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Product type operators on vector valued derivative Besov spaces

Sepideh Nasresfahani[a], Ebrahim Abbasi[b,*], Daryoush Molaei[c]

[a]Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, Iran
[b]Department of Mathematics, Mahabad Branch, Islamic Azad University, Mahabad, Iran
[c]Department of Mathematics, Urmia Branch, Islamic Azad University, Urmia, Iran

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Abstract

In this paper, we characterize the boundedness and compactness of product type operators, including Stević-Sharma operator $T_{\nu_1,\nu 2,\varphi}$, from weak vector valued derivative Besov space $w\mathcal{E}^p_\beta(X)$ into weak vector-valued Besov space $w\mathcal{B}^p_\beta(X)$. As an application, we obtain the boundedness and compactness characterizations of the weighted composition operator on the weak vector valued derivative Besov space.

Keywords: Derivative Besov spaces, weighted composition operator, boundedness, compactness
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1 Introduction

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions $h:\mathbb{D}\to\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of all bounded analytic functions with the norm $\|f\|_\infty=\sup_{z\in\mathbb{D}}|f(z)|$. Let $\mathcal{S}(\mathbb{D})$ be the class of all analytic self-maps on $\mathbb{D}$. For $\nu\in\mathcal{H}(\mathbb{D})$ and $\varphi\in\mathcal{S}(\mathbb{D})$, the weighted composition operator $M_\nu C_\varphi$ is given by $M_\nu C_\varphi h(z)=\nu(z)(h\circ\varphi)(z)$, for $h\in\mathcal{H}(\mathbb{D})$ and $z\in\mathbb{D}$. Then for $\nu=1$, we have the composition operator C_φ and $\varphi(z)=z$, gives us the multiplication operator M_ν. So $M_\nu C_\varphi$, is a product-type operator. This operator plays an important role in the isometry theory of Banach spaces. An extensive study concerning the theory of weighted composition operators has been established during the past four decades on various settings. We refer to standards references [4, 16, 24] for various aspects about the theory of composition operators acting on analytic function spaces, especially the problems of relating operator-theoretic properties of C_φ to function theoretic properties of φ.

The differentiation operator D, is defined by $Dh=h'$, for $h\in\mathcal{H}(\mathbb{D})$. Note that D is typically unbounded on many familiar spaces of analytic functions. The differential operator plays an important role in various fields such as dynamical system theory and operator theory. In six ways, we can consider the products of any three operators of C_φ, M_ν and D, i.e.,

$$M_\nu C_\varphi, C_\varphi M_\nu, M_\nu D, DM_\nu, C_\varphi D, DC_\varphi. \tag{1.1}$$

*Corresponding author
Email addresses: sepide.nasr@gmail.com (Sepideh Nasresfahani), ebrahimabbasi81@gmail.com (Ebrahim Abbasi), d.molaei57@gmail.com (Daryoush Molaei)

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Similarly, in six ways we obtain the products of three operators M_ν, C_φ and D, i.e.,

$$M_\nu C_\varphi D, C_\varphi M_\nu D, M_\nu DC_\varphi, C_\varphi DM_\nu, DM_\nu C_\varphi, DC_\varphi M_\nu. \tag{1.2}$$

During recent years, there has been a great interest in studying these product-type operators between analytic function spaces. The boundedness and compactness of the products DC_φ and $C_\varphi D$ of composition operators and differentiation operators between Bergman spaces and Hardy spaces were first studied by Hibschweiler and Portnoy in [9] and then on Hardy spaces by Onho [15]. Also the authors [20, 21] characterized the boundedness and compactness of the products DM_ν and $M_\nu D$ from H^∞ and mixed norm spaces to Zygmund spaces and Bloch type spaces. For more information on these operators, we refere to [9, 12, 13, 15, 17].

In order to treat above product-type operators in a unified manner, Stević and co-workers for the first time in [17], introduced the so called Stević-Sharma operator $T_{\nu_1,\nu_2,\varphi}$ as follows:

$$T_{\nu_1,\nu_2,\varphi}h = M_{\nu_1}C_\varphi h + M_{\nu_2}C_\varphi Dh = \nu_1 h\circ\varphi + \nu_2 h'\circ\varphi,\ \ h\in\mathcal{H}(\mathbb{D}). \tag{1.3}$$

One of the reasons that the Stević-Sharma operator is important for investigation is that, this operator includes many product-type operators and we can obtain all operators in (1.1) and (1.2) from $T_{\nu_1,\nu_2,\varphi}$ by fixing ν_1, ν_2 and φ. More specially:

$$\begin{aligned}
&M_\nu C_\varphi = T_{\nu,0,\varphi}, \quad C_\varphi M_\nu = T_{\nu\circ\varphi,0,\varphi}, \quad M_\nu D = T_{0,\nu,id}, \quad C_\varphi D = T_{0,1,\varphi},\\
&DC_\varphi = T_{0,\varphi',\varphi}, \quad M_\nu C_\varphi D = T_{0,\nu,\varphi}, \quad C_\varphi M_\nu D = T_{0,\nu\circ\varphi,\varphi}, \quad M_\nu DC_\varphi = T_{0,\nu\varphi',\varphi},\\
&C_\varphi DM_\nu = T_{\nu'\circ\varphi,\nu\circ\varphi,\varphi'}, \quad DM_\nu C_\varphi = T_{\nu',\nu\varphi',\varphi}, \quad DC_\varphi M_\nu = T_{(\nu'\circ\varphi)\varphi',(\nu\circ\varphi)\varphi',\varphi}.
\end{aligned} \tag{1.4}$$

Under some conditions, Stević et al. [17] characterized the boundedness and compactness of the Stević-Sharma operator on the weighted Bergman space. Quite recently, Zhang and Liu [23] presented the boundedness and compactness of the operator $T_{\nu_1,\nu_2,\varphi}$ from Hardy spaces to Zygmund-type spaces. Liu and Yu [21] gave the complete characterizations for the boundedness and compactness of the operator $T_{\nu_1,\nu_2,\varphi}$ from Hardy spaces to the logarithmic Bloch spaces. Liu and co-workers [22] investigated the compactness of the operator $T_{\nu_1,\nu_2,\varphi}$ on logarithmic Bloch spaces. For further results about the Stević-Sharma operator on various holomorphic function spaces, we refer to [1, 18, 19] and references therein.

However, to the best of our knowledge, there are very few investigations about the Stević-Sharma operator in the setting of spaces of vector-valued holomorphic functions. The investigation of holomorphic functions spaces in the vector-valued framework always brings new insights and it often requires the development of entirely new techniques compared to the scalar-valued setting. To this end, we first recall our function spaces to work on. Let dm be the normalized area measure on $\mathbb{D}$, to have the total mass 1. Then for $1 \le p < \infty$ and $-1 < \beta < \infty$, the weighted Bergman space $\mathcal{L}^p_\beta(\mathbb{D})$, consists of all analytic functions $h \in H(\mathbb{D})$, such that

$$\|h\|^p_{\mathcal{L}^p_\beta} := \int_{\mathbb{D}} |h(z)|^p(1-|z|^2)^\beta dm(z) < \infty.$$

In addition, the analytic Besov type space $\mathcal{B}^p_\beta(\mathbb{D})$, is the space of all functions $h \in \mathcal{H}(\mathbb{D})$, for which

$$\|h\|^p_{\mathcal{B}^p_\beta(\mathbb{D})} = \int_{\mathbb{D}} |h'(z)|^p(1-|z|^2)^\beta dm(z) + |h(0)| < \infty.$$

Furthermore, the analytic function $h \in \mathcal{H}(\mathbb{D})$, is considered to be in derivative Besov space $\mathcal{E}^p_\beta(\mathbb{D})$, if the norm

$$\|h\|^p_{\mathcal{E}^p_\beta(\mathbb{D})} = \|h'\|^p_{\mathcal{B}^p_\beta} + |h(0)| = \|h''\|^p_{\mathcal{L}^p_\beta} + |h'(0)| + |h(0)|$$

be finite. Let X be a complex Banach space. The corresponding weak version vector-valued derivative Besov space $w\mathcal{E}^p_\beta(X)$ consists of the analytic functions $h : \mathbb{D} \to X$ for which

$$\|h\|_{w\mathcal{E}^p_\beta(X)} = \sup_{\delta^*\in B_{X^*}} \|\delta^*\circ h\|_{\mathcal{E}^p_\beta(\mathbb{D})} < \infty.$$

Meanwhile, the weak vector-valued Besov space $w\mathcal{B}^p_\beta(X)$ consists of the analytic functions $h : \mathbb{D} \to X$ for which

$$\|h\|_{w\mathcal{B}^p_\beta(X)} = \sup_{\delta^*\in B_{X^*}} \|\delta^*\circ h\|_{\mathcal{B}^p_\beta(\mathbb{D})} < \infty.$$

Here and in the sequel, X^* is the dual space of X and $B_{X^*} = \{\delta^* \in X^* : \|\delta^*\|_{X^*} \leq 1\}$ is the closed unit ball of X^*. In fact, such weak version spaces $wE(X)$ can be introduced under more general conditions on any Banach spaces E consisting of holomorphic functions, see [2, 3, 6, 10, 11, 14] and references therein.

The main concern of the present paper is to discuss the boundedness and compactness of the operator $T_{\nu_1,\nu_2,\varphi}$ from weak vector valued derivative Besov spaces into weak vector valued Besov type spaces. Then as conclusions, according to (1.4), we have characterizations for the boundedness and compactness of product-type operators in (1.1) and (1.2) on these spaces. Also as another interesting result, we have a characterization for the boundedness and compactness of weighted composition operator $M_\nu C_\varphi$ from $w\mathcal{E}^p_{\beta+p}(X)$ into $w\mathcal{E}^p_\beta(X)$.

Throughout this paper, we use $A \preceq B$ or $B \succeq A$ for non-negative quantities A and B to mean $A \leq CB$ for some inessential constant $C > 0$. Similarly, we use the notation $A \asymp B$ if both $A \preceq B$ and $B \succeq A$ hold.

2 Boundedness of product type operators from $\mathcal{E}^p_{\beta+p}(X)$ into $\mathcal{B}^p_\beta(X)$

In this section, our first plan is to find some equivalent statements for the boundedness of some product type operators include Stević-Sharma operator, from $w\mathcal{E}^p_\beta(X)$ into $w\mathcal{L}^p_\beta(X)$. Then we consider the boundedness of two operators $M_{\nu_1}C_\varphi$ and $M_{\nu_2}C_\varphi D$, from $\mathcal{E}^p_{\beta+p}(\mathbb{D})$ into $\mathcal{B}^p_\beta(\mathbb{D})$, to obtain the conditions for studying $T_{\nu_1,\nu_2,\varphi} = M_{\nu_1}C_\varphi + M_{\nu_2}C_\varphi D$ between these two spaces. Finally as a conclusion, we characterize the boundedness of $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \mapsto w\mathcal{E}^p_\beta(X)$.

For a point ξ in $\partial\mathbb{D}$, the boundary of the unit disk and $\epsilon > 0$, we define the Carleson set $\rho(\xi, \epsilon) := \{z \in \mathbb{D} : |\xi - z| < \epsilon\}$. Let $\beta > -1$ and ζ be a positive Borel measure on $\mathbb{D}$. For $0 < p < \infty$, it is well known (see [24, Section 2.4]) that

$$\text{the embedding } \; \mathcal{L}^p_\beta(\mathbb{D}) \subset L^p(\mathbb{D}, d\zeta) \text{ is bounded} \Leftrightarrow \sup_{\xi\in\partial\mathbb{D},\epsilon>0} \frac{\zeta(\rho(\xi,\epsilon))}{\epsilon^{\beta+2}} < \infty \tag{2.1}$$

and

$$\text{the embedding } \; \mathcal{L}^p_\beta(\mathbb{D}) \subset L^p(\mathbb{D}, d\zeta) \text{ is compact} \Leftrightarrow \lim_{\epsilon\to 0} \sup_{\partial\mathbb{D}} \frac{\zeta(\rho(\xi,\epsilon))}{\epsilon^{\beta+2}} = 0. \tag{2.2}$$

We say that ζ is an β-Carleson measure if either side of (2.1) holds. Also, we say that ζ is a compact β-Carleson measure if either side of (2.2) holds.

The connection between composition operators and Carleson measures comes from the standard identity (see [7, P. 163])

$$\int_{\mathbb{D}} (h \circ \varphi)(z)(1 - |z|^2)^\beta dm(z) = \int_{\mathbb{D}} h(z)\; d\zeta(z) \tag{2.3}$$

valid for $\varphi \in \mathcal{S}$ and Borel functions $h \geq 0$ on $\mathbb{D}$. Here, $d\zeta$ denotes the pullback measure defined by

$$\zeta(E) = \int_{\varphi^{-1}(E)} (1 - |z|^2)^\beta dm(z)$$

for Borel sets $E \subset \mathbb{D}$. In particular, one can easily observe from (2.3) that $C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is (compact) bounded if and only if ζ is (compact) β-Carleson measure for $\mathcal{L}^p_\beta(\mathbb{D})$.

Assume $1 \leq p < \infty$ and $-1 < \beta$. Let φ be a self map on $\mathbb{D}$, and $\nu \in H(\mathbb{D})$. Denote by $Z_{\varphi,v}$ the number of zeros of $\varphi(z) - v$ on $\mathbb{D}$. Then $K_{\beta,\nu}(\varphi)$ on $\mathbb{D}$, is defined as follows:

$$K_{\beta,\nu}(\varphi, v) = \sum (1 - |z|^2)^\beta |\nu(z)|^p |\varphi'(z)|^{p-2},$$

when, repeated by multiplicity, the sum extends over the zeros of $\varphi - v$. In particular, for $v \notin \varphi(\mathbb{D})$, we have that $K_{\beta,\nu}(\varphi) = 0$. Also for $\nu = 1$, $\beta = 0$ and $p = 2$, we get $Z_{\varphi,v}$. Now define the measure $\zeta_{\beta,\nu}$ on $\mathbb{D}$, by $d\zeta_{\beta,\nu}(v) =: K_{\beta,\nu}(\varphi) dm(v)$.

Lemma 2.1. [8] Let $p \geq 1$, $\beta > -1$ and $i \in \mathbb{N}$. Then for any $h \in \mathcal{L}^p_\beta(\mathbb{D})$, we have that

$$|h^{(i)}(z)| \preceq \frac{\|h\|_{\mathcal{L}^p_\beta(\mathbb{D})}}{(1 - |z|^2)^{\frac{2+\beta}{p}+i}}.$$

Next lemma, will hepl us to provide easier conditions for consider the boundedness property of product type operators, on desired spaces.

Lemma 2.2. Suppose that $2+\beta<p<\infty$. Then for any $h\in\mathcal{E}^p_\beta(\mathbb{D})$, we get that

$$|h^{(i)}(z)|\preceq \|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}\quad i=0,1. \tag{2.4}$$

$$|h^{(i+2)}(z)|\preceq \frac{\|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}}{(1-|z|^2)^{\frac{2+\beta}{p}+i}},\quad i\geq 0. \tag{2.5}$$

Proof . For any $h\in\mathcal{E}^p_\beta(\mathbb{D})$, $h''\in\mathcal{L}^p_\beta(\mathbb{D})$, it follows from [24] that $|h''(z)|\preceq \frac{\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}}{(1-|z|^2)^{\frac{2+\beta}{p}}}$. Then for $p>2+\beta$, we have

$$|h'(z)-h'(0)|=\left|\int_0^z h''(w)dw\right|\preceq\int_0^1\frac{|z|\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}}{(1-|tz|^2)^{\frac{2+\beta}{p}}}dt\preceq\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}.$$

Thus

$$|h'(z)|\preceq\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}+|h'(0)| \tag{2.6}$$

It follows that $|h'(z)|\preceq\|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}$. On the other hand, (2.6) yields that

$$|h(z)-h(0)|\preceq\int_0^1|z||h'(tz)|dt\preceq\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}+|h'(0)|.$$

Therefore

$$|h(z)|\preceq\|h''\|_{\mathcal{L}^p_\beta(\mathbb{D})}+|h'(0)|+|h(0)|$$

and this gives us $|h(z)|\preceq\|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}$. For any $h\in\mathcal{E}^p_\beta(\mathbb{D})$, we have that $f''=f^{(2)}\in\mathcal{L}^p_\beta(\mathbb{D})$. So inequality (2.5) holds from Lemma 2.1. □

As a result of the above lemma, one can see that $\mathcal{E}^p_\beta(\mathbb{D})\subset H^\infty\subset\mathcal{B}^p_\beta(\mathbb{D})\subset\mathcal{L}^p_\beta(\mathbb{D})$. Also we obtain $\|h\|_\infty\preceq\|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}$ and $\|h'\|_\infty\preceq\|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}$.

Theorem 2.3. Suppose that $\beta>-1$, $p>\beta+2$, $\nu\in\mathcal{H}(\mathbb{D})$ and $\varphi\in\mathcal{S}(\mathbb{D})$. Then the following statements are equivalent:

(a) Operator $M_\nu C_\varphi: w\mathcal{E}^p_\beta(X)\to w\mathcal{L}^p_\beta(X)$ is bounded.
(b) Operator $M_\nu C_\varphi:\mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$ is bounded.
(c) $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$.

Proof . (a) ⇒ (b). Suppose that $M_\nu C_\varphi: w\mathcal{E}^p_\beta(X)\to w\mathcal{L}^p_\beta(X)$ be bounded and $h\in\mathcal{Z}^p_\beta(\mathbb{D})$. If $\delta\in X$ with $\|\delta\|=1$ and consider the function $g:\mathbb{D}\to X$, $g(z)=\delta h(z)$ for $z\in\mathbb{D}$ then we have

$$\begin{aligned}(\delta^*\circ g)'(z)=(\delta^*\circ\delta h)'(z)&=\lim_{w\to z}\frac{\delta^*(\delta h(w))-\delta^*(\delta h(z))}{w-z}\\&=\lim_{w\to z}\frac{h(w)\delta^*(\delta)-h(z)\delta^*(\delta)}{w-z}=h'(z)\delta^*(\delta).\end{aligned}$$

It follows that $(\delta^*\circ g)''(z)=(h'(z)\delta^*\delta)'(z)=h''(z)\delta^*(\delta)$. Then

$$\begin{aligned}\sup_{||\delta^*||_{X^*}\leq 1}\|\delta^*\circ g\|^p_{\mathcal{E}^p_\beta(\mathbb{D})}&=\sup_{||\delta^*||_{X^*}\leq 1}\int_{\mathbb{D}}(|(\delta^*\circ g)''(z)|^p(1-|z|^2)^\beta dm(z)+|(\delta^*\circ g)'(0)|+|(\delta^*\circ g)(0)|\\&=\sup_{||\delta^*||_{X^*}\leq 1}(\int_{\mathbb{D}}|h''(z)\delta^*(\delta)|^p(1-|z|^2)^\beta dm(z)+|\delta^*(\delta)h'(0)|+|\delta^*(\delta)h(0)|\\&=\int_{\mathbb{D}}|h''(z)|^p(1-|z|^2)^\beta dm(z)+|h'(0)|+|h(0)|=\|h\|^p_{\mathcal{Z}^p_\beta(\mathbb{D})}<\infty.\end{aligned}$$

Consequently, $\|g\|^p_{w\mathcal{E}^p_\beta(X)} = \sup_{||\delta^*||_{X^*}\le 1} \|\delta^*\circ g\|^p_{\mathcal{Z}^p_\beta(\mathbb{D})} = \|h\|^p_{\mathcal{E}^p_\beta(\mathbb{D})}$. This implies that $g\in w\mathcal{E}^p_\beta(X)$, therefore the boundedness of $M_\nu C_\varphi : w\mathcal{E}^p_\beta(X)\to w\mathcal{L}^p_\beta(X)$, gives us

$$||M_\nu C_\varphi g||_{w\mathcal{L}^p_\beta(X)} \preceq \|g\|_{w\mathcal{E}^p_\beta(X)} = \|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}. \tag{2.7}$$

In a similar way, we obtain

$$\begin{aligned}
\|M_\nu C_\varphi g\|^p_{w\mathcal{L}^p_\beta(X)} &= \sup_{||\delta^*||_{X^*}\le 1} \|\delta^*\circ M_\nu C_\varphi g\|^p_{\mathcal{L}^p_\beta(\mathbb{D})}\\
&= \sup_{||\delta^*||_{X^*}\le 1} (\int_{\mathbb{D}} |(\delta^*\nu C_\varphi g)(z)|^p(1-|z|^2)^\beta dm(z))\\
&= \sup_{||\delta^*||_{X^*}\le 1} (\int_{\mathbb{D}} |(\delta^*\nu C_\varphi(\delta h))(z)|^p(1-|z|^2)^\beta dm(z)\\
&= \sup_{||\delta^*||_{X^*}\le 1} (\int_{\mathbb{D}} |\delta^*(\delta)(\nu C_\varphi h)(z)|^p(1-|z|^2)^\beta dm(z)\\
&= \int_{\mathbb{D}} |(\nu C_\varphi h)(z)|^p(1-|z|^2)^\beta dm(z) = \|M_\nu C_\varphi h\|^p_{\mathcal{L}^p_\beta(\mathbb{D})}.
\end{aligned} \tag{2.8}$$

So, from (2.7) and (2.8) we get that $\|M_\nu C_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}$. Then $M_\nu C_\varphi : \mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(b)$\Rightarrow$ (c). Let $M_\nu C_\varphi : \mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$ be bounded. Then if we choose function $h=1$, we have that $\|\nu\|_{\mathcal{L}^p_\beta(\mathbb{D})}<\infty$. So $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$.

(c)$\Rightarrow$ (a). Suppose that $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$, then $\|\nu\|_{\mathcal{L}^p_\beta(\mathbb{D})}<\infty$ and for any $h\in\mathcal{E}^p_\beta(\mathbb{D})$, Lemma 2.2 gives us

$$||M_\nu C_\varphi h||_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_\infty\|\nu\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}. \tag{2.9}$$

But for $h\in w\mathcal{E}^p_\beta(X)$ and $\delta^*\in X^*$ such that $\|\delta^*\|\le 1$, we have that $\delta^*\circ h\in\mathcal{E}^p_\beta(\mathbb{D})$. Then (2.9) gives us

$$\|M_\nu C_\varphi(\delta^*\circ h)\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|\delta^*\circ h\|_{\mathcal{E}^p_\beta(\mathbb{D})} \preceq \sup_{||\delta^*||_{X^*}\le 1} \|\delta^*\circ h\|_{\mathcal{E}^p_\beta(\mathbb{D})} = \|h\|_{w\mathcal{E}^p_\beta(X)}.$$

Therefore,

$$\begin{aligned}
||M_\nu C_\varphi h||_{w\mathcal{L}^p_\beta(X)} &= \sup_{\|\delta^*||\le 1} \|\delta^*\circ(M_\nu C_\varphi(h)||_{\mathcal{L}^p_\beta(\mathbb{D})} = \sup_{||\delta^*||\le 1} ||M_\nu C_\varphi(\delta^*\circ h)||_{\mathcal{L}^p_\beta(\mathbb{D})}\\
&\preceq \|h\|_{w\mathcal{E}^p_\beta(X)}.
\end{aligned}$$

This completes the proof. □

Theorem 2.4. Suppose that $\beta>-1$, $p>\beta+2$, $\nu\in\mathcal{H}(\mathbb{D})$ and $\varphi\in\mathcal{S}(\mathbb{D})$. Then the following statements are equivalent:

(a) Operator $M_\nu C_\varphi D : w\mathcal{E}^p_\beta(X)\to w\mathcal{L}^p_\beta(X)$ is bounded.
(b) Operator $M_\nu C_\varphi D : \mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$ is bounded.
(c) $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$.

Proof . (a)$\Leftrightarrow$(b). It is similar to the proof of Theorem 2.3.

(b)$\Rightarrow$(c). If $M_\nu C_\varphi D : \mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$ be bounded, by choosing $h=z$, we get that $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$.

(c)$\Rightarrow$(b). Let $\nu\in\mathcal{L}^p_\beta(\mathbb{D})$, then by using Lemma 2.2, for any analytic function $h\in\mathcal{E}^p_\beta(\mathbb{D})$ we get that

$$\|M_\nu C_\varphi Dh\|_{\mathcal{L}^p_\beta(\mathbb{D})} = \|\nu(z)h'(\varphi(z))\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h'\|_\infty\|\nu\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_\beta(\mathbb{D})}.$$

This implies the boundedness of $M_\nu C_\varphi D : \mathcal{E}^p_\beta(\mathbb{D})\to\mathcal{L}^p_\beta(\mathbb{D})$. □

The following lemma comes from [8], which is vital to prove our main results.

Lemma 2.5. For any $\beta > -1$ and $p > 0$, we have that

$$\int_{\mathbb{D}} |h(z)|^p (1-|z|^2)^\beta dm(z) \preceq \left[|h(0)|^p + \int_{\mathbb{D}} |k(z)|^p (1-|z|^2)^\beta dm(z)\right]$$

and

$$|h(0)|^p + \int_{\mathbb{D}} |k(z)|^p (1-|z|^2)^\beta dm(z) \preceq \int_{\mathbb{D}} |h(z)|^p (1-|z|^2)^\beta dm(z)$$

for all analytic functions $h \in H(\mathbb{D})$, where

$$k(z) = (1-|z|^2)h'(z) \qquad z \in \mathbb{D}.$$

Remark 2.6. As a result of the above lemma, we can observe that $k \in \mathcal{L}^p_\beta(\mathbb{D})$ if and only if $k' \in \mathcal{L}^p_{\beta+p}(\mathbb{D})$.

Lemma 2.7. [5] Let $\varphi \in \mathcal{S}(\mathbb{D})$, $\nu \in \mathcal{H}(\mathbb{D})$, $-1 < \beta < \infty$ and $1 < p < \infty$. Then we have the following equivalent statements:

(a) $M_\nu C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(b) $\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\alpha+2)}} |\nu(z)|^p dm(z) < \infty$.

Theorem 2.8. Let $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then the following statements are equivalent:

(a) Operator $M_\nu DC_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(b) Operator $M_{\nu\varphi'} C_\varphi : \mathcal{B}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(c) Operator $M_{\nu\varphi'} C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(d) $\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}} |\nu\varphi'(z)|^p dm(z) < \infty$.

Proof .(a)$\Rightarrow$(b). If $M_\nu DC_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ be bounded, then for any $h \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$ we have that $\|\nu(z)\varphi'(z)h' \circ \varphi(z)\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}$. This implies that $\|M_{\nu\varphi'} C_\varphi Dh\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}$. Now for an arbitrary analytic function $g \in \mathcal{B}^p_{\beta+p}(\mathbb{D})$, such that $g(0) = 0$, define the function $k(z) := \int_0^z g(w)dw$. Then $k'(z) = g(z) \in \mathcal{B}^p_{\beta+p}(\mathbb{D})$ and $k(0) = k'(0) = 0$, so $\|M_{\nu\varphi'} C_\varphi Dk\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|k\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}$. It follows that $\|M_{\nu\varphi'} C_\varphi g\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|g\|_{\mathcal{B}^p_{\beta+p}(\mathbb{D})}$, therefore $M_{\nu\varphi'} C_\varphi : \mathcal{B}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(b)$\Rightarrow$(c). If $M_{\nu\varphi'} C_\varphi : \mathcal{B}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ be bounded, then by Lemma 2.5, for any $g \in \mathcal{B}^p_{\beta+p}(\mathbb{D})$,

$$\|M_{\nu\varphi'} C_\varphi g\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|g\|_{\mathcal{B}^p_{\beta+p}(\mathbb{D})} = \|g'\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} + |g(0)| \preceq \|g\|_{\mathcal{L}^p_\beta(\mathbb{D})}.$$

(c)$\Leftrightarrow$(d). It is clear according to Lemma 2.7.

(c)$\Rightarrow$(a). For an arbitrary $h \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$, we have that $h' \in \mathcal{B}^p_{\beta+p}(\mathbb{D})$, then $h'' \in \mathcal{L}^p_{\beta+p}(\mathbb{D})$, then by Lemma 2.5, $h' \in \mathcal{L}^p_\beta(\mathbb{D})$. So the boundedness of $M_{\nu\varphi'} C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$, and Lemma 2.5, give us

$$\|\nu\varphi' h' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h'\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h''\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} + |h'(0)| \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.$$

Therefore

$$\|M_\nu DC_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} = ||\nu\varphi' h' \circ \varphi||_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.$$

The proof is complete. □

The previous theorems help us to consider the boundedness of Stević-Sharma operator $T_{\nu_1,\nu_2,\varphi}$ from weak vector valued derivative Besov space into weak vector valued weighted Bergman space.

Theorem 2.9. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu_1, \nu_2 \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$.Then the following statements are equivalent:

(a) Operator $T_{\nu_1,\nu_2,\varphi} : w\mathcal{E}^p_\beta(X) \to w\mathcal{L}^p_\beta(X)$ is bounded.

(b) Operator $T_{\nu_1,\nu_2,\varphi} : \mathcal{E}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

(c) $\nu_1, \nu_2 \in \mathcal{L}^p_\beta(\mathbb{D})$.

Proof . (a)$\Leftrightarrow$ (b). It's similar to the proof of Theorem 2.3.

(b)$\Rightarrow$ (c). By choosing $h_1 = 1$, we get that

$$\|T_{\nu_1,\nu_2,\varphi}(h_1)\|_{\mathcal{E}^p_\beta(\mathbb{D})} = \|\nu_1\|_{\mathcal{L}^p_\beta(\mathbb{D})} < \infty. \tag{2.10}$$

Also by putting $h_2 = z$, we obtain

$$\|T_{\nu_1,\nu_2,\varphi}(h_2)\|_{\mathcal{L}^p_\beta(\mathbb{D})} = \|\nu_1\varphi + \nu_2\|_{\mathcal{L}^p_\beta(\mathbb{D})} < \infty. \tag{2.11}$$

But (2.10) gives us $\|\nu_1\varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|\varphi\|_\infty \|\nu_1\|_{\mathcal{L}^p_\beta(\mathbb{D})} < \infty$. Hence, by the triangle inequality, (2.10) and (2.11), we have that

$$\|\nu_2\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|\nu_2 \pm \nu_1\varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|\nu_2 + \nu_1\varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|\nu_1\varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} < \infty. \tag{2.12}$$

(c)$\Rightarrow$ (b). By an application of Theorems 2.3 and 2.4 and the triangle inequality, it is obvious. □

As an application of the above theorem, we obtain characterizations for the boundedness of all product type operators in (1.1) and (1.2) between the mentioned spaces.

Corollary 2.10. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then the following statements are equivalent:

(a) Operator $DM_\nu C_\varphi : w\mathcal{E}^p_\beta(X) \to w\mathcal{L}^p_\beta(X)$ is bounded.
(b) Operator $DM_\nu C_\varphi : \mathcal{E}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.
(c) $\nu', \nu\varphi' \in \mathcal{L}^p_\beta(\mathbb{D})$.

By using Theorems 2.3 and 2.8, we can characterize the boundedness of $M_\nu C_\varphi$ from derivative Besov space $\mathcal{E}^p_{\beta+p}(\mathbb{D})$ into Besov type space $\mathcal{B}^p_\beta(\mathbb{D})$.

Theorem 2.11. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then we have the following equivalent statements:

(a) The operator $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded
(b) $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and

$$\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|^2)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}} |\nu(z)\varphi'(z)|^p dm(z) < \infty. \tag{2.13}$$

(c) $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and operator $M_{\nu\varphi'} C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded.

Proof . (a)$\Rightarrow$(b). Let $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ be bounded, then for $h = 1$, we get that $||\nu||_{\mathcal{B}^p_\beta(\mathbb{D})} < \infty$. So $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$, and by using Theorem 2.3, for any $h \in \mathcal{Z}^p_{\beta+p}(\mathbb{D})$, we have that

$$\begin{aligned}
\|M_\nu DC_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} &\preceq \|M_\nu DC_\varphi h \pm M_{\nu'} C_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&\preceq \|M_{\nu'} C_\varphi h + M_\nu DC_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|M_{\nu'} C_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&\preceq \|M_\nu C_\varphi h\|_{\mathcal{B}^p_\beta(\mathbb{D})} + \|M_{\nu'} C_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.
\end{aligned}$$

Then the equivalent parts (a) and (c) of Theorem 2.8, completes the proof.

(b)$\Rightarrow$(a). Let $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$. Then according to Theorem 2.3, $M_{\nu'} C_\varphi : \mathcal{Z}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded. Also by assuming (2.13), and applying Theorem 2.8, we obtain the boundedness of $M_\nu DC_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$. In addition, we know that the point evalution map at $\varphi(0)$ on $\mathcal{E}^p_{\beta+p}(\mathbb{D})$ is a linear bounded functional, hence

$$\begin{aligned}
\|M_\nu C_\varphi h\|_{\mathcal{B}^p_\beta(\mathbb{D})} &= \|(\nu h \circ \varphi)'(z)\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)h(\varphi(0))| \\
&\preceq \|M_{\nu'} C_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|M_\nu DC_\varphi h\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)||h(\varphi(0))| \\
&\preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} + \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} + \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} \preceq \|k\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.
\end{aligned}$$

(b)$\Leftrightarrow$(c). According to Lemma 2.7, it is obvious. □

Theorem 2.12. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then operator $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded, if and only if $\nu \in \mathcal{B}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu} = \zeta_{\beta,\nu} \circ \varphi^{-1}$ be a $(\beta+p)$- Carleson measure.

Proof . Let $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded. Then by setting $h = z$, we have that $\nu \in \mathcal{B}^p_\beta(\mathbb{D})$. Now for any $g \in \mathcal{L}^p_{\beta+p}(\mathbb{D})$, define functions $l(z) = \int_0^z g(v)dv$ and $h(z) = \int_0^z l(v)dv$. Then $h(z) \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$ and $h'(0) = h(0) = 0$, so

$$\|M_\nu C_\varphi Dh\|_{\mathcal{B}^p_\beta(\mathbb{D})} = \|\nu' h' \circ \varphi + \nu\varphi' h'' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)h'(\varphi(0))| \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}. \tag{2.14}$$

Then by using the triangle inequality, (2.14), Lemma 2.2 and change of variable formula $v = \varphi(z)$, we get that

$$\begin{aligned}\int_{\mathbb{D}} |h''(v)|^p d\eta_{\beta,\nu}(v) &= \int_{\mathbb{D}} |h''(v)|^p K_{\beta,\nu}(\varphi) dm(v)\\ &= \int_{\mathbb{D}} |h''(\varphi(z)|^p |\nu(z)|^p |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha |\varphi'^2(z)| dm(z)\\ &= \|\nu\varphi' h'' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} = \|\nu\varphi' h'' \circ \varphi \pm \nu' h' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})}\\ &\preceq ||M_\nu C_\varphi Dh||_{\mathcal{B}^p_\beta(\mathbb{D})} + \|h'\|_\infty ||\nu'||_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}\\ &= \int_{\mathbb{D}} |h''(z)|^p (1-|z|^2)^{\beta+p} dm(z).\end{aligned} \tag{2.15}$$

It follows that

$$\int_{\mathbb{D}} |g(z)|^p d\eta_{\beta,\nu} \preceq \|g\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})}. \tag{2.16}$$

So, (2.1) completes the proof.

Conversely, let $g \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$, and $\eta_{\beta,\nu}$ be a measure of $(\beta+p)$-Carleson. Then $g'' \in \mathcal{L}^p_{\beta+p}(\mathbb{D})$ and from the details in (2.1), we get that

$$\|\nu\varphi' g'' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} = \int_{\mathbb{D}} |g''(z)^p d\eta_{\beta,\nu}(z) \preceq \|g''\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} \preceq \|g\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}. \tag{2.17}$$

So (2.14), (2.17), Lemma 2.2 and assumption $\nu \in \mathcal{B}^p_\beta(\mathbb{D})$, give us

$$\begin{aligned}\|M_\nu C_\varphi Dg\|_{\mathcal{B}^p_\beta(\mathbb{D})} &\preceq \|\nu' g' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|\nu\varphi' g'' \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)g'(\varphi(0))|\\ &\preceq \|g'\|_\infty \|\nu'\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|g\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} + |\nu(0)|\|g'\|_\infty\\ &\preceq \|g\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.\end{aligned}$$

□

Theorem 2.13. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu_1, \nu_2 \in \mathcal{H}(\mathbb{D})$, $\varphi \in \mathcal{S}(\mathbb{D})$ and also

$$\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}} |\nu_1 \varphi'(z)|^p dm(z) < \infty.$$

Then we have the following equivalent statements:

(a) Operator $T_{\nu_1,\nu_2,\varphi} : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{B}^p_\beta(X)$ is bounded.
(b) Operator $T_{\nu_1,\nu_2,\varphi} : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded.
(c) $\nu_1, \nu_2 \in \mathcal{B}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu_2} = \zeta_{\beta,\nu_2} \circ \varphi^{-1}$ be a $(\beta+p)$-Carleson measure.

Proof . (a)⇔(b). It is similar to the proof of Theorem 2.3.

(b)⇒(c). By choosing $h_1 = 1$, we obtain

$$||T_{\nu_1,\nu_2,\varphi}(h_1)||_{\mathcal{B}^p_\beta(\mathbb{D})} = ||\nu_1||_{\mathcal{B}^p_\beta(\mathbb{D})} < \infty.$$

Also we assumed that $\sup_{x\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|x|^2)^{\beta+2}(1-|z|^2)^{\beta}}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}}|\nu_1\varphi'(z)|^p dm(z) < \infty$, therefore Theorem 2.11, gives us the boundedness of $M_{\nu_1}C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$. Then with the boundedness of $T_{\nu_1,\nu_2,\varphi} : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$, for any analytic function $h \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$, we get that

$$\begin{aligned}\|M_\nu C_\varphi Dh\|_{\mathcal{B}^p_\beta(\mathbb{D})} &\preceq \|M_\nu C_\varphi Dh \pm M_\nu C_\varphi h\|_{\mathcal{B}^p_\beta(\mathbb{D})}\\ &\preceq \|T_{\nu_1,\nu_2,\varphi}h\|_{\mathcal{B}^p_\beta(\mathbb{D})} + \|M_\nu C_\varphi h\|_{\mathcal{B}^p_\beta(\mathbb{D})}\\ &\preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}.\end{aligned}$$

Therefore, the results follows from Theorem 2.12.

(c)⇒(a). By the hypothesis, Theorem 2.12 and Theorem 2.11, we have the boundedness of operators $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ and $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$. Then for any analytic function $h \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$ we get that $\|T_{\nu_1,\nu_2,\varphi}h\|_{\mathcal{E}^p_\beta(\mathbb{D}} \preceq \|M_\nu C_\varphi h\|_{\mathcal{E}^p_\beta(\mathbb{D})} + \|M_\nu C_\varphi Dh\|_{\mathcal{E}^p_\beta(\mathbb{D})} \preceq \|h\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}$. Hence the proof is complete. □

Remark 2.14. As an application of Theorem 2.13, one can investigate boundedness of the differences of the product-type operators from $w\mathcal{E}^p_{\beta+p}(X)$ into $w\mathcal{B}^p_\beta(X)$, as we have the following relations:

$$\begin{aligned}&M_{\nu_1}C_\varphi D - M_{\nu_2}DC_\varphi = T_{0,\nu_1-\nu_2\varphi',\varphi}, && M_{\nu_1}C_\varphi D - C_\varphi DM_{\nu_2} = T_{-\nu_2'\circ\varphi,\nu_1-\nu_2\circ\varphi,\varphi}\\ &M_{\nu_1}C_\varphi D - C_\varphi M_{\nu_2}D = T_{0,\nu_1-\nu_2\circ\varphi,\varphi}, && M_{\nu_1}DC_\varphi - C_\varphi M_{\nu_2}D = T_{0,\nu_1\varphi'-\nu_2\circ\varphi,\varphi},\\ &M_{\nu_1}C_\varphi D - DC_\varphi M_{\nu_2} = T_{-\varphi'\nu_2'\circ\varphi,\nu_1-\varphi'\nu_2\circ\varphi,\varphi}, && M_{\nu_1}C_\varphi D - DM_{\nu_2}C_\varphi = T_{-\nu_2',\nu_1-\nu_2\varphi',\varphi}.\end{aligned}$$

By using (1.2) and applying Theorem 2.13, we get the following result for the boundedness of operator weighted composition operator $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{E}^p_\beta(X)$.

Theorem 2.15. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$, $\varphi \in \mathcal{S}(\mathbb{D})$ and also

$$\sup_{x\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|x|^2)^{\beta+2}(1-|z|^2)^{\beta}}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}}|\nu'\varphi'(z)|^p dm(z) < \infty.$$

Then we have the following equivalent statements:

(a) Operator $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{E}^p_\beta(X)$ is bounded.
(b) Operator $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{E}^p_\beta(\mathbb{D})$ is bounded.
(c) Operator $DM_\nu C_\varphi = T_{\nu',\nu\varphi',\varphi} : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded.
(d) $\nu', \nu\varphi', \in \mathcal{B}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu\varphi'} = \zeta_{\beta,\nu\varphi'} \circ \varphi^{-1}$ be a $(\beta+p)$-Carleson measure.

3 Compactness of product-type operators from $w\mathcal{E}^p_\beta(X)$ into $w\mathcal{B}^p_\beta(X)$

In this section we are going to characterize the compactness of Stević-Sharma operator $T_{\nu_1,\nu_2,\varphi} : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{B}^p_\beta(X)$. Also as a result we have the compactness of operator weighted composition operator $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{E}^p_\beta(X)$.

Lemma 3.1. [4] Let $2 < p < \infty$, $-1 < \beta$, $\varphi \in \mathcal{S}(\mathbb{D})$ and $\nu \in \mathcal{H}(\mathbb{D})$. Suppose that $M_\nu C_\varphi$ is a bounded operator weighted composition operator on $\mathcal{L}^p_\beta$(*or* $\mathcal{B}^p_\beta$, *or* $\mathcal{Z}^p_\beta$). Then $M_\nu C_\varphi$ is compact on $\mathcal{L}^p_\beta$(*or* $\mathcal{B}^p_\beta$, *or* $\mathcal{E}^p_\beta$) if and only if for any bounded sequence $\{h_n\}_0^\infty$ in $\mathcal{L}^p_\beta$(*or* $\mathcal{B}^p_\beta$, *or* $\mathcal{E}^p_\beta$) such that $\{h_n\}_0^\infty \to 0$ uniformly on compact subsets on $\mathbb{D}$ as $n \to \infty$, we have $||M_\nu C_\varphi(h_n)||_{\mathcal{L}^p_\beta} \to 0$, $(||M_\nu C_\varphi(h_n)||_{\mathcal{B}^p_\beta} \to 0$, or $||M_\nu C_\varphi h_n||_{\mathcal{E}^p_\beta} \to 0)$.

Lemma 3.2. [5] Let $2 < p < \infty$, $\varphi \in \mathcal{S}(\mathbb{D})$ and $\nu \in \mathcal{H}(\mathbb{D})$. Then $M_\nu C_\varphi$ is compact on $\mathcal{L}^p_\beta$ if and only if

$$\lim_{|a|\to 1^-}\int_{\mathbb{D}}(\frac{1-|a|^2}{|1-\overline{a}\varphi(v)|^2})^2|\nu(v)|^p dm(v) = 0.$$

Theorem 3.3. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Then we have the following equivalent statements.

(a) Operator $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is compact.
(b) $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and $M_{\nu\varphi'}C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is compact.
(c) $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and $\lim_{|a|\to 1^-} \int_{\mathbb{D}} (\frac{1-|a|^2}{|1-\bar{a}\varphi(v)|^2})^2 |\nu(v)|^p dm(v) = 0$.

Proof . (a)⇒ (b). Let $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ be compact. Then $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded and according to Theorem 2.11, operators $M_\nu DC_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{L}_\beta(\mathbb{D})$ and $M_{\nu\varphi'}C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ are bounded. Now assume that the bounded sequence $\{h_n\}_{n=0}^\infty \subset \mathcal{L}^p_\beta(\mathbb{D})$ converges uniformly to 0, on compact subsets of $\mathbb{D}$. Then $\{h'_n\}_{n=0}^\infty \subset \mathcal{L}^p_{\beta+p}(\mathbb{D})$, so $\{h_n\}_{n=0}^\infty \subset \mathcal{B}^p_{\beta+p}(\mathbb{D})$. By Define $s_n(z) := \int_0^z h_n(t)dt$, we have that $s'_n \in \mathcal{B}^p_{\beta+p}(\mathbb{D})$, $s_n(0) = s'_n(0) = 0$ and $s_n \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$ and $\{s_n\}_{n=0}^\infty \to 0$ uniformly on compact subsets of $\mathbb{D}$. Therefore

$$\begin{aligned}
\|M_{\nu\varphi'}C_\varphi(h_n)\|_{\mathcal{L}^p_\beta(\mathbb{D})} &= \|\nu\varphi' h_n \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&= \|\nu\varphi' s'_n \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} = \|M_\nu DC_\varphi s_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&\preceq \|s_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} = \|s'_n\|_{\mathcal{B}^p_{\beta+p}(\mathbb{D})} + |s_n(0)| \\
&= \|h_n\|_{\mathcal{B}^p_{\beta+p}(\mathbb{D})} = \|h'_n\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} + |h_n(0)| \preceq \|h_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} \to 0.
\end{aligned}$$

This implies that $\|M_{\nu\varphi'}C_\varphi(h_n)\|_{\mathcal{L}^p_\beta(\mathbb{D})} \to 0$. Also by Theorem 2.11, we get $\|\nu\|_{\mathcal{B}^p_\beta(\mathbb{D})} < \infty$, and this completes the proof.

(b)⇔ (c). According to Lemma 3.2, it is clear.

(b) ⇒ (a). If $M_{\nu\varphi'}C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ be compact. Then $M_{\nu\varphi'}C_\varphi : \mathcal{L}^p_\beta(\mathbb{D}) \to \mathcal{L}^p_\beta(\mathbb{D})$ is bounded. Now let $\{h_n\}_{n=0}^\infty \subset \mathcal{E}^p_{\beta+p}(\mathbb{D})$ converges uniformly to 0, on compact subsets of $\mathbb{D}$. Then $\{h'_n\}_{n=0}^\infty \subset \mathcal{B}^p_{\beta+p}(\mathbb{D})$, and for any $n \in \mathbb{N}$, $h''_n \in \mathcal{L}^p_{\beta+p}(\mathbb{D})$ so by Lemma 2.5, $\{h'_n\}_{n=0}^\infty \subset \mathcal{L}^p_\beta(\mathbb{D})$ converges uniformly to 0, on compact subsets of $\mathbb{D}$. So

$$\begin{aligned}
\|M_\nu DC_\varphi h_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} &= \|\nu\varphi' h'_n \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&= \|M_{\nu\varphi'}C_\varphi h'_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|h'_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} \\
&\preceq ||h''_n||_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} + |h'(0)| \preceq \|h_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})}. \qquad (3.1)
\end{aligned}$$

Also we have assumed that $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$. So

$$||M_{\nu'}C_\varphi h_n||_{\mathcal{L}^p_\beta(\mathbb{D})} = \|\nu' h_n \circ \varphi\|_{\mathcal{L}^p_\beta(\mathbb{D})} \preceq \|\nu'\|_{\mathcal{L}^p_\beta(\mathbb{D})}\|h_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} \preceq \|h_n\|_{\mathcal{E}_{\beta+p}(\mathbb{D})}. \qquad (3.2)$$

On the other hand, standard estimates show that evaluation at $\varphi(0)$, is a linear bounded functional on $\mathcal{E}^p_{\beta+p}(\mathbb{D})$, hence

$$|\nu(0)h_n(\varphi(0))| \preceq \|h_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} \qquad (3.3)$$

Therefore (3.1), (3.2) and (3.3), give us

$$\begin{aligned}
||M_\nu C_\varphi h_n||_{\mathcal{B}^p_\beta(\mathbb{D})} &= \|M_{\nu'}C_\varphi h_n + M_\nu DC_\varphi h_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)h_n(\varphi(0)| \\
&\preceq \|M_{\nu'}C_\varphi h_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} + \|M_\nu DC_\varphi h_n\|_{\mathcal{L}^p_\beta(\mathbb{D})} + |\nu(0)h_n(\varphi(0)| \\
&\preceq \|h_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} \to 0.
\end{aligned}$$

This implies compactness of $M_\nu C_\varphi : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$. □

Theorem 3.4. Let Suppose that $\beta > -1$, $p > \beta + 2$, $\nu \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$.Then operator $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is compact if and only if $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu} = \zeta \circ \varphi^{-1}$ be a compact $(\beta + p)$-Carleson measure.

Proof . Let $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ be compact. Then $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded and by Theorem 2.12, $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu} = \zeta \circ \varphi^{-1}$ is a $(\beta + p)$-Carleson measure. Now if $\{g_n\}$ be a bounded sequence in $\mathcal{L}^p_{\beta+p}(\mathbb{D})$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then similar to the proof of Theorem 2.12, we can see that

$$\int_{\mathbb{D}} |g(z)|^p d\eta_{\beta,\nu} \preceq \|g\|_{\mathcal{L}^p_{\beta+p}(\mathbb{D})} \to 0.$$

So (2.2) yields that $\eta_{\beta,\nu} = \zeta \circ \varphi^{-1}$ is a compact $(\beta + p)$-Carleson measure.
For the converse part, assume that $\eta_{\beta,\nu} = \zeta \circ \varphi^{-1}$ is a compact $(\beta + p)$-Carleson measure and $\nu' \in \mathcal{L}^p_\beta(\mathbb{D})$. Then by Theorem 2.12, $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is bounded. Also similar to the details of Theorem 2.12 (proof of the converse part of this theorem), for any bounded sequence $(h_n)_{n=0}^\infty \in \mathcal{E}^p_{\beta+p}(\mathbb{D})$, that converges uniformly to 0 on compact subsets of $\mathbb{D}$, we get that $||M_\nu C_\varphi D h_n||_{\mathcal{B}^p_\beta(\mathbb{D})} \preceq \|h_n\|_{\mathcal{E}^p_{\beta+p}(\mathbb{D})} \to 0$. This gives us compactness of operator $M_\nu C_\varphi D : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$. □

Similar to the proof of the previous theorems and by using Theorem 3.3 and Theorem 3.4, we obtain the following equivalent statements for compactness of Stević-Sharma type operator $T_{\nu_1,\nu_2,\varphi} : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{B}^p_\beta(X)$.

Theorem 3.5. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu_1, \nu_2 \in \mathcal{H}(\mathbb{D})$, $\varphi \in \mathcal{S}(\mathbb{D})$. Let

$$\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}} |\nu_1\varphi'(z)|^p dm(z) \to 0.$$

Then we have the following equivalent statements:

(a) Operator $T_{\nu_1,\nu_2,\varphi} : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{B}^p_\beta(X)$ is compact.
(b) Operator $T_{\nu_1,\nu_2,\varphi} : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is compact.
(c) $\nu_1, \nu_2 \in \mathcal{B}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu_2} = \zeta \circ \varphi^{-1}$ be an $(\beta + p)$ compact Carleson measure.

As an application of Theorem 3.5 and by using (1.2), we obtain the following theorem.

Theorem 3.6. Suppose that $\beta > -1$, $p > \beta + 2$, $\nu, \in \mathcal{H}(\mathbb{D})$, $\varphi \in \mathcal{S}(\mathbb{D})$. Let

$$\sup_{x\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|x|^2)^{\beta+2}(1-|z|^2)^\beta}{|1-\overline{x}\varphi(z)|^{2(\beta+2)}} |\nu'\varphi'(z)|^p dm(z) \to 0.$$

Then we have the following equivalent statements:

(a) Operator $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{E}^p_\beta(X)$ is compact.
(b) Operator $M_\nu C_\varphi : w\mathcal{E}^p_{\beta+p}(X) \to w\mathcal{B}^p_\beta(X)$ is compact.
(c) Operator $DM_\nu C_\varphi = T_{\nu',\nu\varphi',\varphi} : \mathcal{E}^p_{\beta+p}(\mathbb{D}) \to \mathcal{B}^p_\beta(\mathbb{D})$ is compact.
(d) $\nu', \nu\varphi' \in \mathcal{B}^p_\beta(\mathbb{D})$ and $\eta_{\beta,\nu\varphi'} = \zeta \circ \varphi^{-1}$ be an $(\beta + p)$ compact Carleson measure.

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