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Solutions of system of split mixed equilibrium and fixed points problems

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Abstract

In this paper, we introduce a new iterative method for system of split mixed equilibrium problems and infinite family of demimetric mappings in a real Hilbert space. Then, we establish that the sequence generated by our proposed algorithm converges strongly to a common element in the solutions set of a system of split mixed equilibrium problems and the common fixed points set of infinite family of demimetric mappings. Our result improve and generalize some well-known recent results in the literature.

Keywords: Split problem, Equilibrium problem, Fixed Points, Demimetric mapping, Strong convergence
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1 Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and $||.||$ induced norm on the inner product and C a nonempty closed and convex subset of H.

For any nonlinear mapping T from C into H, denote the set of fixed points of T by $F(T) := \{x \in C : Tx = x\}$. Let $x_n \to x$ and $x_n \rightharpoonup x$ denote respectively the strong and weak convergence of the sequence $\{x_n\}$ to x. We shall also use the symbols $\mathbb{N}$ and $\mathbb{R}$ for the set of natural and real numbers respectively.

Definition 1.1. Let $U : C \to H$ be a mapping, then U is said to be

1. $L-$Lipschitz if there exists $L > 0$ such that $||Ux - Uy|| \leq L||x - y||$, for all $x, y \in C$ and T is nonexpansive if $L = 1$.
2. $k-$strictly pseudocontraction in the sense of Browder and Petryshyn [2] if there exists $k \in [0, 1)$ such that
$$||Ux - Uy||^2 \leq ||x - y||^2 + k||x - Ux - (y - Uy)||^2, \quad \text{for all } \ x, y \in C; \tag{1.1}$$
3. k-strictly pseudononspreading in the sense of Osilike and Isiogugu [11] if there exists $k \in [0, 1)$ such that
$$\begin{aligned} ||Ux - Uy||^2 \quad &\leq \quad ||x - y||^2 + k||x - Ux - (y - Uy)||^2 \\ &\quad +2\langle x - Ux, y - Uy\rangle \ \ \forall\ x, y \in C. \end{aligned}$$

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4. a generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that, for all $x, y \in C$

$$\alpha||Ux - Uy||^2 + (1-\alpha)||x - Uy||^2 \leq \beta||Ux - y||^2 + (1-\beta)||x - y||^2.$$

5. firmly nonexpansive if $||Ux - Uy||^2 \leq \langle Ux - Uy, x - y\rangle, \quad \forall\ x, y \in C.$
6. monotone if $\langle Ux - Uy, x - y\rangle \geq 0,\ \forall x, y \in C.$
7. $\alpha-$inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||Ux - Uy||^2, \quad \forall\ x, y \in C.$$

Recently, Takahashi [22] introduced a new class of nonlinear mapping known as k-demimetric mapping in Hilbert space. A mappping $U : C \to H$ with $F(U) \neq \emptyset$ is called k-demimetric, if there exists a $k \in (-\infty, 1)$ such that

$$\langle x - q, x - Ux\rangle \geq \frac{1-k}{2}\left\|x - Ux\right\|^2$$

for any $x \in C$ and $q \in F(U)$. The study of nonexpansive mapping and its generalization plays an important role in nonlinear analysis and optimization. Finding fixed points of such mapping can be applied to nonlinear problems such zero solution of some monotone operators, equilibrium problem, split feasibility problems, convex minimization problems and variational inequality problems (see [1, 12, 21, 23, 15, 18, 14, 16] for more details).

The classical equilibrium problem (EP for short) is defined as follows: Let $g : C \times C \to \mathbb{R}$ be a bi-function, then

$$\text{find} \quad p \in C \ \text{ such that } \ g(p, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of problem (EP) is defined as

$$EP(g) = \{p \in C : g(p, y) \geq 0,\ \forall\ y \in C\}.$$

This problem was introduced by Blum and Oettli [1] in 1994. They studied existence theorems and variational principle for equilibrium problems which had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown in [1] that the theory of equilibrium problem provides a natural, novel and unified framework for solving several problems arising in pure and applied sciences. Later in 2012, He [6] introduced split equilibrium problem (SEP) as follows:

Let C and Q be nonempty closed and conves subsets of real Hilbert spaces H_1 and H_2 respectively, $g_1 : C \times C \to \mathbb{R}$ and $g_2 : Q \times Q \to \mathbb{R}$ be nonlinear bi-functions and let $A : H_1 \to H_2$ be a bounded linear operator. Then, the split equilibrium problem is to find $p \in C$ such that

$$g_1(p, x) \geq 0, \quad \forall x \in C \text{ and } q = Ap \in Q \text{ solves } g_2(q, y) \geq 0,\ \forall\ y \in Q.$$

The set of solutions of this problem is denoted by SEP(g, φ). Recently, a lot of research effort are devoted to finding a solution of split equilibrium problem and system of split equilibrium problems (see [7, 9, 8, 19, 24] and the references therein). In this paper, we also consider the mixed equilibrium problem (MEP) which is to find $p \in C$ such that

$$g(p, x) + \varphi(x) - \varphi(p) \geq 0, \forall\ x \in C.$$

In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem. The set of solutions for problem is denoted by MEP(g, φ). In 2017, Onjai-uea and Phuengrattana [10] first studied split mixed equilibrium problems as follows: find $p \in C$ such that

$$g_1(p, x) + \varphi_1(x) - \varphi_1(p) \geq 0,\ \forall\ x \in C$$

and such that

$$q = Ap \text{ solves } g_2(q, y) + \varphi_2(y) - \varphi_2(q) \geq 0,\ \forall`y \in Q,$$

where $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions such that $C \cap dom(\varphi_1) \neq \emptyset$ and $Q \cap dom(\varphi_2) \neq \emptyset$. Then the solution of split mixed equilibrium problems is defined as follows:

$$SMEP(g_1, \varphi_1, g_2, \varphi_2) = \{p \in C : p \in MEP(g_1, \varphi_1) \text{ and } Ap \in MEP(g_2, \varphi_2)\}.$$

In 2016, Ugwunnadi and Ali [24] introduced and studied the following algorithm for finding a common fixed point of a finite family of continuous pseudocontractive mappings $\{T_i\}_{i=1}^N$ which is a unique solution of some variational inequality problem and whose image under some bounded linear operator A with its adjoint A^* is a common solution of some system of equilibrium problems in a real Hilbert space as follows:

$$\begin{cases} y_n = P_C\left(x_n + \lambda B\left(\mathcal{J}_n^M - I\right) Ax_n\right), \\ z_n = \beta_n y_n + (1-\beta_n) T_{[n]r_n} y_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n) I - \alpha_n \mu G) z_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\mathcal{J}_n^M = T^{g_M}_{s_{M,n}} T^{g_{M-1}}_{s_{M-1,n}} \cdots T^{g_2}_{s_{2,n}} T^{g_1}_{s_{1,n}}$, and $\mathcal{J}_n^0 = I$ for all $n \in \mathbb{N}$ and $T_{[n]r_n} x := \left\{z \in C : \langle y - z, T_{[n]} z\rangle - \frac{1}{r_n}\langle y - z, (1+r_n) z - x\rangle \le 0, \forall y \in C\right\}$, $T_{[n]} := T_{nmodN}$, $\beta \in (0,1)$, $0 < \liminf_{n\to\infty} \delta_n < \limsup_{n\to\infty} \delta_n < 1$, $\{r_n\} \in (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\lambda \in \left(0, \frac{1}{\|A^*\|^2}\right)$ and $\{s_{k,n}\}_{k=1}^M \in (0,\infty)$ with $\liminf_{n\to\infty} s_{k,n} > 0$, for every $k \in \{1,2,3,\ldots,M\}$, G is an η-strongly monotone and μ-strictly pseudocontractive with $\eta + \mu > 1$, f is a contraction with $\alpha \in (0,1)$ with some condition on $\{\alpha_n\}$ and $\{r_n\}$. They proved the sequence generated by (1.2) converges strongly to a common solution of the system of equilibrium problem and common fixed points of the family of continuous pseudocontractive mappings.

In 2018, Rizvi [17], studied a modified Mann iterative and Halpern iterative method for find a common solution of split mixed equilibrium problem and fixed point for a nonexpansive mapping and prove the following theorem:

Theorem 1.2. Let H_1 and H_2 be two real Hilbert spaces and $K_1 \subset H_1$ and $K_2 \subset H_2$ be nonempty closed and convex subsets. Let $B : H_1 \to H_2$ be a bounded linear operator. Assume that $F : K_1 \times K_2 \to \mathbb{R}$ and $G : K_2 \times K_2 \to \mathbb{R}$ are the bifumctions satisfying some conditions and G is upper semicontinuous in first argument. Let $f : K_1 \to K_1$ and $g : K_2 \to K_2$ be θ_1 and θ_2-inverse strongly monotone mappings respectively and let $S : K_1 \to K_2$ be a nonexpansive mapping such that $\Omega : SMEP(F,G) \cap F(T) \neq \emptyset$. For a given $x_0 \in K_1$ arbitrary, let the iterative sequences $\{x_n\}, \{y_n\}, \{v_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} y_n = T^F_{r_n}(x_n - r_n f x_n), \\ v_n = T^G_{r_n}(I - r_n g) B y_n, \\ z_n = P_{K_1}[y_n + \delta B^*(v_n - B y_n)], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) z_n], \ n \ge 1. \end{cases}$$

where $\{r_n\} \subset (0, 2\theta)$; $\theta = \min\{\theta_1, \theta_2\}$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega u$.

Motivated and inspired by the results, in this paper we introduce and study a new iterative method for finding an element of the set of system of split mixed equilibrium problem and common fixed points of infinite family of demimetric mappings in a real Hilbert space. Our result improve and generalized some recent results in the literature.

2 Preliminaries

Let H be a real Hilbert space. Then, it is well known that following inequalities hold for all $x, y \in H$ and $\lambda \in [0,1]$

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle,$$

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle,$$

and

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

which can be extended to the more general situation: for all $x_1, x_2, \ldots, x_n \in H, \lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$, we have

$$\left\|\sum\nolimits_{i=1}^n \lambda_i x_i\right\|^2 = \sum\nolimits_{i=1}^n \lambda_i \|x_i\|^2 - \sum_{1\le i\le j\le n} \lambda_i \lambda_J \|x_i - x_j\|^2.$$

The metric projection from a Hilbert space H onto a nonempty closed and convex subset C of H is the mapping $P_C : H \to C$ for each $x \in H$, there exists a unique point $z = P_C(x)$ such that

$$||x - z|| = \inf_{y \in C} ||x - y||.$$

Lemma 2.1. Let $x \in H$ and $z \in C$ be any point. Then we have

(i) $z = P_C(x)$ if and only if the following relation holds

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

(ii) There holds the relation

$$\langle P_C(x) - P_C(y), x - y \rangle \geq ||P_C(x) - P_C(y)||^2, \ \forall x, y \in H.$$

(iii) For $x \in H$ and $y \in K$

$$||y - P_C(x)||^2 + ||x - P_C(x)||^2 \leq ||x - y||^2.$$

Lemma 2.2. [4]Let $\{\alpha_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{n_i} < \alpha_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:

$$\alpha_{m_k} \leq \alpha_{m_{k+1}} \text{ and } \alpha_k \leq \alpha_{m_{k+1}}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}$.

Lemma 2.3. [25]Let $\{x_n\}$ be a sequence of nonnegative numbers satisfying the property:

$$x_{n+1} \leq (1 - \alpha_n) x_n + b_n + \alpha_n c_n$$

where $\{\alpha_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the restrictions:

1. $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,
2. $b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n < \infty$,
3. $\limsup_{n\to\infty} c_n \leq 0$.
   Then, $\lim_{n\to\infty} x_n = 0$.

Lemma 2.4. [5]Let C be a nonempty closed convex subset of a real Hilbert space H. Let $k \in (-\infty, 1)$ and T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let t be a real number with $t \in (0, \infty)$ and define $S = (1 - t) I + tT$. Then there holds that

1. $F(S) = F(T)$ if $t \neq 0$,
2. S is a quasi-nonexpansive mapping for $t \in (0, 1 - k]$,
3. $F(T)$ is a closed convex subset of H.

Lemma 2.5. [20] Let H be a Hilbert space and C be nonempty convex subset of H. Assume that $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of $k_i -$ *demimetric* mappings with $\sup\{k_i : i \in \mathbb{N}\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\sum_{i=1}^{\infty} \eta_i T_i : C \to H$ is a k-demimetric mapping with $k = \sup\{k_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \eta_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

The demiclosedness principle for mappings plays an important role in our proof in the subsequent section.

Definition 2.6. [3] A self-mapping T on a Banach space is said to be demiclosed at y, if for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to y, then $T(x) = y$. In particular, if $y = 0$, then T is demiclosed at 0.

Definition 2.7. Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, then the Hilbert adloint operator $A^* : H_2 \to H_1$ of A is defined for all $x \in H_1$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $y \in H_2$.

The Hilbert adjoint operators have the following well known properties. Let H_1, H_2 be Hilbert spaces, $A : H_1 \to H_2$ and $B : H_1 \to H_2$ be bounded linear operators and α any scalar, then

(a) $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x \in H_1$ and $y \in H_2$;

(b) $(A + B)^* = A^* + B^*$;

(c) $(\alpha A) = \bar{\alpha} A$, where $\bar{\alpha}$ is the conjugate of α;

(d) $(A^*)^* = A$;

(e) $A^*A = 0 \Leftrightarrow A = 0$;

(f) $||A^*A|| = ||AA^*|| = ||A||^2$.

We need the following assumptions to solve a mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$ and a mapping $\varphi : C \to \mathbb{R} \cup \{+\infty\}$, then

(A1) $F(x, x) = 0, \forall x \in C$,
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$,
(A3) $\lim_{\lambda \to 0} F(\lambda z + (1 - \lambda) x, y) \leq F(x, y)$ for all $x, y, z \in C$
(A4) $\forall x \in C$, $y \longmapsto F(x, y)$ is convex and lower semicontinuous,
(A5) For each $x \in C$, $\lambda \in (0, 1]$ and $r > 0$, there exist a bounded subset $D \subseteq C$ and $a \in C$ such that for any $z \in C \backslash D$,

$$F(z, a) + \varphi(a) - \varphi(z) + \frac{1}{r}\langle a - z, z - x\rangle < 0.$$

(A6) C is a bounded set

Lemma 2.8. [13] Let C be a nonempty closed convex subset of a Hilbert space H_1 and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex mapping such that $C \cap dom\varphi = \emptyset$. Suppose that bifunction $F : C \times C \to \mathbb{R}$ and a mapping φ satisfy the conditions (A1)-(A6). For $r > 0$ and $x \in H_1$, let $T_r^F : H_1 \to C$ be a mapping defined by

$$T_r^F(x) = \left\{ z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{2.2}$$

Assume that either (A5) or (A6) holds. Then:
(i) For each $x \in H_1$, $T_r^F x \neq \emptyset$,
(ii) T_r^F is single valued,
(iii) T_r^F is firmly nonexpansive,
(iv) $F\left(T_r^F\right) = MEP(F, \varphi)$ and it is closed and convex.

Let $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex mapping such that $Q \cap dom\phi = \emptyset$. Suppose that bifunction $G : Q \times Q \to \mathbb{R}$ and a mapping ϕ satisfy the conditions (A1)-(A6). For $s > 0$ and $u \in H_2$. Let $T_s^G : H_2 \to Q$ be a mapping defined by

$$T_s^G(u) = \left\{ v \in Q : G(v, w) + \phi(w) - \phi(v) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q \right\}. \tag{2.3}$$

Then clearly T_s^G satisfies (i)-(iv) of Lemma 2.8, and $F\left(T_s^G\right) = MEP(G, \phi)$. We introduce the system of split mixed equilibrium problem by the following way.

Definition 2.9. Let C_i and Q_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}, 1 \leq i \leq N$, nonlinear bifunctions and let $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $C_i \cap dom\varphi_i \neq \emptyset$ and $Q_i \cap dom\phi_i \neq \emptyset$. The system of split mixed equilibrium problem is to find $x^* \in C = \cap_{i=1}^N C_i$ such that

$$F_i\left(x^*, x\right) + \varphi_i\left(x\right) - \varphi_i\left(x^*\right) \geq 0, \ \forall x \in C_i, \tag{2.4}$$

and such that $y^* = Ax^* \in Q = \cap_{i=1}^N Q_i$ solves

$$G_i\left(y^*, y\right) + \phi_i\left(y\right) - \phi_i\left(y^*\right) \geq 0, \ \forall y \in Q_i. \tag{2.5}$$

The solution set of system of split mixed equilibrium problem (2.4) and (2.5) is denoted by

$$SSMEP(F_i, \varphi_i, G_i, \phi_i) \quad = \quad \Big\{x^* \in C : x^* \in \cap_{i=1}^N MEP\left(F_i, \varphi_i\right) \text{ and } Ax^* \in \cap_{i=1}^N MEP\left(G_i, \phi_i\right)\Big\},$$

where $MEP\left(F_i, \varphi_i\right)$ is the set of solutions of mixed equilibrium problem, i.e.,

$$MEP\left(F_i, \varphi_i\right) := \left\{x^* \in C_i : F_i\left(x^*, x\right) + \varphi_i\left(x\right) - \varphi_i\left(x^*\right) \geq 0, \ \forall x \in C_i\right\}.$$

3 Main Results

Theorem 3.1. Let C_i and Q_i, $1 \leq i \leq N$, be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator and $\{S_m\}$ be an infinite family of k_m-demimetric and demiclosed mappings from $C = \cap_{i=1}^N C_i$ to H_1. Let $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be nonlinear bifunctions satisfying assumptions (A1)-(A6), $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ proper lower semicontinuous and convex functions such that $C_i \cap dom\varphi_i \neq \emptyset$ and $Q_i \cap dom\phi_i \neq \emptyset$ and let G_i be upper semicontinuous in the first argument. Assume that $\Gamma = \cap_{m=1}^\infty F\left(S_m\right) \cap SSMEP\left(F_i, \varphi_i, G_i, \phi_i\right) \neq \emptyset$ and u is a fixed vector in C. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + \left(1 - \alpha_n\right)\left(1 - \delta\right) x_n + \left(1 - \alpha_n\right) \delta y_n, \\ y_n = \left(1 - \beta_n\right) u_n + \beta_n \sum_{m=1}^\infty \eta_m S_m u_n, \\ u_n = T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{G_1}\right) A\right) u_{n,1}, \\ u_{n,1} = T_{r_n}^{F_2}\left(I - \gamma A^*\left(I - T_{r_n}^{G_2}\right) A\right) u_{n,2}, \\ \vdots \\ u_{n,N-2} = T_{r_n}^{F_{N-1}}\left(I - \gamma A^*\left(I - T_{r_n}^{G_{N-1}}\right) A\right) u_{n,N-1}, \\ u_{n,N-1} = T_{r_n}^{F_N}\left(I - \gamma A^*\left(I - T_{r_n}^{G_N}\right) A\right) x_n, \forall n \in \mathbb{N} \end{cases} \tag{3.1}$$

where $a_n, \beta_n, \eta_m, \delta \in (0,1)$, $r_n \in (0, \infty)$ and $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(i) $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < a \leq \beta_n \leq b < 1 - k$ where $k = \sup\{k_i : i \in \mathbb{N}\}$;
(iii) $\sum_{m=1}^\infty \eta_m = 1$;
(iv) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $p = P_\Gamma u$.

Proof . We divide our proof into six steps.

Step 1. In first step, we show that $A^*\left(I - T_{r_n}^{Gi}\right) A$ is a $\frac{1}{L}$-ism for all $i = 1, 2, \ldots, N$. Since $T_{r_n}^{Gi}$ is firmly nonexpansive and $I - T_{r_n}^{Gi}$ is 1-ism, by using that A^* is adjoint of A, we have

$$\begin{aligned} \left\|A^*\left(I - T_{r_n}^{Gi}\right) Ax - A^*\left(I - T_{r_n}^{Gi}\right) Ay\right\|^2 &= \left\langle A^*\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right), A^*\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right)\right\rangle \\ &= \left\langle\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right), AA^*\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right)\right\rangle \\ &\leq L\left\langle\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right), \left(I - T_{r_n}^{Gi}\right) A\left(x - y\right)\right\rangle \\ &= L\left\|\left(I - T_{r_n}^{Gi}\right) A\left(x - y\right)\right\|^2 \\ &\leq L\left\langle A\left(x - y\right), \left(I - T_{r_n}^{Gi}\right) A\left(x - y\right)\right\rangle \\ &= L\left\langle x - y, A^*\left(I - T_{r_n}^{Gi}\right) Ax - A^*\left(I - T_{r_n}^{Gi}\right) Ay\right\rangle \end{aligned}$$

for all $x, y \in H_1$. So, $A^*\left(I - T_{r_n}^{G_i}\right)A$ is a $\frac{1}{L}$-ism for all $i = 1, 2, \ldots, N$. On the other hand, since $0 < \gamma < \frac{1}{L}$, we get $I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A$ is a nonexpansive mapping.

Step 2. In second step, we show that sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are bounded. Let $q \in \Gamma$. It means that q is a fixed point of the mappings $S_m, T_{r_n}^{Fi}$ and $I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A$. Let $J_i = T_{r_n}^{F_i}\left(I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A\right)$. Since $T_{r_n}^{Fi}$ and $I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A$ are nonexpansive mappings, we have J_i is nonexpansive mapping and $q \in F(J_i)$ for all $i = 1, 2, \ldots, N$. So, we get

$$\|u_{n,N-1} - q\| = \|J_N x_n - J_N q\| \leq \|x_n - q\| \tag{3.2}$$

and

$$\begin{aligned}
\|u_n - q\| &= \|J_1 u_{n,1} - J_1 q\| \leq \|u_{n,1} - q\| \\
&\leq \|J_2 u_{n,2} - J_2 q\| \leq \|u_{n,2} - q\| \\
&\leq \cdots \leq \|u_{n,N-1} - q\| \leq \|x_n - q\|.
\end{aligned} \tag{3.3}$$

Let $V = \sum_{m=1}^{\infty} \eta_m S_m$ and $W_n = (1 - \beta_n) I + \beta_n V$. Then, it is easy to see by Lemma 2.5, 2.4 that W_n is a quasi nonexpansive mapping with $F(W_n) = F(V) = \cap_{m=1}^{\infty} F(S_m)$ .Using (3.3), we obtain

$$\|y_n - q\| = \|W_n u_n - q\| \leq \|u_n - q\| \leq \|x_n - q\|. \tag{3.4}$$

So, we have from (3.3) and (3.4) that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n u + (1 - \alpha_n)(1 - \delta) x_n + (1 - \alpha_n) \delta y_n - q\| \\
&\leq \alpha_n \|u - q\| + (1 - \alpha_n)(1 - \delta) \|x_n - q\| + (1 - \alpha_n) \delta \|y_n - q\| \\
&\leq \alpha_n \|u - q\| + (1 - \alpha_n) \|x_n - q\| \\
&\leq \max\{\|u - q\|, \|x_n - q\|\}.
\end{aligned} \tag{3.5}$$

Therefore, we have $\|x_n - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\}$ for all $n \in \mathbb{N}$. Hence, it follows from (3.5) that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{u_n\}$.

Step 3. In this step, we show that $\|u_n - x_n\| \to 0$. Using algorithm (3.1) we have

$$\begin{aligned}
\|u_n - q\|^2 &= \|J_1 u_{n,1} - q\|^2 \\
&= \left\|T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{G_1}\right)A\right) u_{n,1} - q\right\|^2 \\
&\leq \left\|u_{n,1} - q - \gamma A^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 \\
&\leq \|u_{n,1} - q\|^2 + \gamma^2 \left\|A^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 - 2\gamma \left\langle u_{n,1} - q, A^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle \\
&= \|u_{n,1} - q\|^2 + \gamma^2 \left\langle A^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}, A^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle + 2\gamma \left\langle A(q - u_{n,1}), \left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle \\
&= \|u_{n,1} - q\|^2 + \gamma^2 \left\langle \left(I - T_{r_n}^{G_1}\right) A u_{n,1}, AA^*\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle \\
&\quad + 2\gamma \left\langle A(q - u_{n,1}) + \left(I - T_{r_n}^{G_1}\right) A u_{n,1}, \left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle - 2\gamma \left\langle \left(I - T_{r_n}^{G_1}\right) A u_{n,1}, \left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\rangle \\
&\leq \|u_{n,1} - q\|^2 + L\gamma^2 \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 + 2\gamma \frac{1}{2} \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 - 2\gamma \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 \\
&= \|u_{n,1} - q\|^2 + \gamma(L\gamma - 1) \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2.
\end{aligned}$$

On the other hand, using (3.4), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n u + (1 - \alpha_n)(1 - \delta) x_n + (1 - \alpha_n) \delta y_n - q\|^2 \\
&\leq \alpha_n \|u_n - q\|^2 + (1 - \alpha_n)(1 - \delta) \|x_n - q\|^2 + (1 - \alpha_n) \delta \|u_n - q\|^2 \\
&\leq \alpha_n \|u_n - q\|^2 + \|x_n - q\|^2 + (1 - \alpha_n) \delta \gamma (L\gamma - 1) \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2,
\end{aligned}$$

and so, we obtain

$$-(1 - \alpha_n) \delta \gamma (L\gamma - 1) \left\|\left(I - T_{r_n}^{G_1}\right) A u_{n,1}\right\|^2 \leq \alpha_n \|u_n - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2. \tag{3.6}$$

Now, there exist two cases.

Case 1: First we assume that there exists an integer $m>0$ such that $\{\|x_n-q\|\}$ is a decreasing sequence for all $n>m$. Since $\lim_{n\to\infty}\alpha_n=0$ and $\{u_n\}$ is a bounded sequence, if we take limit from both side of inequality (3.6), we have $\left\|\left(I-T_{r_n}^{G_1}\right)Au_{n,1}\right\|\to 0$. Similarly, since $u_n=J_1u_{n,1}=J_1J_2u_{n,2}=\cdots=J_1J_2\ldots J_{N-1}u_{n,N-1}=J_1J_2\ldots J_{N-1}J_Nx_n$, we see that

$$\lim_{n\to\infty}\left\|\left(I-T_{r_n}^{G_i}\right)Au_{n,i}\right\|=0 \tag{3.7}$$

for $i=1,2,\ldots,N-1$ and

$$\lim_{n\to\infty}\left\|\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|=0. \tag{3.8}$$

Also, since $T_{r_n}^{F_N}$ is firmly nonexpansive, we get

$$\begin{aligned}
\|u_{n,N-1}-q\|^2 &= \left\|T_{r_n}^{F_N}\left(I-\gamma A^*\left(I-T_{r_n}^{G_N}\right)A\right)x_n-T_{r_n}^{F_N}q\right\|^2\\
&\le \left\langle T_{r_n}^{F_N}\left(I-\gamma A^*\left(I-T_{r_n}^{G_N}\right)A\right)x_n-T_{r_n}^{F_N}q,\left(I-\gamma A^*\left(I-T_{r_n}^{G_N}\right)A\right)x_n-q\right\rangle\\
&= \left\langle u_{n,N-1}-q,\left(I-\gamma A^*\left(I-T_{r_n}^{G_N}\right)A\right)x_n-q\right\rangle\\
&= \frac{1}{2}\Big(\|u_{n,N-1}-q\|^2+\left\|\left(I-\gamma A^*\left(I-T_{r_n}^{G_N}\right)A\right)x_n-q\right\|^2\\
&\quad -\left\|u_{n,N-1}-x_n-\gamma A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|^2\Big)\\
&\le \frac{1}{2}\Big(\|u_{n,N-1}-q\|^2+\|x_n-q\|^2-\|u_{n,N-1}-x_n\|^2-\gamma^2\left\|A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|^2\\
&\quad +2\gamma\left\langle u_{n,N-1}-x_n,A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\rangle\Big).
\end{aligned}$$

So, it follows from Cauchy-Schwarz inequality that

$$\begin{aligned}
\|u_{n,N-1}-q\|^2 &\le \|x_n-q\|^2-\|u_{n,N-1}-x_n\|^2+2\gamma\left\langle u_{n,N-1}-x_n,A^*\left(I-T_{r_n}^{G_n}\right)Ax_n\right\rangle\\
&\le \|x_n-q\|^2-\|u_{n,N-1}-x_n\|^2+2\gamma\|u_{n,N-1}-x_n\|\left\|A^*\left(I-T_{r_n}^{G_n}\right)Ax_n\right\|
\end{aligned}$$

Last inequality with (3.3) and (3.4) implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \|\alpha_nu+(1-\alpha_n)(1-\delta)x_n+(1-\alpha_n)\delta y_n-q\|^2\\
&\le \alpha_n\|u-q\|^2+(1-\alpha_n)(1-\delta)\|x_n-q\|^2+(1-\alpha_n)\delta\|u_n-q\|^2\\
&\le \alpha_n\|u-q\|^2+(1-\alpha_n)(1-\delta)\|x_n-q\|^2+(1-\alpha_n)\delta\|u_{n,N-1}-q\|^2\\
&\le \alpha_n\|u-q\|^2+(1-\alpha_n)(1-\delta)\|x_n-q\|^2+(1-\alpha_n)\delta\Big(\|x_n-q\|^2-\|u_{n,N-1}-x_n\|^2\\
&\quad +2\gamma\|u_{n,N-1}-x_n\|\left\|A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|\Big)\\
&\le \alpha_n\|u-q\|^2+\|x_n-q\|^2-(1-\alpha_n)\delta\|u_{n,N-1}-x_n\|^2+2\gamma(1-\alpha_n)\delta M\left\|A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|.
\end{aligned}$$

where $M=\sup_{n\in\mathbb{N}}\{\|u_{n,N-1}-x_n\|\}$. Hence, we obtain

$$(1-\alpha_n)\delta\|u_{n,N-1}-x_n\|^2 \le \alpha_n\|u-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2+2\gamma(1-\alpha_n)\delta M\left\|A^*\left(I-T_{r_n}^{G_N}\right)Ax_n\right\|.$$

Therefore, it follows from (3.8) that

$$\lim_{n\to\infty}\|u_{n,N-1}-x_n\|=0. \tag{3.9}$$

Similarly, we have

$$\lim_{n\to\infty}\|u_{n,i}-u_{n,i+1}\|=0 \text{ and } \lim_{n\to\infty}\|u_n-u_{n,1}\|=0. \tag{3.10}$$

Since

$$\|u_n-x_n\|\le\|u_n-u_{n,1}\|+\|u_{n,1}-u_{n,2}\|+\cdots+\|u_{n,N-2}-u_{n,N-1}\|+\|u_{n,N-1}-x_n\|,$$

using (3.9) and (3.10), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.11}$$

On the other hand, since

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \|\alpha_n u+(1-\alpha_n)(1-\delta)x_n+(1-\alpha_n)\delta y_n-q\|^2\\ &\leq \alpha_n\|u-q\|^2+(1-\alpha_n)(1-\delta)\|x_n-q\|^2+(1-\alpha_n)\delta\|y_n-q\|^2-(1-\alpha_n)^2(1-\delta)\delta\|x_n-y_n\|^2\\ &\leq \alpha_n\|u-q\|^2+\|x_n-q\|^2-(1-\alpha_n)^2(1-\delta)\delta\|x_n-y_n\|^2\end{aligned}$$

we get

$$(1-\alpha_n)^2(1-\delta)\delta\|x_n-y_n\|^2 \leq \alpha_n\|u-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2.$$

By taking limit from both side we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.12}$$

Also, since

$$\|y_n-u_n\| \leq \|x_n-y_n\|+\|u_n-x_n\|,$$

we have

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.13}$$

Since

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n u+(1-\alpha_n)(1-\delta)x_n+(1-\alpha_n)\delta y_n-x_n\|\\ &= \|\alpha_n(u-x_n)+(1-\alpha_n)\delta(y_n-x_n)\|\\ &\leq \alpha_n\|u-x_n\|+(1-\alpha_n)\delta\|x_n-y_n\|,\end{aligned}$$

from (3.12), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.14}$$

Step 4. Now, we show that $\lim_{n\to\infty}\|u_n-S_m u_n\|=0$. Since $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup z\in C$. Without loss of generality, we can assume that

$$\lim_{i\to\infty}\langle u-p, x_{n_i}-p\rangle=\limsup_{n\to\infty}\langle u-p, x_n-p\rangle. \tag{3.15}$$

So, it follows from (3.15) that

$$\limsup_{n\to\infty}\langle u-p, x_n-p\rangle=\lim_{i\to\infty}\langle u-p, x_{n_i}-p\rangle=\langle u-P_\Gamma u, z-P_\Gamma u\rangle\leq 0. \tag{3.16}$$

Also, it is obvious that

$$\begin{aligned}\langle u_n-p, u_n-y_n\rangle &= \beta_n\langle u_n-p, u_n-Vu_n\rangle\\ &= \beta_n\sum_{m=1}^{\infty}\eta_m\langle u_n-p, u_n-S_m u_n\rangle\\ &\geq \beta_n\sum_{m=1}^{\infty}\eta_m\frac{1-k_m}{2}\|u_n-S_m u_n\|\\ &\geq \beta_n\frac{1-k}{2}\sum_{m=1}^{\infty}\eta_m\|u_n-S_m u_n\|.\end{aligned}$$

From (3.13) and condition (ii), we get

$$\lim_{n\to\infty}\|u_n-S_m u_n\|=0$$

for all $m\in\mathbb{N}$. So, it follows from demiclosed principle of S_m and (3.11) that $z\in\cap_{m=1}^{\infty}F(S_m)$.

Step 5. Next, we show that $z \in \cap_{i=1}^{N} MEP(F_i, \varphi_i)$. Since $u_n = T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{G_1}\right)A\right)u_{n,1}$, we get

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\left\langle y - u_n, u_n - u_{n,1} + \gamma A^*\left(I - T_{r_n}^{G_1}\right)Au_{n,1}\right\rangle \geq 0$$

for all $y \in C_1$. So, we can write

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - u_{n,1}\rangle + \frac{1}{r_n}\left\langle y - u_n, \gamma A^*\left(I - T_{r_n}^{G_1}\right)Au_{n,1}\right\rangle \geq 0, \forall y \in C_1.$$

Since F_1 is a monotone mapping, we have

$$\varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - u_{n,1}\rangle + \frac{1}{r_n}\left\langle y - u_n, \gamma A^*\left(I - T_{r_n}^{G_1}\right)Au_{n,1}\right\rangle \quad \geq \quad F_1(y, u_n), \forall y \in C_1,$$

and hence

$$\varphi_1(y) - \varphi_1(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - u_{n_i,1}\rangle + \frac{1}{r_{n_i}}\left\langle y - u_{n_i}, \gamma A^*\left(I - T_{r_{n_i}}^{G_1}\right)Au_{n_i,1}\right\rangle \quad \geq \quad F_1(y, u_{n_i}), \forall y \in C_1.$$

It follows from weakly convergence of u_{n_i} to z, condition (iv), (3.7), (3.10) and the proper lower semicontinuity of φ_1 that

$$F_1(y, z) + \varphi_1(z) - \varphi_1(y) \leq 0, \forall y \in C_1.$$

Let $y_\lambda = \lambda y + (1-\lambda)z$, for all $\lambda \in (0,1]$ and $y \in C_1$. It is clear that $y_\lambda \in C_1$. So, last inequality holds for $y = y_\lambda$, that is,

$$F_1(y_\lambda, z) + \varphi_1(z) - \varphi_1(y_\lambda) \leq 0.$$

From assumptions (A1)-(A6) and last inequality, we have

$$\begin{aligned} 0 &= F_1(y_\lambda, y_\lambda) + \varphi_1(y_\lambda) - \varphi_1(y_\lambda) \\ &\leq \lambda F_1(y_\lambda, y) + (1-\lambda)F_1(y_\lambda, z) + \lambda\varphi_1(y) + (1-\lambda)\varphi_1(z) - \lambda\varphi_1(y_\lambda) - (1-\lambda)\varphi_1(y_\lambda) \\ &= \lambda(F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda)) + (1-\lambda)(F_1(y_\lambda, z) + \varphi_1(z) - \varphi_1(y_\lambda)) \\ &\leq \lambda(F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda)) \end{aligned}$$

Therefore, we have

$$F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda) \geq 0, \forall y \in C_1.$$

By taking limit as $\lambda \to 0$, we get

$$F_1(z, y) + \varphi_1(y) - \varphi_1(z) \geq 0, \forall y \in C_1,$$

that is, $z \in MEP(F_1, \varphi_1)$. Similarly, since $u_{n,i} = J_{i+1}u_{n,i+1}$ for $1 \leq i \leq N-2$ and $u_{n,N-1} = J_N x_n$, it follows from (3.7), (3.8), (3.9) and (3.10) that $z \in MEP(F_i, \varphi_i)$ for $1 \leq i \leq N$. So, we obtain that $z \in \cap_{i=1}^{N} MEP(F_i, \varphi_i)$ for $y \in C = \cap_{i=1}^{N} C_i$. On the other hand, since A is a bounded linear operator, we get $Ax_{n_i} \rightharpoonup Az$. Then, from (3.7), (3.8) and (3.11), we have $T_{r_{n_i}}^{G_k}Ax_{n_i} \rightharpoonup Az$, for $k = 1, 2, \ldots, N$. So, from definition of $T_{r_{n_i}}^{G_k}Ax_{n_i}$, we get

$$G_k\left(T_{r_{n_i}}^{G_k}Ax_{n_i}, y\right) + \phi_k(y) - \phi_k\left(T_{r_{n_i}}^{G_k}Ax_{n_i}\right) + \frac{1}{r_{n_i}}\left\langle y - T_{r_{n_i}}^{G_k}Ax_{n_i}, T_{r_{n_i}}^{G_k}Ax_{n_i} - Ax_{n_i}\right\rangle \quad \geq \quad 0, \forall y \in Q_k.$$

It follows from weakly convergence of $T_{r_{n_i}}^{G_k}Ax_{n_i}$ to Az and upper semicontinuity in the first argument of G_k that

$$G_k(Az, y) + \phi_k(y) - \phi_k(Az) \geq 0, \forall y \in Q_k.$$

This implies that $Az \in MEP(G_i, \phi_i)$ and so $Az \in \cap_{i=1}^{N} MEP(G_i, \phi_i)$ for $y \in Q = \cap_{i=1}^{N} Q_i$. Hence, $z \in SSMEP(F_i, \varphi_i, G_i, \phi_i)$ and so $z \in \Gamma$.

Step 6. Finally, we show that $x_n \to p \in \Gamma$. Since we assume that $\{\|x_n - q\|\}$ is decreasing mapping for $n > m$, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \langle \alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta y_n - p, x_{n+1}-p\rangle \\
&= \langle \alpha_n (u-p) + (1-\alpha_n)(1-\delta)(x_n-p) + (1-\alpha_n)\delta(y_n-p), x_{n+1}-p\rangle \\
&\le \alpha_n \langle u-p, x_{n+1}-p\rangle + (1-\alpha_n)(1-\delta)\|x_n-p\|\|x_{n+1}-p\| + (1-\alpha_n)\delta\|y_n-p\|\|x_{n+1}-p\| \\
&\le (1-\alpha_n)\|x_n-p\|\|x_{n+1}-p\| + \alpha_n\langle u-p, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\langle u-p, x_{n+1}-p\rangle .
\end{aligned}$$

So, it follows from Lemma 2.3 and (3.16) that $\{x_n\}$ converges strongly to $p = P_\Gamma u$.

Case 2: Let assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i} - p\| < \|x_{n_{i+1}} - p\|$ for all $i \in \mathbb{N}$. Then, it follows from Lemma 2.2 that there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that $\|x_{m_j} - p\| \le \|x_{m_{j+1}} - p\|$ and $\|x_j - p\| \le \|x_{m_{j+1}} - p\|$. Now, we show that

$$\limsup_{j\to\infty}\langle u-p, x_{m_j}-p\rangle \le 0.$$

Without loss of generality, we can assume that there exists a subsequence $\{x_{m_{j_k}}\}$ of $\{x_{m_j}\}$ such that $x_{m_j} \rightharpoonup s \in C$ and

$$\lim_{k\to\infty}\left\langle u-p, x_{m_{j_k}}-p\right\rangle = \limsup_{j\to\infty}\langle u-p, x_{m_j}-p\rangle .$$

So, we have

$$\begin{aligned}
\lim_{k\to\infty}\left\langle u-p, x_{m_{j_k}}-p\right\rangle &= \limsup_{j\to\infty}\langle u-p, x_{m_j}-p\rangle \\
&= \langle u - P_\Gamma u, s - P_\Gamma u\rangle \le 0. \qquad (3.17)
\end{aligned}$$

In a similar way as in the Case 1, we get

$$\lim_{j\to\infty}\|x_{m_j} - u_{m_j}\| = \lim_{j\to\infty}\|u_{m_j} - S_m u_{m_j}\| = 0.$$

Since S_m is demiclosed, we have $s \in \cap_{m=1}^{\infty} F(S_m)$. Similarly, we can obtain $s \in SSMEP(F_i, \varphi_i, G_i, \phi_i)$,

$$\lim_{j\to\infty}\|x_{m_{j+1}} - x_{m_j}\| = 0. \qquad (3.18)$$

On the other hand, since $x_{m_j} = \alpha_{m_j}u + (1-\alpha_{m_j})(1-\delta)x_{m_j} + (1-\alpha_{m_j})\delta y_{m_j}$, we have

$$\begin{aligned}
\|x_{m_{j+1}}-p\|^2 &= \langle \alpha_{m_j}u + (1-\alpha_{m_j})(1-\delta)x_{m_j} + (1-\alpha_{m_j})\delta y_{m_j} - p, x_{m_{j+1}}-p\rangle \\
&= \langle \alpha_{m_j}(u-p) + (1-\alpha_{m_j})(1-\delta)(x_{m_j}-p) + (1-\alpha_{m_j})\delta(y_{m_j}-p), x_{m_{j+1}}-p\rangle \\
&\le \alpha_{m_j}\langle u-p, x_{m_{j+1}}-p\rangle + (1-\alpha_{m_j})(1-\delta)\|x_{m_j}-p\|\|x_{m_{j+1}}-p\| \\
&\quad + (1-\alpha_{m_j})\delta\|y_{m_j}-p\|\|x_{m_{j+1}}-p\| \\
&\le (1-\alpha_{m_j})\|x_{m_j}-p\|\|x_{m_{j+1}}-p\| + \alpha_{m_j}\langle u-p, x_{m_{j+1}}-p\rangle \\
&\le (1-\alpha_{m_j})\|x_{m_{j+1}}-p\|^2 + \alpha_{m_j}\langle u-p, x_{m_j}-p\rangle + \alpha_{m_j}\|u-p\|\|x_{m_{j+1}}-x_{m_j}\| .
\end{aligned}$$

Therefore, we get

$$\|x_{m_{j+1}}-p\|^2 \le \|u-p\|\|x_{m_{j+1}}-x_{m_j}\| + \langle u-p, x_{m_j}-p\rangle .$$

So, it follows from (3.17) and (3.18) that

$$\lim_{j\to\infty}\|x_{m_{j+1}}-p\| = 0.$$

Since we know from Lemma 2.2 that $\|x_j - p\| \leq \|x_{m_{j+1}} - p\|$, we get that $\{x_n\}$ converges strongly to $p = P_\Gamma u$. This completes the proof. □

Since the class of demimetric mappings contains the class of generalized hybrid mappings with nonempty fixed point (see [22]), the following result follows from Theorem 3.1.

Corollary 3.2. Let C_i and Q_i, $1 \leq i \leq N$, be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator and $\{S_m\}$ be an infinite family of generalized hybrid mappings from $C = \cap_{i=1}^N C_i$ to H_1. Let $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be nonlinear bifunctions satisfying assumptions (A1)-(A6), $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ proper lower semicontinuous and convex functions such that $C_i \cap dom\varphi_i \neq \emptyset$ and $Q_i \cap dom\phi_i \neq \emptyset$ and let G_i be upper semicontinuous in the first argument. Assume that $\Lambda = \cap_{m=1}^\infty F(S_m) \cap SSMEP(F_i, \varphi_i, G_i, \phi_i) \neq \emptyset$ and u is a fixed vector in C. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta y_n, \\ y_n = (1-\beta_n)u_n + \beta_n \sum_{m=1}^\infty \eta_m S_m u_n, \\ u_n = J_1 J_2 \dots J_N x_n, \forall n \in \mathbb{N} \end{cases} \tag{3.19}$$

where $a_n, \beta_n, \eta_m, \delta \in (0,1)$, $r_n \in (0, \infty)$, $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A and $J_i = T_{r_n}^{F_i}\left(I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A\right), 1 \leq i \leq N$. Assume that the following conditions hold:

1. $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
2. $0 < a \leq \beta_n \leq b < 1$;
3. $\sum_{m=1}^\infty \eta_m = 1$;
4. $0 < \liminf_{n\to\infty} r_n$.

   Then, the sequence $\{x_n\}$ generated by (3.19) converges strongly to $p = P_\Lambda u$.

Remark 3.3. In Theorem 3.1, if we take $\varphi_i = \phi_i = 0$ for all $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ generated by our iterative algorithm (3.1) converges strongly to common solution of system of split equilibrium problems and fixed point problem of infinite family of demimetric mappings.

4 Applications

In this section, we give some applications of proposed method and problem to split variational inequality problems and convex minimization problems.

4.1 Application to System of Split Variational Inequality Problems

Let $F : C \times C \to \mathbb{R}$ be a bifunction. As stated in the first section, the classical equilibrium problem is understood to be that of finding $x^* \in C$ such that $F(x^*, y) \geq 0$ for all $y \in C$. On the other hand, the classical variational inequality problem for a monotone mapping $A : C \to H$ is to find a point $x^* \in C$ such that $\langle A(x^*), x - x^* \rangle \geq 0$ for all $x \in C$. It is trivial that these two problems are equivalent if $F(x, y) = \langle A(x), y - x \rangle$. So, we can give the following theorem for the system of split variational inequality problems.

Theorem 4.1. Let C_i and Q_i, $1 \leq i \leq N$, be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator and $\{S_m\}$ be an infinite family of k_m-demimetric and demiclosed mappings from $C = \cap_{i=1}^N C_i$ to H_1. Let $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be nonlinear bifunctions satisfying assumptions (A3)-(A6) difened by $F_i(x, y) = \langle A_i(x^*), y - x^* \rangle$ and $G_i(u, v) = \langle B_i(u^*), v - u^* \rangle$, respectively where $A_i : C_i \to H_1$ and $B_i : Q_i \to H_2$ are monotone mappings and let G_i be upper semicontinuous in the first argument. Assume that $\Pi = \cap_{m=1}^\infty F(S_m) \cap SSVIP(A_i, B_i) \neq \emptyset$ where $SSVIP(A_i, B_i) := \{x^* \in C : \langle A_i(x^*), x - x^* \rangle \geq 0, \forall x \in C_i$ and $\langle B_i(Ax^*), y - Ax^* \rangle \geq 0, \forall y \in Q_i\}$ and u is a fixed vector in C. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta y_n, \\ y_n = (1-\beta_n)u_n + \beta_n \sum_{m=1}^\infty \eta_m S_m u_n, \\ u_n = J_1 J_2 \dots J_N x_n, \forall n \in \mathbb{N} \end{cases} \tag{4.1}$$

where $a_n, \beta_n, \eta_m, \delta \in (0,1)$, $r_n \in (0, \infty)$, $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A and $J_i = T_{r_n}^{F_i}\left(I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A\right), 1 \leq i \leq N$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < a \le \beta_n \le b < 1 - k$ where $k = \sup\{k_i : i \in \mathbb{N}\}$;
(iii) $\sum_{m=1}^{\infty} \eta_m = 1$;
(iv) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (4.1) converges strongly to $p = P_\Pi u$.

4.2 Application to System of Convex Minimization Problems

Let f be a convex and differentiable function. A convex minimization problem is to find a point x^* such that

$$f(x^*) = \min_{x\in C} f(x). \tag{4.2}$$

A point x^* is a solution of the problem (4.2) if and only if it is a solution of the following variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle \nabla f x^*, x - x^* \rangle \ge 0, \forall x \in C. \tag{4.3}$$

where ∇f is the gradient of f. Also, it is well known that x^* is a solution of (4.3) if and only if x^* is a fixed point of $P_C(I - \lambda\nabla f)$. A mapping $P_C(I - \lambda A)$ is a nonexpansive mapping if the mapping A is a inverse strongly monotone mapping. Since every Lipschitz continuous mapping is inverse strongly monotone and every demimetric mapping is nonexpansive, we can give the following theorem for the solutions of convex minimization problems without proof.

Theorem 4.2. Let C_i and Q_i, $1 \le i \le N$, be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator and f be a convex and differentiable function on an open set D containing the set $C = \cap_{i=1}^N C_i$. Assume that ∇f is a κ-Lipschitz continuous operator on D and minimizers of f relative to the set C exist. Let $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be nonlinear bifunctions satisfying assumptions (A1)-(A6), $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ proper lower semicontinuous and convex functions such that $C_i \cap dom\varphi_i \ne \emptyset$ and $Q_i \cap dom\phi_i \ne \emptyset$ and let G_i be upper semicontinuous in the first argument. Assume that $\Omega = SSMEP(F_i, \varphi_i, G_i, \phi_i) \ne \emptyset$ and u is a fixed vector in C. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)(1-\delta)x_n + (1-\alpha_n)\delta y_n, \\ y_n = (1-\beta_n)u_n + \beta_n P_C(I - \lambda\nabla f)u_n, \\ u_n = J_1 J_2 \dots J_N x_n, \forall n \in \mathbb{N} \end{cases} \tag{4.4}$$

where $a_n, \beta_n, \eta_m, \delta \in (0,1)$, $r_n \in (0,\infty)$, $\lambda \in (0, 2/\kappa)$, $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A, and $J_i = T_{r_n}^{F_i}\left(I - \gamma A^*\left(I - T_{r_n}^{G_i}\right)A\right), 1 \le i \le N$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < a \le \beta_n \le b < 1 - k$ where $k = \sup\{k_i : i \in \mathbb{N}\}$;
(iii) $\sum_{m=1}^{\infty} \eta_m = 1$;
(iv) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (4.4) converges strongly to the minimizers of f and $p = P_\Omega u$.

Example 4.3. Let $H_1 = H_2 = \mathbb{R}, C_i = [-i-2, 0], Q_i = [-5-i, 0], \varphi_i(x) = \phi_i(x) = 0, F_i : C_i \times C_i \to H_1, F_i(x,y) = ixy - ix^2, G_i : Q_i \times Q_i \to H_2, G_i(x,y) = (5+i)xy - (5+i)x^2$, $1 \le i \le N, S_m : C \to H_1, S_m x = -3x$, $A : H_1 \to H_2, Ax = \frac{x}{3}$ where $C = \bigcap_{i=1}^N C_i = [-3, 0]$ and $Q = \bigcap_{i=1}^N Q_i = [-6, 0]$. It is clear that F_i and G_i satisfy assumptions (A1)-(A6), the mapping S_m is $\frac{1}{2}$-demimetric mapping with $\cap_{m=1}^{\infty} F(S_m) = \{0\}$, the adjoint operator A^* of A is defined by $A^*x = \frac{x}{3}$ from H_2 to H_1 and the spectral radius of A^*A is $L = \frac{1}{9}$. It is clear that $x^* = 0$ satisfies the following system of equilibrium problems:

$$F_i(x^*, x) = ix^*x - (x^*)^2 \ge 0,$$

for all $x \in [-i-2, 0]$ and for $1 \le i \le N$. It is easy to see that the point $y^* = Ax^* = 0$ solves the following system of equilibrium problems:

$$G_i(y^*, y) = (5+i)y^*y - (5+i)(y^*)^2 \ge 0,$$

for all $y \in [-5-i, 0]$ and for $1 \le i \le N$. So, this implies that $x^* = 0$ is a solution for the system of split equilibrium problems and fixed point problem, i.e., $0 \in \Gamma = \cap_{m=1}^{\infty} F(S_m) \cap SSEP(F_i, \varphi_i, G_i, \phi_i)$. Next, with a simple calculation, we obtain that

$$T_{r_n}^{Gi} Ax = \frac{x}{3 + 3(5+i)r_n}, \left(I - \gamma A^*\left(I - T_{r_n}^{Gi}\right)A\right)x = \frac{8x}{9} + \frac{x}{9 + 9(5+i)r_n}$$

x_n	$x_1 = -0.7,\ N = 10$	$x_1 = -0.4,\ N = 8$
x_{100}	$-200E-03$	$-200E-03$
x_{200}	$-100E-03$	$-99E-04$
x_{2500}	$-79E-05$	$-79E-05$
x_{20000}	$-9E-06$	$-9E-06$

Table 1:

and

$$T_{r_n}^{F_i}\left(I-\gamma A^*\left(I-T_{r_n}^{Gi}\right)A\right)x=\frac{1}{1+ir_n}\left(\frac{8x}{9}+\frac{x}{9+9\left(5+i\right)r_n}\right)$$

for $\gamma = 1$. Now, we show that the sequence $\{x_n\}$ generated by (3.1) converges strongly to the common solution $x^* = 0$. Let $\alpha_n = \frac{1}{2n+3}, \beta_n = \frac{n}{3n+1}, \delta = \frac{1}{4}, r_n = \frac{n}{4n+1}$ and $u = -0,1$. It is clear that $\alpha_n, \beta_n, \eta_n, r_n, \delta$ and γ satisfy the conditions (i)-(iv) of Theorem 3.1. Then, algorithm (3.1) becomes

$$\begin{cases} x_{n+1} = \frac{-1}{20n+30} + \frac{3n+3}{4n+6}x_n + \frac{n+1}{4n+6}y_n, \\ y_n = \frac{-n+1}{3n+1}u_n, \\ u_{n,i} = \frac{4n+1}{(4+i)n+1}\left[\frac{8}{9} + \frac{(4n+1)}{(81+9i)n+9}\right]u_{n,i+1}, \\ u_{n,N-1} = \frac{4n+1}{(4+N)n+1}\left[\frac{8}{9} + \frac{(4n+1)}{(81+9N)n+9}\right]x_n, \forall n \in \mathbb{N} \end{cases} \tag{4.5}$$

where $1 \leq i \leq N-2$. By using Mathematica software, we see that the sequence $\{x_n\}$ generated by alghorithm (4.5) converges strongly to common solution $x^* = 0$. Below, we give some steps of alghorithm (4.5) for some special initial values x_1 and special N.

From the Table 1, it can be seen that there is only a small difference between the iteration values that starts from the different initial values x_1 for different N.

5 Conclusion

In this paper, we studied a new system of split mixed equilibrium problem, which includes split and mixed equilibrium as special cases. We established that the sequence generated by our proposed algorithm converges strongly to a common element in the solutions set of a system of split mixed equilibrium problems and the common fixed points set of infinite family of demimetric mappings. Our result unify, extend and generalize the results in [7, 6, 9, 8, 10, 17, 19, 24].

References

[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Stud. **63** (1994), 123–145.

[2] F.E. Browder and W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces,* J. Math. Anal. Appl. **20** (1967), 197–228.

[3] C.E. Chidume and S. Maruster, *Iterative methods for the computation of fixed points of demicontractive mappings,* J. Comput. Appl. Math. **234** (2010), 861–882.

[4] P. E. Maingé, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **325** (2007), 469–479.

[5] S. Akashi and W. Takahashi, *Weak convergence theorem for an infinite family of demimetric mappings in a Hilbert space,* J. Nonlinear Convex Anal. **10** (2016), 2159–2169.

[6] Z. He, *The split equilibrium problem and its convergence algorithms,* J. Inequal Appl. **2012** (2012), 162.

[7] D.V. Hieua, *Parallel extragradient-proximal methods for split equilibrium problems*, Math. Modell. Anal. **21** (2016), no. 4, 478–501.

[8] K.R. Kazmi and S.H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem,* J. Egypt. Math. Soc. **21** (2013), no. 1, 44–51.

[9] A. Moudafi, *The split common fixed-point problem for demicontractive mappings,* Inverse Prob. **26** (2010), no. 5, Article ID 055007.

[10] N. Onjai-uea and W. Phuengrattana, *On solving split mixed equilibrium problems and a fixed point problems of hybrid-type multivalued mappings in Hilbert spaces,* J. Inequal. Appl. **2017** (2017), 137.

[11] M.O. Osilike and F.O. Isiogugu, *Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces,* Nonlinear Anal. **74** (2011), 1814–1822.

[12] H. Piri, *Hybrid pseudo-viscosity approximation schemes for systems of equilibrium problems and fixed point problems of infinite family and semigroup of non-expansive mappings,* Nonlinear Anal. **74** (2011), 6788–6804.

[13] M. Rahaman, YC. Liou, R. Ahmad, and I. Ahmad, *Convergence theorems for split equality generalized mixed equilibrium problems for demi-contractive mappings,* J. Inequal. Appl. **2015** (2015), 418.

[14] A. Razani, *Double sequence iteration for a strongly contractive mapping in the modular space,* Iran. J. Math. Sci. Inform. **11** (2016), no. 2, 119–130.

[15] A. Razani and M. Bagherboum, *A modified Mann iterative scheme for a sequence on nonexpansive mappings and a monotone mapping with applications,* Bull. Iran. Math. Soc. **40** (2014), no. 4, 823–849.

[16] A. Razani and Z. Goodarzi, *Iteration by Cesaro means for quasi-contractive mappings,* Filomat **28** (2014), no. 8, 1575–1584

[17] S.H. Rizvi, *A strong convergence theorem for split mixed equilibrium and fixed point problems for nonexpansive mappings,* J. Fixed Point Theory Appl. (2018) 20:8. doi.org/10.1007/s11784-018-0487-8.

[18] M. Shahsavari, A. Razani, and Gh. Abbasi, *Some random iteration processes in modular function space,* Ital. J. Pure Appl. Math. **47** (2022), 929–949.

[19] Y. Shehu and F.U. Ogbuisi, *An iterative algorithm for approximating a solution of split common fixed point problem for demi-cotractive maps,* Dyn. Contin. Discrete Impuls. Syst. B: Appl. Algorithms **23** (2016), 205–216.

[20] Y. Song, *Iterative methods for fixed point problems and generalized split feasibility problems in Banach spaces,* J. Nonlinear Sci. Appl. **11** (2018), 198–217.

[21] T. Suzuki, *Strong convergence of Krasnoselskii and Mannâs type sequences for oneparameter nonexpansive semigroups without Bochner integrals,* J. Math. Anal. Appl. **305** (2005), no. 1, 227–239.

[22] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces,* J. Convex Anal. **24** (2017), no. 3, 1015–1028.

[23] W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces,* Nonlinear Anal. **70** (2009), no. 1, 45–57.

[24] G.C. Ugwunnadi and B. Ali, *Approximation methods for solutions of system of split equilibrium problems,* Adv. Oper. Theory **1** (2016), no. 2, 164–183.

[25] H.-K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. **66** (2002), 240–256.
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