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Star ordering of range symmetric matrices in indefinite inner product space

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Abstract

In this paper, we obtain some characterizations of range symmetric matrices and utilize them to study the partial ordering of range symmetric matrices with respect to the indefinite inner product. As a consequence of this, different characterizations of partial orders on range symmetric matrices are obtained.

Keywords: Indefinite inner product, range symmetric, partial order
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1 Introduction

An indefinite inner product is a conjugate symmetric sesquilinear form $[x, y] = \langle x, Jy\rangle$, where $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product. Investigations of linear maps on indefinite inner product utilize the usual multiplication of matrices which is induced by the Euclidean inner product of vectors ([3],[22]). This causes a problem as there are two different values for dot product of vectors. To beat this trouble, a new indefinite matrix product studied and introduced by Ramanathan et. al [22] in 2004. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, individually, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is characterized to be the matrix $J_nA^*J_m$, where J_m and J_n are weights. Indefinite matrix product concept was discussed further by many researcher in ([7], [8], [9], [12], [13], [14], [20], [21]). Summarizing the equivalent conditions for the definition of a range symmetric matrix form [7, 10, 12], the following equivalent conditions will be used in the forthcoming results:

[RS-1] $\mathbf{N}$ is range symmetric,

[RS-2] $Nu(\mathbf{N}) = Nu(\mathbf{N}^{[*]})$,

[RS-3] $\mathbf{N} \circ \mathbf{N}^{[\dagger]} = \mathbf{N}^{[\dagger]} \circ \mathbf{N}$,

[RS-4] $Ra(\mathbf{N}) = Ra(\mathbf{N}^{[*]})$,

[RS-5] their exist a J-unitary matrix $\mathbf{U}$ such that $\mathbf{N} = \mathbf{U}_\mathbf{N}(\mathbf{D}_\mathbf{N} \bigoplus \mathbf{0})\mathbf{U}_\mathbf{N}^{[*]}$.

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Partial orders on matrices has remained the topic of interest for many authors in the area of matrix theory and generalized inverse. Almost all authors who have worked on partial ordering of matrices have formulated the definition involving different kinds of generalized inverses and in particular the Moore-Penrose Inverse. Results involving partial orders on matrices in relation with their generalized inverse are scattered in the literature of the matrix theory and generalized inverses for instance see ([1], [2], [6], [11], [16], [17], [19]). Different kinds of partial orders on matrices have been studied which include Star partial ordering $\leq^*$ introduced by Drazin [4], minus partial order $\leq^-$ introduced by Hartwig [5], Sharp partial $\leq^\#$ order introduced by Mitra [18]. In [7], Jayaraman introduced the partial ordering on matrices in indefinite inner product space. He also established some equivalent conditions for the reverse order law to hold in relation to the partial ordering with respect to indefinite matrix product. For any two matrices $\mathbf{N}, \mathbf{S} \in M_{(m,n)}(\mathbb{C})$, $\mathbf{N}$ is said to be below $\mathbf{S}$ under the partial order with respect to the adjoint, denoted by $\mathbf{N} \leq^{[*]} \mathbf{S}$, if one of the following equivalent conditions is satisfied:

[PO-1] $\mathbf{N}^{[*]} \circ \mathbf{N} = \mathbf{N}^{[*]} \circ \mathbf{S}$ and $\mathbf{N} \circ \mathbf{N}^{[*]} = \mathbf{S} \circ \mathbf{N}^{[*]}$,

[PO-2] $\mathbf{N}^{[\dagger]} \circ \mathbf{N} = \mathbf{N}^{[\dagger]} \circ \mathbf{S}$ and $\mathbf{N} \circ \mathbf{N}^{[\dagger]} = \mathbf{S} \circ \mathbf{N}^{[\dagger]}$,

In any of the above cases, we say $\mathbf{N}$ is predecessor of $\mathbf{S}$ or $\mathbf{S}$ is successor of $\mathbf{N}$. We will use the notation $M_n^k(\mathbb{C}) = M_n^k$ to denote the set of all the matrices of index k.

2 Preliminaries

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. [22] A matrix $A \in M_n(\mathbb{C})$ is said to be J-invertible if there exists $X \in M_n(\mathbb{C})$, such that $A \circ X = X \circ A = J_n$ such an X is denoted by $A^{[-1]} = JA^{-1}J$.

Definition 2.2. [15] A matrix $A \in M_n(\mathbb{C})$ is said to be J-unitary if $A \circ A^{[*]} = A^{[*]} \circ A = J$.

Definition 2.3. [9] A matrix $A \in M_n(\mathbb{C})$ is said to be J-symmetric if $A = A^{[*]}$.

Definition 2.4. [13] For $A \in M_{(m,n)}(\mathbb{C})$, a matrix X satisfying $A \circ X \circ A = A$ is called a generalized inverse of A relative to the weight J. $A_J\{1\}$ is the set of all generalized inverses of A relative to the weight J.

Remark 2.5. [13] For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\} = A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if J_nXJ_m is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X : J_nXJ_m \text{ is a generalized inverse of A}\}$.

Definition 2.6. [7] For $A \in M_{(m,n)}(\mathbb{C})$, and $X \in M_{(n,m)}(\mathbb{C})$ is called the Moore-Penrose inverse of A if it satisfies the following equations:
(i) $A \circ X \circ A = A$.
(ii) $X \circ A \circ X = X$.
(iii) $(A \circ X)^{[*]} = A \circ X$.
(iv) $(X \circ A)^{[*]} = X \circ A$.
such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^\dagger J_m$.

Definition 2.7. [22] The range space of $A \in M_{(m,n)}(\mathbb{C})$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$. The null space of $A \in M_{(m,n)}(\mathbb{C})$ is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.

Property 2.8. [12] Let $A \in M_{(m,n)}(\mathbb{C})$. Then
(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$.
(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv) $Ra(A^{[*]}) = Ra(A^{[\dagger]})$.
(v) $Ra(A \circ A^{[*]}) = Ra(A),\ Ra(A^{[*]} \circ A) = Ra(A^{[*]})$.
(vi) $Nu(A \circ A^{[*]}) = Nu(A^{[*]}),\ Nu(A^{[*]} \circ A) = Nu(A)$.

Definition 2.9. [12] $A \in M_n(\mathbb{C})$ is range symmetric(J-EP) in $\mathscr{I}$ if and only if $Ra(A) = Ra(A^{[*]})$ $(A \circ A^{[\dagger]} = A^{[\dagger]} \circ A)$.

Definition 2.10. [7] $A \in M_n(\mathbb{C})$ is $J - EP$ in $\mathscr{I}$ if and only if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.

Remark 2.11. [12] In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.

3 Properties of Range Symmetric matrices

In this section we develop some properties of range symmetric matrices by utilizing the representation obtained in [10]. Let $\mathbf{N}, \mathbf{S} \in M_n(\mathbb{C})$ be non-zero range symmetric matrices of rank q and s respectively. Then $\mathbf{N}$ and $\mathbf{S}$, according to the above mentioned result, can be written as

$$\mathbf{N} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_N^{[*]}}, \tag{3.1}$$

and

$$\mathbf{S} = \mathbf{U_S} \begin{pmatrix} \mathbf{D_S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}}, \tag{3.2}$$

where $\mathbf{U_N}$ and $\mathbf{U_S}$ are J-unitary and $\mathbf{D_N}$ and $\mathbf{D_S}$ are invertible matrices of order $r \times r$.

Theorem 3.1. Let $\mathbf{N}, \mathbf{S} \in M_n(\mathbb{C})$ be such that $\mathbf{N}$ is range symmetric. Then, the following statements are equivalent:

(i) $\mathbf{N} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{N}$,

(ii) If $\mathbf{N}$ is given by (3.1), then there exists $\mathbf{G} \in M_r(\mathbb{C})$ and $\mathbf{M} \in M_{n-r}(\mathbb{C})$ such that $\mathbf{S} = \mathbf{U_N} \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$ with $\mathbf{D_N G} = \mathbf{G D_N}$.

Proof . We consider the decomposition of the matrix $\mathbf{S}$, according to the size of blocks of $\mathbf{N}$, as:

$$\mathbf{S} = \mathbf{U_N} \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}.$$

From the statement (i) of the Theorem, we get

$$\begin{pmatrix} \mathbf{D_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} = \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{D_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

This gives $\mathbf{D_N G} = \mathbf{G D_N}, \mathbf{L} = \mathbf{0}$ and $\mathbf{K} = \mathbf{0}$. Hence, the result follows. □

Theorem 3.2. Let $\mathbf{N}, \mathbf{S} \in M_n(\mathbb{C})$ be range symmetric matrices. If $\mathbf{U_N^{[*]}} \circ \mathbf{U_S} = \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix}$. Then, the following statements are equivalent:

(i) $\mathbf{N} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{N}$,

(ii) $(\mathbf{G^{[*]} D_N G})\mathbf{D_S} = (\mathbf{D_S G^{[*]} D_N})\mathbf{G}$,

(iii) $\mathbf{D_N}(\mathbf{G D_S G^{[*]}}) = (\mathbf{G D_S G^{[*]}})\mathbf{D_N}$.

Proof . $(i) \Leftrightarrow (ii)$ Consider the representations of $\mathbf{N}$ and $\mathbf{S}$ given by (3.1) and (3.2) respectively. With given $\mathbf{U_N^{[*]}} \circ \mathbf{U_S} = \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix}$, we have

$$\mathbf{N} \circ \mathbf{S} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{D_S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N G D_S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}}. \tag{3.3}$$

Also

$$\mathbf{U_S^{[*]}} \circ \mathbf{U_N} = (\mathbf{U_N^{[*]}} \circ \mathbf{U_S})^{[*]} = \begin{pmatrix} \mathbf{G^{[*]}} & \mathbf{L^{[*]}} \\ \mathbf{K^{[*]}} & \mathbf{M^{[*]}} \end{pmatrix}.$$

Therefore,

$$\mathbf{S}\circ\mathbf{N}=\mathbf{U_S}\begin{pmatrix}\mathbf{D_S} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{G}^{[*]} & \mathbf{L}^{[*]}\\ \mathbf{K}^{[*]} & \mathbf{M}^{[*]}\end{pmatrix}\begin{pmatrix}\mathbf{D_N} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_N}^{[*]}=\mathbf{U_S}\begin{pmatrix}\mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_N}^{[*]}. \tag{3.4}$$

From equations (3.3) and (3.4), we have

$$\mathbf{U_N}\begin{pmatrix}\mathbf{D_N}\mathbf{G}\mathbf{D_S} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_S}^{[*]}=\mathbf{U_S}\begin{pmatrix}\mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_N}^{[*]}. \tag{3.5}$$

Premultiplying and postmultiplying (3.5) by $\mathbf{U_S}^{[*]}$ and $\mathbf{U_S}$ respectively and substituting the matrix representation of $\mathbf{U_S}^{[*]}\circ\mathbf{U_N}$ and $\mathbf{U_N}^{[*]}\circ\mathbf{U_S}$ in (3.5), we get

$$\begin{pmatrix}\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{G}\mathbf{D_S} & \mathbf{0}\\ \mathbf{K}^{[*]}\mathbf{D_N}\mathbf{G}\mathbf{D_S} & \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{G} & \mathbf{0}\\ \mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{K} & \mathbf{0}\end{pmatrix}.$$

From this equality, on using the fact that $\mathbf{D_N}$ and $\mathbf{D_S}$ are nonsingular, we have $(\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{G})\mathbf{D_S}=\mathbf{D_S}(\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{G})$, $\mathbf{K}^{[*]}\mathbf{D_N}\mathbf{G}=\mathbf{0}$ and $\mathbf{G}^{[*]}\mathbf{D_N}\mathbf{K}=\mathbf{0}$. Hence, the equivalence follows.

$(i)\Leftrightarrow(iii)$ From $\mathbf{U_N}^{[*]}\circ\mathbf{U_S}=\begin{pmatrix}\mathbf{G} & \mathbf{K}\\ \mathbf{L} & \mathbf{M}\end{pmatrix}$, using the fact that $\mathbf{U_N}$ is J-unitary, we have $\mathbf{U_S}=\mathbf{U_N}\begin{pmatrix}\mathbf{G} & \mathbf{K}\\ \mathbf{L} & \mathbf{M}\end{pmatrix}$. Therefore, $\mathbf{U_S}^{[*]}=\begin{pmatrix}\mathbf{G}^{[*]} & \mathbf{L}^{[*]}\\ \mathbf{K}^{[*]} & \mathbf{M}^{[*]}\end{pmatrix}\mathbf{U_N}^{[*]}$. Substituting the representations of $\mathbf{U_S}$ and $\mathbf{U_S}^{[*]}$ in the block representation of $\mathbf{S}$ given by (3.2), we have

$$\mathbf{S}=\mathbf{U_N}\begin{pmatrix}\mathbf{G} & \mathbf{K}\\ \mathbf{L} & \mathbf{M}\end{pmatrix}\begin{pmatrix}\mathbf{D_S} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{G}^{[*]} & \mathbf{L}^{[*]}\\ \mathbf{K}^{[*]} & \mathbf{M}^{[*]}\end{pmatrix}\mathbf{U_N}^{[*]}=\mathbf{U_N}\begin{pmatrix}\mathbf{G}\mathbf{D_S}\mathbf{G}^{[*]} & \mathbf{G}\mathbf{D_S}\mathbf{L}^{[*]}\\ \mathbf{L}\mathbf{D_S}\mathbf{G}^{[*]} & \mathbf{L}\mathbf{D_S}\mathbf{L}^{[*]}\end{pmatrix}\mathbf{U_N}^{[*]}.$$

Furthermore, doing some algebra, we have

$$\mathbf{N}\circ\mathbf{S}=\mathbf{U_N}\begin{pmatrix}\mathbf{D_N}\mathbf{G}\mathbf{D_S}\mathbf{G}^{[*]} & \mathbf{D_N}\mathbf{G}\mathbf{D_S}\mathbf{L}^{[*]}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_N}^{[*]}\text{ and }\mathbf{S}\circ\mathbf{N}=\mathbf{U_N}\begin{pmatrix}\mathbf{G}\mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N} & \mathbf{0}\\ \mathbf{L}\mathbf{D_S}\mathbf{G}^{[*]}\mathbf{D_N} & \mathbf{0}\end{pmatrix}\mathbf{U_N}^{[*]}.$$

Therefore, the equality $\mathbf{N}\circ\mathbf{S}=\mathbf{S}\circ\mathbf{N}$, on using the fact that $\mathbf{D_N},\mathbf{D_S}$ are nonsingular, gives

$$\mathbf{D_N}(\mathbf{G}\mathbf{D_S}\mathbf{G}^{[*]})=(\mathbf{G}\mathbf{D_S}\mathbf{G}^{[*]})\mathbf{D_N},\mathbf{L}\mathbf{D_S}\mathbf{G}^{[*]}=\mathbf{0}\text{ and }\mathbf{G}\mathbf{D_S}\mathbf{L}^{[*]}=\mathbf{0}.$$

Hence, the equivalence follows. □

Theorem 3.3. Let $\mathbf{N}\in M_n(\mathbb{C})$ be such that $\mathbf{N}^{[\dagger]}$ exists. Then, the following statements are equivalent:

(i) $\mathbf{N}$ is range symmetric in $\mathscr{I}$,
(ii) $Ra(\mathbf{N})=Ra(\mathbf{N}^{[\dagger]})$,
(iii) $Nu(\mathbf{N})=Nu(\mathbf{N}^{[\dagger]})$.

Proof . $(i)\Leftrightarrow(ii)$ Since $\mathbf{N}\circ\mathbf{N}^{[\dagger]}$ and $\mathbf{N}^{[\dagger]}\circ\mathbf{N}$ are J-projectors, on using [RS-3], we have $\mathbf{N}$ is range symmetric if and only if $\mathbf{N}\circ\mathbf{N}^{[\dagger]}=\mathbf{N}^{[\dagger]}\circ\mathbf{N}$. Also from the first two conditions in the definition of Moore-Penrose inverse, we have $Ra(\mathbf{N})=Ra(\mathbf{N}\circ\mathbf{N}^{[\dagger]})$ and $Ra(\mathbf{N}^{[\dagger]})=Ra(\mathbf{N}^{[\dagger]}\circ\mathbf{N})$. Therefore, $Ra(\mathbf{N})=Ra(\mathbf{N}^{[\dagger]})$. Hence the equivalence follows.

$(i)\Leftrightarrow(iii)$ Since $(\mathbf{I_n}-\mathbf{N}\circ\mathbf{N}^{[\dagger]})$ and $(\mathbf{I_n}-\mathbf{N}^{[\dagger]}\circ\mathbf{N})$ are idempotents such that $Nu(\mathbf{N}^{[\dagger]})=Ra(\mathbf{I_n}-\mathbf{N}\circ\mathbf{N}^{[\dagger]})$ and $Nu(\mathbf{N})=Ra(\mathbf{I_n}-\mathbf{N}^{[\dagger]}\circ\mathbf{N})$. Again using [RS-3], the result follows. □

Theorem 3.4. Let $\mathbf{N}\in M_n(\mathbb{C})$ be a non zero matrix. Then, the following statements are equivalent:

(i) $\mathbf{N}$ is range symmetric in $\mathscr{I}$
(ii) There exists an invertible matrix $\mathbf{M}\in M_{n-r}(\mathbb{C})$ and $\mathbf{L}\in M_{(n-r,r)}(\mathbb{C})$ such that $\mathbf{N}=\mathbf{N}^{[*]}\circ\mathbf{E}$ with $\mathbf{E}=\mathbf{U_N}\begin{pmatrix}(\mathbf{D_N}^{[*]})^{[-1]}\mathbf{D_N} & \mathbf{0}\\ \mathbf{L} & \mathbf{M}\end{pmatrix}\mathbf{U_N}^{[*]}$,

(iii) There exists an invertible matrix $\mathbf{M} \in M_{n-r}(\mathbb{C})$ and $\mathbf{L} \in M_{(n-r,r)}(\mathbb{C})$ such that $\mathbf{N} = \mathbf{N}^{[\dagger]} \circ \mathbf{E}$ with $\mathbf{E} = \mathbf{U_N} \begin{pmatrix} (\mathbf{D_N})^{\mathbf{2}} & \mathbf{0} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$.

Proof . $(i) \Leftrightarrow (ii)$ Using [RS-4], there exists an invertible matrix $\mathbf{E} \in M_n(\mathbb{C})$ such that $\mathbf{N} = \mathbf{N}^{[*]} \circ \mathbf{E}$. We partition $\mathbf{E}$ according to the blocks of $\mathbf{N}$ such that

$$\mathbf{E} = \mathbf{U_N} \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}.$$

Now, using the fact that $\mathbf{D_N}$ is invertible and J-unitary, the equality $\mathbf{N} = \mathbf{N}^{[*]} \circ \mathbf{E}$, gives

$$\mathbf{E} = \mathbf{U_N} \begin{pmatrix} (\mathbf{D_N^{[*]}})^{[-\mathbf{1}]}\mathbf{D_N} & \mathbf{0} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}.$$

$(i) \Leftrightarrow (iii)$ From statement (ii) of the Theorem 3.3 and [RS-4], we have $Ra(\mathbf{N}^{[\dagger]}) = Ra(\mathbf{N}^{[*]})$, and the equivalence follows on the same lines as above. □

Theorem 3.5. Let $\mathbf{N} \in M_n(\mathbb{C})$ be a nonzero matrix. Then, the following statements are equivalent:

(i) $\mathbf{N}$ is range symmetric,

(ii) There exists an invertible matrix $\mathbf{M} \in M_{n-r}(\mathbb{C})$ and $\mathbf{K} \in M_{(r,n-r)}(\mathbb{C})$ such that $\mathbf{N} = \mathbf{N}^{[*]} \circ \mathbf{F}$ with $\mathbf{F} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N}(\mathbf{D_N^{[*]}})^{[-\mathbf{1}]} & \mathbf{K} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$,

(iii) There exists an invertible matrix $\mathbf{M} \in M_{n-r}(\mathbb{C})$ and $\mathbf{K} \in M_{(r,n-r)}(\mathbb{C})$ such that $\mathbf{N} = \mathbf{N}^{[\dagger]} \circ \mathbf{F}$ with $\mathbf{F} = \mathbf{U_N} \begin{pmatrix} (\mathbf{D_N})^{\mathbf{2}} & \mathbf{K} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$.

Proof . The proof follows on the same lines as in the above Theorem, using the fact that two matrices $\mathbf{N}$ and $\mathbf{S}$ are row equivalent if and only if $Nu(\mathbf{N}) = Nu(\mathbf{S})$ and utilizing the statement (iii) of Theorem 3.3 and [RS-2]. □

4 Partial ordering of range symmetric matrices with respect to indefinite inner product

In this section some characterizations of predecessors of range symmetric matrices under the partial ordering with respect to adjoint are obtained. Using the equivalences of the definition of partial ordering with respect to adjoint, that is, [PO-1] and, [PO-2], it can be easily verified that $\mathbf{N}^{[*]} \circ \mathbf{S}$, $\mathbf{S} \circ \mathbf{N}^{[*]}$, $\mathbf{N}^{[\dagger]} \circ \mathbf{S}$ and $\mathbf{S} \circ \mathbf{N}^{[\dagger]}$ are J-symmetric.

Theorem 4.1. Let $\mathbf{N},\ \mathbf{S} \in M_n(\mathbb{C})$ such that $\mathbf{S}$ is a nonzero range symmetric matrix. Then, the following statements are equivalent:

(i) $\mathbf{N} \leq^{[*]} \mathbf{S}$,

(ii) There exists $\mathbf{G} \in M_r(\mathbb{C})$ such that

$$\mathbf{N} = \mathbf{U_S} \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}} \text{ with } \mathbf{G} \leq^{[*]} \mathbf{D_S}. \tag{4.1}$$

Proof . $(i) \Leftrightarrow (ii)$ We consider the following block representation of $\mathbf{N}$ according to the block size of $\mathbf{S}$ as:

$$\mathbf{N} = \mathbf{U_S} \begin{pmatrix} \mathbf{G} & \mathbf{K} \\ \mathbf{L} & \mathbf{M} \end{pmatrix} \mathbf{U_S^{[*]}}.$$

Then,

$$\mathbf{N}^{[*]} \circ \mathbf{N} = \mathbf{U_S} \begin{pmatrix} \mathbf{G}^{[*]}\mathbf{G} + \mathbf{L}^{[*]}\mathbf{L} & \mathbf{G}^{[*]}\mathbf{K} + \mathbf{L}^{[*]}\mathbf{M} \\ \mathbf{K}^{[*]}\mathbf{G} + \mathbf{M}^{[*]}\mathbf{L} & \mathbf{K}^{[*]}\mathbf{K} + \mathbf{M}^{[*]}\mathbf{M} \end{pmatrix} \mathbf{U_S^{[*]}},$$

and

$$\mathbf{N}^{[*]} \circ \mathbf{S} = \mathbf{U_S} \begin{pmatrix} \mathbf{G}^{[*]}\mathbf{D_S} & \mathbf{0} \\ \mathbf{L}^{[*]}\mathbf{D_S} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}}.$$

Therefore, the equality $\mathbf{N}^{[*]} \circ \mathbf{N} = \mathbf{N}^{[*]} \circ \mathbf{S}$ gives

$$\mathbf{G}^{[*]}\mathbf{G} + \mathbf{L}^{[*]}\mathbf{L} = \mathbf{G}^{[*]}\mathbf{D_S} \text{ and } \mathbf{K}^{[*]}\mathbf{K} + \mathbf{M}^{[*]}\mathbf{M} = \mathbf{0} \Rightarrow \mathbf{K} = \mathbf{0} \text{ and } \mathbf{M} = \mathbf{0}.$$

Also computing $\mathbf{N} \circ \mathbf{N}^{[*]}$ and $\mathbf{S} \circ \mathbf{N}^{[*]}$ and using the equality $\mathbf{N} \circ \mathbf{N}^{[*]} = \mathbf{S} \circ \mathbf{N}^{[*]}$, we get $\mathbf{G}\mathbf{G}^{[*]} + \mathbf{K}\mathbf{K}^{[*]} = \mathbf{D_S}\mathbf{G}$ and $\mathbf{L} = \mathbf{0}$. Thus, $\mathbf{G}^{[*]}\mathbf{G} = \mathbf{G}\mathbf{D_S}$ and $\mathbf{G}\mathbf{G}^{[*]} = \mathbf{D_S}\mathbf{G}$, that is, $\mathbf{G} \leq^{[*]} \mathbf{D_S}$. □

Remark 4.2. If both the matrices $\mathbf{N}$, $\mathbf{S} \in M_n(\mathbb{C})$ are range symmetric and $\mathbf{N} \leq^{[*]} \mathbf{S}$, then using the statements [PO-1], [PO-2] and [RS-3], it can be easily observed that $\mathbf{N}^{[\dagger]} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{N}^{[\dagger]}$. Using the representations (3.2) and (4.1) of $\mathbf{S}$ and $\mathbf{N}$ respectively, we have another equivalent condition for the partial ordering of range symmetric matrices with respect to the adjoint given by $\mathbf{N} \leq^{[*]} \mathbf{S} \Leftrightarrow \mathbf{N} \circ \mathbf{N}^{[\dagger]} = \mathbf{S}^{[\dagger]} \circ \mathbf{N}$ and $\mathbf{N}^{[\dagger]} \circ \mathbf{N} = \mathbf{N} \circ \mathbf{S}^{[\dagger]}$. Furthermore, $\mathbf{N}$ is range symmetric, we have $\mathbf{S}^{[\dagger]} \circ \mathbf{N} = \mathbf{N} \circ \mathbf{S}^{[\dagger]}$.

The next result gives some equivalent conditions for a matrix $\mathbf{N}$ to be range symmetric when $\mathbf{S}$ is range symmetric and $\mathbf{S}$ is the successor of $\mathbf{N}$.

Theorem 4.3. Let $\mathbf{N}$, $\mathbf{S} \in M_n(\mathbb{C})$ such that $\mathbf{S}$ is a nonzero range symmetric matrix and $\mathbf{N} \leq^{[*]} \mathbf{S}$, where $\mathbf{S}$ is given by (3.2) and $\mathbf{N}$ is given by (4.1). Then, the following statements are equivalent:

(i) $\mathbf{N}$ is range symmetric in $\mathscr{I}$,
(ii) $\mathbf{G}\mathbf{D_S} = \mathbf{D_S}\mathbf{G}$,
(iii) $\mathbf{G}^{[\dagger]}\mathbf{D_S} = \mathbf{D_S}\mathbf{G}^{[\dagger]}$,
(iv) $\mathbf{G}(\mathbf{G}^{[*]} - \mathbf{D_S}) = \mathbf{D_S}(\mathbf{G}^{[*]} - \mathbf{G})$,
(v) $(\mathbf{G}^{[*]} - \mathbf{D_S})\mathbf{G} = (\mathbf{G}^{[*]} - \mathbf{G})\mathbf{D_S}$,
(vi) $\mathbf{G}$ is range symmetric in $\mathscr{I}$.

Proof . *(i)* ⇔ *(ii)* From Remark 4.2, we have $\mathbf{S}^{[\dagger]}\mathbf{N} = \mathbf{N}\mathbf{S}^{[\dagger]}$. Now using the facts that $\mathbf{D_S}^{[\dagger]} = \mathbf{D_S}^{[-1]}$; $\mathbf{D_S}$ being invertible and $\mathbf{U_S}$ is J-unitary and substituting the representations of $\mathbf{S}$ and $\mathbf{N}$ from (3.2) and (4.1) respectively in the above equality and doing some simple algebra leads to $\mathbf{G}\mathbf{D_S} = \mathbf{D_S}\mathbf{G}$.

(ii) ⇔ *(iii)* For $\mathbf{N} = \mathbf{U_S}\begin{pmatrix}\mathbf{G} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_S}^{[*]}$, $\mathbf{N}^{[\dagger]} = \mathbf{U_S}\begin{pmatrix}\mathbf{G}^{[\dagger]} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U_S}^{[*]}$. Again using Remark 4.2 and substituting the respective representations of $\mathbf{N}^{[\dagger]}$ and $\mathbf{S}$, the equivalence follows.

(ii) ⇔ *(iv)* Using [PO-1] and substituting the representations of $\mathbf{N}$, $\mathbf{N}^{[*]}$ and $\mathbf{S}$, the equivalence follows after some computation.

On the same lines the equivalences *(ii)* ⇔ *(v)* and *(iii)* ⇔ *(vi)* follow by using the Remark 4.2 and statements [PO-1] and [PO-2]. □

The next result similar to Theorem 4.1 holds if we consider $\mathbf{N}$ to be range symmetric and decompose $\mathbf{S}$ in terms of representation for $\mathbf{N}$

Theorem 4.4. Let $\mathbf{N}$, $\mathbf{S} \in M_n(\mathbb{C})$ such that $\mathbf{N}$ is a nonzero range symmetric matrix. Then, the following statements are equivalent:

(i) $\mathbf{N} \leq^{[*]} \mathbf{S}$,
(ii) There exists $\mathbf{M} \in M_{n-r}(\mathbb{C})$ such that

$$\mathbf{S} = \mathbf{U_N}\begin{pmatrix}\mathbf{D_N} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{pmatrix}\mathbf{U_N}^{[*]}. \tag{4.2}$$

Proof . The proof follows on the same line as in Theorem 4.1. □

We again note that if $\mathbf{N} \leq^{[*]} \mathbf{S}$ and $\mathbf{N}$ is range symmetric then, $\mathbf{S}$ need not be range symmetric.

Theorem 4.5. Let $\mathbf{N}$, $\mathbf{S} \in M_n(\mathbb{C})$ be given by (3.1) and (4.2) respectively such that $\mathbf{N}$ is a nonzero range symmetric matrix and $\mathbf{N} \leq^{[*]} \mathbf{S}$. Then, the following statements are equivalent:

(i) $\mathbf{S}$ is range symmetric in $\mathscr{I}$,
(ii) $\mathbf{M}$ is range symmetric in $\mathscr{I}$,
(iii) $\mathbf{S} \circ (\mathbf{N}^{[\dagger]} - \mathbf{S}^{[\dagger]}) = (\mathbf{N}^{[\dagger]} - \mathbf{S}^{[\dagger]}) \circ \mathbf{S}$,

(iii) $\mathbf{S}^{[\dagger]} \circ (\mathbf{N} - \mathbf{S}) = (\mathbf{N} - \mathbf{S}) \circ \mathbf{S}^{[\dagger]}$.

Proof . $(i) \Leftrightarrow (ii)$ For $\mathbf{S} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$, since $\mathbf{D_N}$ is nonsingular and $\mathbf{U_N}$ is J-unitary, direct verification shows that $\mathbf{S}^{[\dagger]} = \mathbf{U_N} \begin{pmatrix} \mathbf{D_N^{[\dagger]}} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{[\dagger]} \end{pmatrix} \mathbf{U_N^{[*]}}$. Therefore, $\mathbf{S} \circ \mathbf{S}^{[\dagger]} = \mathbf{U_N} \begin{pmatrix} \mathbf{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \circ \mathbf{M}^{[\dagger]} \end{pmatrix} \mathbf{U_N^{[*]}}$ and $\mathbf{S}^{[\dagger]} \circ \mathbf{S} = \mathbf{U_N} \begin{pmatrix} \mathbf{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{[\dagger]}\mathbf{M} \end{pmatrix} \mathbf{U_N^{[*]}}$. $\mathbf{S}$ being range symmetric, by [RS-3], we have $\mathbf{S}\circ\mathbf{S}^{[\dagger]} = \mathbf{S}^{[\dagger]}\circ\mathbf{S}$. This gives, $\mathbf{M}\circ\mathbf{M}^{[\dagger]} = \mathbf{M}^{[\dagger]}\circ\mathbf{M}$ and the equivalence follows.

$(i) \Rightarrow (iii)$ Since $\mathbf{N} \leq^{[*]} \mathbf{S}$ and $\mathbf{N}$ and $\mathbf{S}$ are range symmetric, using the observation mentioned in Remark 4.2, that is, $\mathbf{N}^{[\dagger]} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{N}^{[\dagger]}$, we have $\mathbf{S} \circ \mathbf{N}^{[\dagger]} - \mathbf{S} \circ \mathbf{S}^{[\dagger]} = \mathbf{N}^{[\dagger]} \circ \mathbf{S} - \mathbf{S}^{[\dagger]} \circ \mathbf{S}$, the equivalence follows.

$(iii) \Rightarrow (i)$ Since $\mathbf{N} \leq^{[*]} \mathbf{S}$ and $\mathbf{N}$ is range symmetric, again by the same fact that $\mathbf{N}^{[\dagger]}$ and $\mathbf{S}$ commute, using (iii), that is, $\mathbf{S} \circ (\mathbf{N}^{[\dagger]} - \mathbf{S}^{[\dagger]}) = (\mathbf{N}^{[\dagger]} - \mathbf{S}^{[\dagger]}) \circ \mathbf{S}$, we get $\mathbf{S}$ is range symmetric.

$(i) \Leftrightarrow (iv)$ From Remark 4.2, we have $\mathbf{N} \circ \mathbf{S}^{[\dagger]} = \mathbf{S}^{[\dagger]} \circ \mathbf{N}$. This gives $\mathbf{N} \circ \mathbf{S}^{[\dagger]} - \mathbf{S} \circ \mathbf{S}^{[\dagger]} = \mathbf{S}^{[\dagger]} \circ \mathbf{N} - \mathbf{S}^{[\dagger]} \circ \mathbf{S}$. Now using the fact that $\mathbf{S}$ is range symmetric, the equivalence follows. □

In the above results we have used the commutativity of $\mathbf{N}$ and $\mathbf{S}^{[\dagger]}$, and $\mathbf{N}^{[\dagger]}$ and $\mathbf{S}$. However, if we assume the conditions given in the above Theorem with an additional assumption that $\mathbf{N}\circ\mathbf{S} = \mathbf{S}\circ\mathbf{N}$, then the conditions obtained by interchanging $\mathbf{N}$ and $\mathbf{S}$ are also equivalent.

Theorem 4.6. Let $\mathbf{N}$, $\mathbf{S} \in M_n(\mathbb{C})$ be range symmetric such that $\mathbf{N}$ is a non zero matrix. Then, the following statements are equivalent:

(i) $\mathbf{N} \leq^{[*]} \mathbf{S}$,

(ii) There exists a J-unitary matrix $\mathbf{U} \in M_n(\mathbb{C})$, $\mathbf{D} \in M_r(\mathbb{C})$ and $\mathbf{M} \in M_{r-s}(\mathbb{C})$ such that $\mathbf{N} = \mathbf{U} \begin{pmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^{[*]}$ and $\mathbf{S} = \mathbf{U} \begin{pmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^{[*]}$.

Proof . $(i) \Rightarrow (ii)$ Consider $\mathbf{S}$ given by (3.2) that is, $\mathbf{S} = \mathbf{U_S} \begin{pmatrix} \mathbf{D_S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}}$. $\mathbf{S}$ is range symmetric. Therefore, by Theorem 4.1 statement (4.1), there exists $\mathbf{G} \in M_r(\mathbb{C})$ such that $\mathbf{N} = \mathbf{U_S} \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_S^{[*]}}$ with $\mathbf{G} \leq^{[*]} \mathbf{D_S}$. Using Theorem 4.3, we have $\mathbf{G}$ is range symmetric. We consider the block representation of $\mathbf{G}$ as $\mathbf{G} = \mathbf{U_G} \begin{pmatrix} \mathbf{D_G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U_G^{[*]}}$, where $\mathbf{U_G}$ is J-unitary and $\mathbf{D_G} \in M_r(\mathbb{C})$ is invertible. Since $\mathbf{G} \leq^{[*]} \mathbf{D_S}$, by Theorem 4.4, we can find $\mathbf{M} \in M_{r-s}(\mathbb{C})$ such that $\mathbf{D_S} = \mathbf{U_G} \begin{pmatrix} \mathbf{D_G} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \mathbf{U_G^{[*]}}$. Thus, $\mathbf{M}$ is nonsingular when $\mathbf{M} \neq \mathbf{0}$. Taking $\mathbf{D_G} = \mathbf{D}$, we have $\mathbf{S} = \mathbf{U} \begin{pmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^{[*]}$, where $\mathbf{U} = \mathbf{U_S} \begin{pmatrix} \mathbf{D_G} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$ is J-unitary.

$(i) \Rightarrow (ii)$ Follows at once by direct verification. □

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