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A note on different conditions of the unique solvability for the absolute value equation

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Abstract

In this study, we compare the different conditions which are used to detect the unique solvability of the absolute value equation (AVE) $Ax - B|x| = b$. Also, analyzing which condition is more suitable to use according to our problem and later, we will see the advantage and disadvantages of different unique solvability conditions. Further, we take an example where existing conditions are invalid to judge the unique solvability of the AVE.

Keywords: Absolute value equation, Necessary condition, Sufficient condition, Unique solution
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1 Introduction

Let us consider the AVE

$$Ax - B|x| = b, \tag{1.1}$$

with $A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ is to be determined.

The AVE have many applications in the field of linear complementarity problem, quadratic programs, linear programs, linear interval equation, bimatrix games, etc. AVE was first introduced by Jiri Rohn [27], who provided an alternative theorem for its unique solvability. Theoretical analysis is a vital part of research for AVE, where authors provided different conditions for its unique solvability (see [18, 19, 26] and references therein). Different numerical methods are developed for AVE based on the unique solvability conditions. For more details about AVE, the reader may refer ([13, 15, 16, 18, 19, 30] and references therein). In the literature, the unique solvability conditions of the AVE (1.1) are extended to the absolute value matrix equations, and one may refer ([7, 12, 14] and references therein).

Based on the different unique solvability conditions of the AVE, various numerical methods are discussed for the solution of the AVE. Mangasarian proposed different schemes for the solution of the AVE, for example, a succession of linear programs [20], a direct generalized Newton method [21], an iterative method by utilizing a dual-complementarity

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condition [22], new linear programming method [23], a hybrid algorithm [24]. Khan et al. proposed the Newton-type technique [11] to solve the AVE. Anane et al. gave the preconditioned conjugate gradient method [6] to evaluate an approximation solution of the AVE. Achache et al. suggest a Picard iterative method [2] and an infeasible path-following interior-point algorithm [1] to solve the AVE. Edalatpour provided a generalized Gauss–Seidel iteration method [8] for solving the large sparse AVE. For more details about the numerical method for AVE, one may refer ([3, 4, 5, 9, 10, 29] and references therein).

2 Discussion on different unique solvability conditions of AVE

In this section, we review different conditions for the unique solvability of the AVE. We see the importance and limitations of different conditions for the unique solvability of the AVE.

We have the following result for AVE based on singular value conditions of matrices.

Theorem 2.1. [28, 31, 32] If any one of the following sufficient singular value conditions satisfy then AVE (1.1) has exactly one solution:
(i) $\sigma_{max}(|B|) < \sigma_{min}(A)$;
(ii) $\sigma_{max}(B) < \sigma_{min}(A)$;
(iii) $\sigma_{max}(A^{-1}B) < 1$ or $\sigma_{min}(B^{-1}A) > 1$, where $\sigma_{max}(.)$ and $\sigma_{min}(.)$ will denotes the maximum and minimum singular values respectively.

Remark 2.2. If we compare the conditions of Theorem 2.1, (iii) conditions is superior to the (ii) [32] and (ii) is superior than (i) [31]. In Example 2.3, condition (iii) of Theorem 2.1 is valid to judge the unique solvability of the AVE, while the other two conditions of Theorem 2.1 are invalid.

Example 2.3. Consider the matrices A and B

$$A = \begin{bmatrix} 4 & -1.5 \\ -3 & 3 \end{bmatrix} \quad and \quad B = \begin{bmatrix} -0.6 & 1.5 \\ 0.5 & -2.2 \end{bmatrix}$$

Here $\sigma_{min}(A) = 1.2746$, $\sigma_{max}(B) = 2.7672$, $\sigma_{max}(|B|) = 2.7672$, $\sigma_{max}(A^{-1}B) = 0.5988 < 1$ and $\sigma_{min}(B^{-1}A) = 1.6701 > 1$. Clearly, conditions (i) and (ii) of Theorem 2.1 are invalid in the judge of the unique solvability of the AVE (1.1), while condition (iii) of Theorem 2.1 is satisfying and hence AVE (1.1) has a unique solution.

Based on spectral radius conditions of matrices, we have the following result for AVE.

Theorem 2.4. [17, 29, 32] If any one of the following spectral radius conditions satisfy then AVE (1.1) has exactly one solution:
(i) $\rho(|A^{-1}||B|) < 1$;
(ii) $\rho(|A^{-1}B|) < 1$;
(iii) $\rho(A^{-1}B\bar{D}) < 1$, or $\rho(B\bar{D}A^{-1}) < 1$, for any diagonal matrix $\bar{D} = diag(\bar{d}_i)$ with $-1 \leq \bar{d}_i \leq 1$, where $\rho(.)$ denotes the spectral radius of a matrix.

Comments on Theorem 2.1 and Theorem 2.4. The conditions given in Theorem 2.1 and Theorem 2.4 are sufficient but not necessary. So there exist some examples such that given conditions not satisfied, still AVE (1.1) has a unique solution. But one advantage of these conditions is that they are easy to use in practice compared to the conditions of Theorem 2.7, 2.8 and 2.12. Condition (iii) of Theorem 2.4 has minimal uses in practicality because to apply that condition, we need to identify all diagonal matrix $\bar{D}$, which is not an easy task. The conditions of Theorem 2.1 and Theorem 2.4 are not easy to use in practice when the matrices are large since they involve computing an inverse and estimating a spectral radius and singular values.

Remark 2.5. Sometimes the conditions of Theorem 2.1 and Theorem 2.4 are not satisfying, but AVE (1.1) still has a unique solution. Let's see the following example.

Example 2.6. Consider the AVE (1.1), where matrix A and B are given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \, and \, B = \begin{bmatrix} -1.8 & 0.4 \\ -0.8 & -1.6 \end{bmatrix}$$

Here AVE (1.1) has unique solution $x = [3.33333, -21.66667]^T$ for $b = [4, -6]^T$, but conditions of Theorem 2.1 and Theorem 2.4 are not satisfying.

Since $\sigma_{min}(A) = 2$, $\sigma_{max}(B)$ =2.0264, $\sigma_{max}(|B|)$ =2.3200 and $\sigma_{min}(B^{-1}A) = 0.9870 \not> 1$, $\sigma_{max}(A^{-1}B) = 1.0132 \not< 1$, $\rho(|A^{-1}||B|) = 1.1372 \not< 1$ and $\rho(|A^{-1}B|) = 1.1372 \not< 1$.

The conditions of Theorem 2.1 and Theorem 2.4 may revise in the future.

Based on the interval matrix, Lotfi et al. [17] provided the following result for AVE.

Theorem 2.7. [17] The AVE (1.1) has exactly one solution if the interval matrix $[A - |B|, A + |B|]$ is regular.

Based on the column $\mathcal{W}$-property, Mezzadri [25] provides the following result for AVE.

Theorem 2.8. [25] Following necessary and sufficient conditions give the surety for the unique solution for the AVE (1.1):
(i) $\{A - B, A + B\}$ holds the column $\mathcal{W}$-property;
(ii) $(A - B)$ is invertible and $\{I, (A - B)^{-1}(A + B)\}$ holds the column $\mathcal{W}$-property;
(iii) $\{(A - B)F_1 + (A + B)F_2\}$ is invertible, where $F_1, F_2 \in R^{n\times n}$ are two any non-negative diagonal matrices with $diag(F_1 + F_2) > 0$.

Comments on Theorem 2.7 and Theorem 2.8. The results of Theorem 2.7 and Theorem 2.8 are helpful to show that AVE (1.1) does not has a unique solution instead of to show that AVE (1.1) has a unique solution. Because to apply the Theorem 2.7, we need to know all the invertible matrices lie between interval matrix $[A-|B|, A+|B|]$ for given matrices A and B, and the number of such matrices may be countable or uncountable. To apply the Theorem 2.8, we need to find determinants of all column representative matrices of a given set of matrices, which is the time taken and lengthy process.

Based on matrix property, Mezzadri [25] provides the following result for AVE.

Theorem 2.9. [25] If each diagonal entry of $A + B$ has the same sign as the corresponding entry of $A - B$. Then, AVE (1.1) has exactly one solution if any one of the following conditions is satisfied:
(i) $A - B$ and $A + B$ are strictly diagonally dominant by columns;
(ii) $A - B$, $A + B$ and all their column representative matrices are irreducible diagonally dominant by columns.

Comments on Theorem 2.9. The result of the Theorem 2.9 gives different insights to check the unique solvability of the AVE (1.1). The above conditions have their significance, and applying them is practically easy. We do not need to calculate the norm and inverse of the involved matrices to apply the Theorem 2.9. Moreover, sometimes conditions of Theorem 2.1 and Theorem 2.4 hold, and those of Theorem 2.9 are not, and vice versa.

Based on the P-matrix property, we have the following results for AVE.

Theorem 2.10. [2, 17, 32] If any one of the following conditions satisfy then AVE (1.1) has a unique solution:
(i) for invertible matrix $(A - B)$, the matrix $(A - B)^{-1}(A + B)$ is a P-matrix;
(ii) $A^TA - |||B|||_2^2 I$ is a P-matrix;
(iii) $A^TA - ||B||_2^2 I$ is a P-matrix.

Remark 2.11. The converse of the first condition of Theorem 2.10 is also true.

Comments on Theorem 2.10. The outcome of Theorem 2.10 is derived from the P-matrix classes. Since every positive definite matrix is also a P-matrix, the result of Theorem 2.10 is also applicable for the positive definite matrices classes. One benefit of the above results is that we do not need to calculate singular values, spectral radius, or any norm of involved matrices. The results given in Theorem 2.10 are sufficient but not necessary, so there exist some examples such that given conditions are not satisfied, still AVE (1.1) has exactly one solution.

Theorem 2.12. [32] The AVE has a unique solution if and only if $A + B\bar{D}$ is invertible for any diagonal matrix $\bar{D} = diag(\bar{d}_i)$ with $-1 \leq \bar{d}_i \leq 1$.

Comments on Theorem 2.12. For applying this condition, we need to find all diagonal matrix $\bar{D}$, which is a complex and time-consuming task because there are infinitely many choices for $\bar{D}$. This result can be helpful for the prove that AVE does not have a unique solution. If we find only one $\bar{D}$ matrix such that matrix $A + B\bar{D}$ is singular, then AVE does not have a unique solution. The same situation also occurs with condition (iii) of Theorem 2.4.

Theorem 2.13. [2] The AVE (1.1) has exactly one solution if the invertible matrix A satisfies the norm condition $||A^{-1}B||_\infty < 1$.

Comments on Theorem 2.13. For applying this condition, we have a restriction on A to have an invertible matrix, which is one drawback. Also, sometimes this condition is not satisfying even if AVE (1.1) has a unique solution. So there is also the possibility of improving this norm condition in the future.

3 Concluding Remarks

In this note, we reviewed different conditions for the unique solvability of the AVE. We found that some conditions have limited practical uses, some conditions have restrictions due to the size of the matrix and some conditions are effective and easy to use to detect the unique solvability of the AVE. Further, we take an example where existing conditions are invalid to judge the unique solvability of the AVE. There are possibilities to revise these conditions in the future.

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