2}\right]$ we have

$$H_q^+(Ux\backslash\{x\}, Uy\backslash\{y\}) = H_q^+(\left(\frac{1}{2}, \frac{3}{4}\right], \left(\frac{1}{2}, \frac{3}{4}\right]) = \frac{3}{16} \leq \frac{1}{4}(\alpha + \beta + 2\gamma)$$

and (1) satisfied.

Case(ii) $x \in \left[0, \frac{1}{2}\right], y \in \left(\frac{1}{2}, 1\right]$, we have

$$H_q^+(Ux\backslash\{x\}, Uy\backslash\{y\}) = H_q^+(\left(\frac{1}{2}, \frac{3}{4}\right], \left[0, \frac{1}{2}\right]) = \frac{3}{16} \leq \frac{1}{4}(\alpha + \beta + \frac{7}{4}\gamma)$$

and (1) satisfied.

Case(iii) $x \in \left(\frac{1}{2}, 1\right], y \in \left[0, \frac{1}{2}\right]$, we have

$$H_q^+(Ux\backslash\{x\}, Uy\backslash\{y\}) = H_q^+(\left[0, \frac{1}{2}\right], \left(\frac{1}{2}, \frac{3}{4}\right]) = \frac{3}{16} \leq \frac{1}{4}(\alpha + \beta + \frac{7}{4}\gamma)$$

and (1) satisfied.

Case(iv) $x \in \left(\frac{1}{2}, 1\right], y \in \left(\frac{1}{2}, 1\right]$, we have

$$H_q^+(Ux\backslash\{x\}, Uy\backslash\{y\}) = H_q^+(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]) = \frac{1}{8} \leq \frac{3}{16}(\alpha + \beta + 2\gamma)$$

and (1) satisfied.

Condition (1) is satisfied in all the possible cases. Further, we shall show that for every $x \in X, y \in U(x)$ and $\epsilon > 0$, there exists $z \in U(y)$ such that $q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon$. Indeed,

(i) For $x \in \left[0, \frac{1}{2}\right], y \in U(x) = \left[0, \frac{3}{4}\right], \epsilon > 0$, there exists $z \in U(y) = \left[0, \frac{3}{4}\right]$ such that

$$\frac{3}{16} = q(y, z) < \frac{3}{16} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(ii) For $x \in \left(\frac{1}{2}, 1\right], y \in U(x) = \left[0, \frac{3}{4}\right], \epsilon > 0$, there exists $z \in U(y) = \left[0, \frac{3}{4}\right]$ such that

$$\frac{3}{16} = q(y, z) < \frac{3}{16} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

Hence, contractive conditions (1) and (2) are satisfied. Also, for all $x \in \left[\frac{1}{2}, 1\right], f(x) \in U(x)$. Therefore $x \in \left[\frac{1}{2}, 1\right]$ are the coincidence points of f and U.

Now, we prove the following theorem for H_q^+- generalized hybrid contraction mapping.

Theorem 3.3. Let (X, q) be a weak partial metric space. $f : X \longrightarrow X$ be a single-valued mapping and $U : X \longrightarrow CB^q(X)$ be a H_q^+- type generalized hybrid contraction mapping. Suppose fX is a complete subspace of X and $Ux \subset fX$. Then f and U have a coincidence point. Furthermore, if f is $T-$ weakly commuting at coincidence points of f and U then f and U have a common fixed point.

Proof . Let x_0 be an arbitrary point of X and fx_0 also let $\lambda = k + 2\epsilon < 1$. We construct sequences $\{x_k\}$ in X. Since $Ux \subset fX$, there exists $x_1 \in X$ such that $fx_1 \in Ux_0$. If $M(x_1, x_0) = 0$, then x_0 is a coincidence point. Hence, assume $M(x_1, x_0) > 0$. Now, there exists $fx_2 \in Ux_1$ such that $q(fx_1, fx_2) \leq H_q^+(Ux_0, Ux_1) + \epsilon.M(x_0, x_1)$ Similarly, assume $M(x_1, x_2) > 0$. Again by (2) and the fact $Ux \subset fX$, there exists $fx_3 \in Ux_2$ such that $q(fx_2, fx_3) \leq H_q^+(Ux_1, Ux_2) + \epsilon.M(x_1, x_2)$, assume $M(x_2, x_3) > 0$.

Proceeding in this way, we can construct a sequence $fx_{n+1} \in Ux_n$, assume $q(fx_n, fx_{n+1}) > 0$ satisfying

$$q(fx_n, fx_{n+1}) \leq H_q^+(Ux_{n-1}, Ux_n) + \epsilon.M(x_{n-1}, x_n)$$

By using (1), we get

$$\begin{aligned}
q(fx_n, fx_{n+1}) &\leq H_q^+(Ux_{n-1}, Ux_n) + \epsilon M(x_{n-1}, x_n) \\
&= H_q^+(Ux_{n-1}\backslash\{x_{n-1}\}, Ux_n\backslash\{x_n\}) + \epsilon M(x_{n-1}, x_n) \\
&\leq \alpha M(x_{n-1}, x_n) + \beta \max\{q(fx_{n-1}, Ux_{n-1}), q(fx_n, Ux_n)\} \\
&\quad + \gamma[q(fx_{n-1}, Ux_n) + q(fx_n, Ux_{n-1})] + \epsilon M(x_{n-1}, x_n) \\
&= (\alpha + \epsilon). \max\{q(fx_{n-1}, fx_n), q(fx_{n-1}, Ux_{n-1}), q(fx_n, Ux_n), \frac{1}{2}[q(fx_{n-1}, Ux_n) + q(fx_n, Ux_{n-1})]\} \\
&\quad + \beta \max\{q(fx_{n-1}, Ux_{n-1}), q(fx_n, Ux_n)\} + \gamma[q(fx_{n-1}, Ux_n) + q(fx_n, Ux_{n-1})] \\
&\leq (\alpha + \epsilon). \max\{q(fx_{n-1}, fx_n), q(fx_{n-1}, fx_n), q(fx_n, fx_{n+1}), \frac{1}{2}[q(fx_{n-1}, fx_{n+1}) + q(fx_n, fx_n)]\} \\
&\quad + \beta. \max\{q(fx_{n-1}, fx_n), q(fx_n, fx_{n+1})\} + \gamma[q(fx_{n-1}, fx_{n+1}) + q(fx_n, fx_n)] \\
&= (\alpha + \epsilon). \max\{q(fx_{n-1}, fx_n), q(fx_n, fx_{n+1}), \frac{1}{2}[q(fx_{n-1}, fx_n) + q(fx_n, fx_{n+1}) + q(fx_n, fx_n)]\} \\
&\quad + \beta. \max\{q(fx_{n-1}, fx_n), q(fx_n, fx_{n+1})\} + \gamma[q(fx_{n-1}, fx_n) + q(fx_n, fx_{n+1}) + q(fx_n, fx_n)]
\end{aligned} \tag{3.1}$$

Now, if $q(fx_n, fx_{n+1}) > q(fx_{n-1}, fx_n)$, then by (2.1) we have

$$\begin{aligned} q(fx_n, fx_{n+1}) &\leq (\alpha+\epsilon). \max\{q(fx_n, fx_{n+1}), \frac{3}{2}q(fx_n, fx_{n+1})\} + \beta q(fx_n, fx_{n+1}) + \gamma 3q(fx_n, fx_{n+1}) \\ &\leq (2\alpha+\beta+3\gamma+2\epsilon).q(fx_n, fx_{n+1}) \\ &= \lambda q(fx_n, fx_{n+1}) \end{aligned}$$

Since $2\alpha+\beta+3\gamma+2\epsilon < 1$, above inequality implies that $q(fx_n, fx_{n+1}) = 0$. Then $fx_n = fx_{n+1}$ but $fx_n \neq fx_{n+1}$. So, a contradiction occurs. Hence

$$q(fx_n, fx_{n+1}) \leq q(fx_{n-1}, fx_n). \tag{3.-1}$$

Thus,

$$\begin{aligned} q(fx_n, fx_{n+1}) &\leq (\alpha+\epsilon). \max\{q(fx_{n-1}, fx_n), \frac{3}{2}q(fx_{n-1}, fx_n)\} + \beta q(fx_{n-1}, fx_n) \\ &\quad + \gamma 3q(fx_{n-1}, fx_n) \\ &\leq (2\alpha+\beta+3\gamma+2\epsilon).q(fx_{n-1}, fx_n) \\ &= \lambda q(fx_{n-1}, fx_n). \end{aligned}$$

Adopting similar process, we obtain

$$q(fx_{n+1}, fx_{n+2}) \leq \lambda q(fx_n, fx_{n+1}).$$

Now, by induction on n, we get

$$q(fx_n, fx_{n+1}) \leq \lambda^n q(fx_0, fx_1).$$

For any $m \in \mathbb{N}$, we have

$$\begin{aligned} q^s(fx_n, fx_{n+m}) &\leq q(fx_n, fx_{n+m}) \\ &\leq q(fx_n, fx_{n+1}) + q(fx_{n+1}, fx_{n+2}) + q(fx_{n+2}, fx_{n+3}) + ... + q(fx_{n+m-1}, fx_{n+m}) \\ &\leq \lambda^n q(fx_0, fx_1) + \lambda^{n+1} q(fx_0, fx_1) + \lambda^{n+2} q(fx_0, fx_1) + ... + \lambda^{n+m-1} q(fx_0, fx_1) \\ &= (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + ... + \lambda^{n+m-1}) q(fx_0, fx_1) \\ &\leq \frac{\lambda^n}{1-\lambda}.q(fx_0, fx_1) \longrightarrow 0 \quad as \quad n \to \infty. \end{aligned}$$

This implies that $\{fx_k\}$ where $k = 1, 2, 3, ...$ is a Cauchy sequence in (X, q^s). Since fX is complete, there exists $w \in X$ such that the sequence fx_n converges to fw as $n \longrightarrow \infty$ w.r.t. the metric q^s, that is, $\lim_{n\to\infty} q^s(fx_n, fw) = 0$. Moreover, we have

$$q(fw, fw) = \lim_{n\to\infty} q(fx_n, fw) = \lim_{n\to\infty} q(fx_n, fx_n) = 0.$$

We now show that $fw \in Uw$. Now, by triangle inequality,

$$\begin{aligned} q(fw, Uw) &\leq q(fw, fx_k) + q(fx_k, Uw) \\ &\leq q(fw, fx_k) + H_q^+(Ux_{k-1}, Uw) \\ &= q(fw, fx_k) + H_q^+(Ux_{k-1}\backslash\{x_{k-1}\}, Uw\backslash\{w\}) \\ &\leq q(fw, fx_k) + \alpha M(x_{k-1}, w) + \beta. \max\{q(fx_{k-1}, Ux_{k-1}), q(fw, Uw)\} \\ &\quad + \gamma[q(fx_{k-1}, Uw) + q(fw, Ux_{k-1})] \\ &= q(fw, fx_k) + \alpha. \max\{q(fx_{k-1}, fw), q(fx_{k-1}, Ux_{k-1}), q(fw, Uw), \frac{1}{2}[q(fx_{k-1}, Uw) + q(fw, Ux_{k-1})]\} \\ &\quad + \beta. \max\{q(fx_{k-1}, fx_k), q(fw, Uw)\} + \gamma[q(fx_{k-1}, Uw) + q(fw, fx_k)]. \end{aligned}$$

Letting $k \to \infty$ we get

$$q(fw, Uw) \leq (\alpha+\beta+\gamma) q(fw, Uw).$$

As $\alpha+\beta+\gamma<1$, therefore $q(fw,Uw)=0$. Since Uw is closed, $fw\in Uw$. Therefore, f and U have a coincidence point $w\in X$. Let $t=fw\in Uw$. It follows from the definition of H_q^+- type Hausdroff metric that

$$\begin{aligned}
q(t,ft)\le q(t,Ut) &= q(fw,Ut)\\
&\le H_q^+(Uw,Ut)\\
&= H_q^+(Uw\backslash\{w\},Ut\backslash\{t\})\\
&\le \alpha M(w,t)+\beta.\max\{q(fw,Uw),q(ft,Ut)\}+\gamma[q(fw,Ut)+q(ft,Uw)]\\
&= \alpha.\max\{q(fw,ft),q(fw,Uw),q(ft,Ut),\frac{1}{2}[q(fw,Ut)+q(ft,Uw)]\}\\
&\quad +\beta.\max\{q(fw,Uw),q(ft,Ut)\}+\gamma[q(fw,Ut)+q(ft,Uw)]\\
&= (\alpha+\gamma).q(fw,ft)\\
&= (\alpha+\gamma).q(t,ft).
\end{aligned}$$

This implies that $q(t,ft)=0$. It follows from $q(ft,Ut)=q(fw,Ut)\le H_q^+(Uw,Ut)=0$. Since Ut is closed, $t=ft\in Ut$. Thus f and U have a common fixed point. □

Example 3.4. Let (X,q) be a weak partial metric space w.r.t. weak partial metric $q:X\times X\to[0,\infty)$ where $X=\{0,1,2\}$ and q is defined by

$$q(0,0)=q(1,1)=0,q(2,2)=\frac{4}{9},q(0,1)=\frac{1}{3},q(0,2)=\frac{11}{24},q(1,2)=\frac{1}{2}\ \ \forall\ \ x,y\in X.$$

Define the maps $U:X\to CB^q(X)$ such that

$$U(x)=\begin{cases}\{0\}, & \text{if } x=\{0,1\}\\ \{0,1\}, & \text{if } x=2\end{cases}$$

and $f:X\to X$ such that

$$f(x)=x \quad \text{for all} \quad x\in X.$$

Since $q(2,2)=\dfrac{4}{9}\ne 0$, q is not a metric on X. Here $Ux\subset fX$. Also, note that Ux is closed and bounded for all $x\in X$ under the given weak partial metric space (X,q). Now, for all $x,y\in X$, we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

(i) $x=0,y=0$. We have
$$H_q^+(U(0)\backslash\{0\},U(0)\backslash\{0\})=H_q^+(\phi,\phi)=0$$
and (1) is satisfied.

(ii) $x=0,y=2$. We have
$$H_q^+(U(0)\backslash\{0\},U(2)\backslash\{2\})=H_q^+(\{0\},\{0,1\})=\frac{1}{6}\le\frac{11}{24}(\alpha+\beta+\gamma)$$
and (1) is satisfied.

(iii) $x=2,y=0$. We have
$$H_q^+(U(2)\backslash\{2\},U(0)\backslash\{0\})=H_q^+(\{0,1\},\{0\})=\frac{1}{6}\le\frac{11}{24}(\alpha+\beta+\gamma)$$
and (1) is satisfied.

(iv) $x=0,y=1$. We have
$$H_q^+(U(0)\backslash\{0\},U(1)\backslash\{1\})=H_q^+(\phi,\{0\})=0$$
and (1) is satisfied.

(v) $x = 1, y = 0$. We have

$$H_q^+(U(1)\backslash\{1\}, U(0)\backslash\{0\}) = H_q^+(\{0\}, \phi) = 0$$

and (1) is satisfied.

(vi) $x = 2, y = 2$. We have

$$H_q^+(U(2)\backslash\{2\}, U(2)\backslash\{2\}) = H_q^+(\{0,1\}, \{0,1\}) = 0$$

and (1) is satisfied.

(vii) $x = 2, y = 1$. We have

$$H_q^+(U(2)\backslash\{2\}, U(1)\backslash\{1\}) = H_q^+(\{0,1\}, \{0\}) = \frac{1}{6} \leq \frac{12\alpha + 11\beta + 11\gamma}{24}$$

and (1) is satisfied.

(viii) $x = 1, y = 2$. We have

$$H_q^+(U(1)\backslash\{1\}, U(2)\backslash\{2\}) = H_q^+(\{0\}, \{0,1\}) = \frac{1}{6} \leq \frac{12\alpha + 11\beta + 11\gamma}{24}$$

and (1) is satisfied.

(ix) $x = 1, y = 1$. We have

$$H_q^+(U(1)\backslash\{1\}, U(1)\backslash\{1\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

Further, we shall show that for every x in X, y in $U(x)$ and $\epsilon > 0$, there exists z in $U(y)$ such that $q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon$. Indeed,

(1) if $x = 0, y \in U(0) = \{0\}, \epsilon > 0,$, there exists $z \in U(y) = \{0\}$ such that

$$0 = q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

(2a) if $x = 2,\ y \in U(2) = \{0,1\}$, say $y = 0, \epsilon > 0,$, there exists $z \in U(y) = \{0\}$, such that

$$0 = q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

(2b) if $x = 2,\ y \in U(2) = \{0,1\}$, say $y = 1, \epsilon > 0,$, there exists $z \in U(y) = \{0\}$, such that

$$\frac{1}{3} = q(y,z) < \frac{1}{6} + \epsilon = H_q^+(U(y), U(x)) + \epsilon.$$

(3) If $x = 1, y \in U(1) = \{0\}, \epsilon > 0$, there exists $z \in U(0) = \{0\}$ such that

$$0 = q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

Here $x = 0$ is the coincidence point of f and U. Also f is T-weakly commuting at coincidence point.

(i) For $x = 0, ff(0) = 0$ and $Uf(0) = \{0\}$. Thus $ff(0) \in Uf(0)$.

(ii) For $x = 1, ff(1) = 1$ and $Uf(1) = \{0\}$. Thus $ff(1) \notin Uf(1)$.

(iii) For $x = 2, ff(2) = 2$ and $Uf(2) = \{0,1\}$. Thus $ff(2) \notin Uf(2)$

Hence, all the conditions of theorem are satisfied. Here $x = 0 = f(0) \in U(0)$ so $x = 0$ is a common fixed point of f and U.

Example 3.5. Let (X, q) be a weak partial metric space w.r.t. weak partial metric $q : X \times X \to [0, \infty)$ where $X = [0, 1]$ and q is defined by $q = \max\{x, y\}$, define the maps $U : X \to CB^q(X)$ such that

$$U(x) = \left[0, \frac{3}{8}\right] \qquad \forall x \in X$$

and $f : X \to X$ such that

$$f(x) = \begin{cases} \frac{3x}{4} \ \ if \ \ x \in \left[0, \frac{1}{2}\right] \\ \\ 1 \ \ if \ \ x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Clearly, we can see that q is weak partial metric but not a metric on X. Here $Ux \subset fX$. Also, note that Ux is closed and bounded for all $x \in X$ under the given weak partial metric space (X, q). Now, for all $x, y \in X$, we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

Case(i) $x \in \left[0, \frac{1}{2}\right], y \in \left[0, \frac{1}{2}\right]$ we have

$$H_q^+(Ux \backslash \{x\}, Uy \backslash \{y\}) = H_q^+(\phi, \phi) = 0$$

and (1) satisfied.

Case(ii) $x \in \left[0, \frac{1}{2}\right], y \in \left(\frac{1}{2}, 1\right]$, we have

$$H_q^+(Ux \backslash \{x\}, Uy \backslash \{y\}) = H_q^+(\phi, \left[0, \frac{3}{8}\right]) = 0$$

and (1) satisfied.

Case(iii) $x \in \left(\frac{1}{2}, 1\right], y \in \left[0, \frac{1}{2}\right]$, we have

$$H_q^+(Ux \backslash \{x\}, Uy \backslash \{y\}) = H_q^+(\left[0, \frac{3}{8}\right], \phi) = 0$$

and (1) satisfied.

Case(iv) $x \in \left(\frac{1}{2}, 1\right], y \in \left(\frac{1}{2}, 1\right]$, we have

$$H_q^+(Ux \backslash \{x\}, Uy \backslash \{y\}) = H_q^+(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]) = \frac{1}{2} \leq \alpha + \beta + 2\gamma$$

and (1) satisfied.

Further, we shall show that for every $x \in X, y \in U(x)$ and $\epsilon > 0$, there exists $z \in U(y)$ such that $q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon$. Indeed,

(i) For $x \in \left[0, \frac{1}{2}\right], y \in U(x) = \left[0, \frac{3}{8}\right], \epsilon > 0$, there exists $z \in U(y) = \left[0, \frac{3}{8}\right]$ such that

$$\frac{3}{8} = q(y, z) < \frac{3}{8} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(ii) For $x \in \left(\frac{1}{2}, 1\right], y \in U(x) = \left[0, \frac{3}{8}\right], \epsilon > 0$, there exists $z \in U(y) = \left[0, \frac{3}{8}\right]$ such that

$$\frac{3}{8} = q(y, z) < \frac{3}{8} + \epsilon = H_q^+(U(y), U(x)) + \epsilon.$$

Hence, Condition (1) and (2) are satisfied. Here $X = 0, 1$ are the coincidence points of f and U. Furthermore, f is $T-$weakly commuting at $x = 0$.

(i) For $x = 0, ff(0) = 0$ and $Uf(0) = \left[0, \frac{3}{8}\right]$. Thus, $ff(0) \in Uf(0)$.

(ii) For $x = 1, ff(1) = 1$ and $Uf(1) = \left[0, \frac{3}{8}\right]$. Thus, $ff(1) \notin Uf(1)$.

Hence, all the conditions of theorem are satisfied. Here $x = 0 = f(0) \in U(0)$, so $x = 0$ is a common fixed point of f and U.

Corollary 3.6. In theorem 2.1, if $M(x, y) = q(fx, fy)$, taking $\beta = \gamma = 0$ we get theorem 1.5 as special case of our result.

In Theorem 2.1, taking $f = I$(Identity map) we get the following corollary.

Corollary 3.7. Let (X, q) be a complete weak partial metric space and U be a multivalued map from X to $CB^q(X)$ such that for all $x, y \in X$,

$$\begin{aligned} H_q^+(Ux\backslash\{x\}, Uy\backslash\{y\}) \leq &\alpha \max\{q(fx, fy), q(fx, Ux), q(fy, Uy), \frac{1}{2}[q(fx, Uy) + q(fy, Ux)]\} \\ &+ \beta \max\{q(fx, Ux), q(fy, Uy)\} + \gamma[q(fx, Uy) + q(fy, Ux)] \end{aligned}$$

where $\alpha + \beta + \gamma \leq k < 1$. Then U has a fixed point.

Again by taking $\beta = 0, \gamma = 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 3.8. Let (X, q) be a weak partial metric space. Let $f : X \to X$ and $U : X \to CB^q(X)$ be a single valued and multivalued maps respectively such that $U(x) \subset fX$ and fX is a complete subspace of X. If for all $x, y \in X$,

1. there exists $\alpha \in (0, 1)$ such that
$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq \alpha M(x, y)$$
where
$$M(x, y) = \max\left\{q(fx, fy), q(fx, Ux), q(fy, Uy), \frac{1}{2}[q(fx, Uy) + q(fy, Ux)]\right\}.$$

2. for every x in X, y in $U(x)$ and $\epsilon > 0$, there exists z in $U(y)$ such that
$$q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon$$

Then f and U have a coincidence point. Furthermore if f is T-weakly commutative then f and U have a common fixed point.

4 Application

In this section, we give an application of our main result on homotopy for weak partial metric spaces. Let (X, q) be a weak partial metric space, $\mathbb{R}$ be endowed with Hausdorff topology and let [0,1] be equipped with subspace topology. First, we observe that

(i) Weak partial metric q on X generates a T_0 topology τ_q on X.

(ii) A topological space X is connected if and only if its only clopen sets are X and ϕ.

Theorem 4.1. Let (X, q) be a weak partial metric space, $f : X \to X$ such that $F(x, t) \subset fX$ and fX be a complete subspace of X, A be an open subset of X and C be a closed subset of X, with $A \subset C$. Let $F : C \times [0, 1] \to CB^q(X)$ be an operator satisfying:

(a) $x \notin F(x,t)$ for every $x \in C\backslash A$ and each $t \in [0,1]$,

(b1) there exists $\alpha,\ \beta,\ \gamma > 0$ such that for each $t \in [0,1]$ and each $x, y \in C$ we have

$H_q^+(F(x,t)\backslash\{x\}, F(y,t)\backslash\{y\}) \leq \alpha M(x,y) + \beta.\max\{q(fx, F(x,t)), q(fy, F(y,t))\} + \gamma[q(fx, F(y,t)) + q(fy, F(x,t))]$.

where $M(x,y) = \max\{q(fx,fy), q(fx, F(x,t)), q(fy, F(y,t)), \frac{1}{2}[q(fx, F(y,t)) + q(fy, F(x,t))]\}$

(b2) for every $x \in C, y \in F(x,t)$ and $\epsilon > 0, \exists\ z \in F(y,t)$ such that

$$q(y,z) \leq H_q^+(F(y.t), F(x,t)) + \epsilon$$

(c) there exists a continuous function $\eta : [0,1] \to \mathbb{R}$ such that

$$H_q^+(F(x,t)\backslash\{x\}, F(x,s)\backslash\{x\}) \leq \alpha|\eta(t) - \eta(s)|$$

for all $t, s \in [0,1]$ and each $x \in C$,

(d) if $x \in F(x,t)$ then $F(x,t) = \{x\}$, then $F(.,0)$ has a fixed point if and only if $F(.,1)$ has a fixed point.

Proof . Let $F(.,0)$ has a fixed point. Consider the set

$$Q := \{t \in [0,1] | x \in F(x,t) \text{ for some } \ x \in A\}.$$

As (a) holds and $F(.,0)$ has a fixed point, we have $0 \in Q$, so Q is a non-empty set. Now we show that Q is both closed and open in $[0,1]$. Thus, by the connectedness of $[0,1]$ we are accomplished since $Q = [0,1]$.

First, let us prove that Q is open in [0,1]. For this, let $t_0 \in Q$ and $x_0 \in F(x_0, t_0)$. As A is open in (X, q), there exists $r > 0$ such that $B_q(fx_0, r) \subseteq A$. Consider $\epsilon = r + q(fx_0, fx_0) - (\alpha+\beta+\gamma)(r + q(fx_0, fx_0)) > 0$. Since η is continuous on t_0, there exists $k(\epsilon) > 0$ such that $|\eta(t) - \eta(t_0)| < \epsilon$ for all $t \in (t_0 - k(\epsilon), t_0 + k(\epsilon))$ for $x \in \overline{B_q(fx_0, r)} = \{fx \in X | q(fx_0, fx) \leq q(fx_0, fx_0) + r\}$, thus

$$\begin{aligned}
q(F(x,t), fx_0) &\leq H_q^+(F(x,t), F(x_0,t_0)) \\
&\leq H_q^+(F(x,t), F(x,t_0)) + H_q^+(F(x,t_0), F(x_0,t_0)) \\
&= H_q^+(F(x,t)\backslash\{x\}, F(x,t_0)\backslash\{x\}) + H_q^+(F(x,t_0)\backslash\{x\}, F(x_0,t_0)\backslash\{x_0\}) \\
&\leq \alpha|\eta(t) - \eta(t_0)| + \alpha M(x,x_0) + \beta.\max\{q(fx, F(x,t)), q(fx_0, F(x_0,t_0))\} \\
&+ \gamma[q(fx, F(x_0,t_0)) + q(fx_0, F(x,t))] \\
&\leq \alpha|\eta(t) - \eta(t_0)| + \alpha q(fx, fx_0) + \beta.q(fx, F(x,t)) + \gamma[q(fx, fx_0) + q(fx_0, F(x,t))] \\
&\leq \alpha\epsilon + \alpha(q(fx, fx_0) + r) + \beta(q(fx, fx_0) + r) + \gamma(q(fx, fx_0) + r) \\
&\leq \alpha\epsilon + (\alpha+\beta+\gamma)(q(fx, fx_0) + r) \\
&\leq \alpha\epsilon + (\alpha+\beta+\gamma)(q(fx_0, fx_0) + r) \\
&\leq \alpha\{r + q(fx_0, fx_0) - (\alpha+\beta+\gamma)(r + q(fx_0, fx_0))\} + (\alpha+\beta+\gamma)(q(fx_0, fx_0) + r) \\
&< r + q(fx_0, fx_0) - (\alpha+\beta+\gamma)(r + q(fx_0, fx_0)) + (\alpha+\beta+\gamma)(q(fx_0, fx_0) + r) \\
&= r + q(fx_0, fx_0).
\end{aligned}$$

Note that (b1) implies (1). It follows that for every $t \in (t_0 - k(\epsilon), t_0 + k(\epsilon))$, $F(.,t) : \overline{B_q(fx_0, r)} \to CB^q(X)$ satisfies all the hypothesis of theorem 2.1 and so $F(.,t)$ has a fixed point in $B_q(fx_0, r) \subset C$. But this fixed point must be in A as (a) holds. Hence $(t_0 - k(\epsilon), t_0 + k\epsilon) \subseteq Q$ and therefore Q is open in $[0,1]$.

Next, we show that Q is closed in $[0,1]$. To prove this, let $\{t_n\}, n \in \mathbb{N}$ be a sequence in Q with $t_n \to t^* \in [0,1]$ as $n \to \infty$. We must prove that $t^* \in Q$. By the definition of Q, for all $n \in \mathbb{N}$, there exists $x_n \in A$ such that $fx_n \in A$ with $fx_n \in F(x_n, t_n)$. Then, for $m, n \in \mathbb{N}$, using (d) and (wh3) we get

$$\begin{aligned}
q(fx_n, fx_m) &= H_q^+(F(x_n,t_n), F(x_m,t_m)) \\
&\leq H_q^+(F(x_n,t_n), F(x_n,t_m)) + H_q^+(F(x_n,t_m), F(x_m,t_m)) \\
&= H_q^+(F(x_n,t_n)\backslash\{x_n\}, F(x_n,t_m)\backslash\{x_n\}) + H_q^+(F(x_n,t_m)\backslash\{x_n\}, F(x_m,t_m)\backslash\{x_m\}) \\
&\leq \alpha|\eta(t_n) - \eta(t_m)| + \alpha M(x_n,x_m) + \beta\max\{q(fx_n, F(x_n,t_n), q(fx_m, F(x_m,t_m))\} \\
&+ \gamma[q(fx_n, F(x_m,t_m)) + q(fx_m, F(x_n,t_n))]
\end{aligned}$$

where,

$$M(x_n, x_m) = \max\{q(fx_n, fx_m), q(fx_n, F(x_n, t_n)), q(fx_m, F(x_m, t_m)), \frac{1}{2}[q(fx_n, F(x_m, t_m)) + q(fx_m, F(x_n, t_n))]\}.$$

It further implies that

$$\begin{aligned} q(fx_n, fx_m) &\leq \alpha|\eta(t_n) - \eta(t_m)| + \alpha q(fx_n, fx_m) + \beta q(fx_m, fx_m) + 2\gamma q(fx_n, fx_m) \\ &\leq \alpha|\eta(t_n) - \eta(t_m)| + (\alpha + \beta + 2\gamma) q(fx_n, fx_m) \end{aligned}$$

and we have

$$q(fx_n, fx_m) \leq \frac{\alpha}{1 - (\alpha + \beta + 2\gamma)} |\eta(t_n) - \eta(t_m)|.$$

Since η is continuous and $t_n, n \in \mathbb{N}$ is convergent, letting $m, n \to \infty$ in the above inequality, we obtain $\lim\limits_{n,m\to\infty} q(fx_n, fx_m) = 0$, that is, $\{fx_n\}, n \in \mathbb{N}$ is a Cauchy sequence in(X, q). Since fX is complete subspace of X, there exists $fx^* \in C$ with $q(fx^*, fx^*) = \lim\limits_{n\to\infty} q(fx^*, fx_n) = \lim\limits_{n,m\to\infty} q(fx_n, fx_m) = 0$. On the other hand, we have

$$\begin{aligned} q(fx_n, F(x^*, t^*)) &= H_q^+(F(x_n, t_n), F(x^*, t^*)) \\ &\leq H_q^+(F(x_n, t_n), F(x_n, t^*)) + H_q^+(F(x_n, t^*), F(x^*, t^*)) \\ &\leq H_q^+(F(x_n, t_n)\backslash\{x_n\}, F(x_n, t^*)\backslash\{x_n\}) + H_q^+(F(x_n, t^*)\backslash\{x_n\}, F(x^*, t^*)\backslash\{x^*\}) \\ &\leq \alpha|\eta(t_n) - \eta(t^*)| + \alpha M(x_n, x^*) + \beta \max\{q(fx_n, F(x_n, t_n)), q(fx^*, F(x^*, t^*))\} \\ &+ \gamma[q(fx_n, F(x^*, t^*)) + q(fx^*, F(x_n, t_n))] \end{aligned}$$

where,

$$M(x_n, x^*) = \max\{q(fx_n, fx^*), q(fx_n, F(x_n, t_n)), q(fx^*, F(x^*, t^*)), \frac{1}{2}[q(fx_n, F(x^*, t^*)), q(fx^*, F(x_n, t_n))]\}.$$

So,

$$\begin{aligned} q(fx_n, F(x^*, t^*)) &\leq \alpha|\eta(t_n) - \eta(t^*)| + \alpha q(fx_n, fx^*) + \beta q(fx_n, fx^*) + 2\gamma q(fx_n, fx^*) \\ &\leq \alpha|\eta(t_n) - \eta(t^*)| + (\alpha + \beta + 2\gamma) q(fx_n, fx^*). \end{aligned}$$

Letting $n \to \infty$ in te above inequality, we get $\lim\limits_{n\to\infty} q(fx_n, F(x^*, t^*)) = 0$ and so

$$q(fx^*, F(x^*, t^*)) = \lim_{n\to\infty} q(fx_n, F(x^*, t^*)) = 0.$$

It follows that $fx^* \in F(x^*, t^*)$. Thus $t^* \in Q$ and hence Q is closed in $[0, 1]$. By the connectedness of $[0, 1]$ we have $Q = [0, 1]$. The reverse implication easily follows by applying the same method. This completes the proof. □

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