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Relationship between nonsmooth vector optimization problem and vector variational inequalities using convexificators

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Abstract

In this article, we examine a nonsmooth vector optimization problem with locally Lipschitz approximately convex mappings in terms of the convexificator and provide some ideas for approximate effective solutions. Additionally, we define the relationship between the convexificator-based solutions of Stampacchia type vector variational inequalities (VVI) and the approximate efficient approximation convex function of nonsmooth vector optimization problems using the locally Lipschitz function. Furthermore, we provide a numerical example to demonstrate the veracity of our findings.

Keywords: Convexificator, nonsmooth vector optimization problem, approximate efficient solutions
2020 MSC: 90C46, 58E17, 49J40, 49J52

1 Introduction

In optimization theory, nonsmooth occurrences frequently occur, which has prompted the development of several subdifferentials and generalized directional derivative notions. A generalization of plenty of well subdifferentials, particularly Mordukhovich, Michel-Penot, and Clarke subdifferentials, is the idea of a convexificator. It has been demonstrated that the idea of convexificators is a helpful tool in the field of nonsmooth optimization. The concept of a convexificator was proposed by Demyanov [3] in the year 1994. Convexificators were recently employed by Golestani and Nobakhtian [6], Li and Zhang [15], Long and Huang [16], and Luu [17] to create the ideal circumstances for non-smooth optimization problems, see for example [5, 4, 11, 13, 14, 18, 23] and its sources for further details on convexificators.

It is sometimes computationally prohibitive or impractical to discover a precise solution in optimization theory, making approximation approaches vital. The challenges created by computational flaws and modelling constraints can thus be overcome with the use of approximate efficient solutions (AES). Utilizing approximate VVI of Minty and Stampacchia form in terms of the Clarke subdifferentials, Mishra and Laha [20] introduced the idea of AES for a vector optimization problem (VOP) using locally Lipschitz approximately convex functions. For further applications and literature on approximation, see [8, 7, 10, 20, 24, 25] and the references therein. In the last three decades, researchers have developed numerous definitions of convex function. In 2013, Bhatia et al. [1] developed four unique classes of approximate convex functions and presented optimality requirements for quasi efficient vector optimization solutions.

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Laha et al. [12] define Stampacchia and Minty VVI in terms of convexificators and use them to identify necessary and sufficient criteria for a point to be a vector minimum point of the VOP. Mishra and Upadhyay [21] and Upadhyay et al. [22] demonstrated links between nonsmooth VOP and VVI. Motivated and inspired by ongoing research work, we describe generalized approximate convex functions in terms of convexificators and demonstrate a relationship between nonsmooth VOP and VVI.

The rest sections of this paper are organized as follows: In the second section, we take a look at some key concepts and terminologies that will come up later on. In the third section, we develop an approximate Stampacchia and Minty type VVI in terms of the convexificators and use it to define an AES to the NVOP. A numerical illustration has also been provided to verify the reliability of the results has been presented in the fourth section.

2 Preliminaries

In this section, we will go over various concepts that pertain to nonsmooth analysis. For more details, see [2]. Let us assume that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^n_+$ is its nonnegative orthant, and $int\mathbb{R}^n_+$ is the positive orthant of $\mathbb{R}^n$. Let the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ signify the extended real line, and the notation $\langle ., . \rangle$ denote the Euclidean inner product. Further, we will assume that $\phi \neq D \subseteq \mathbb{R}^n$ that also contains the Euclidean norm $\|.\|$.

The convention for equality and inequalities is as follows:
If $\omega, \upsilon \in \mathbb{R}^n$, then
$\omega \geqq \upsilon \;\Leftrightarrow\; \omega_j \geq \upsilon_j,\; j = 1, 2, 3, ..., n \;\Leftrightarrow\; \omega - \upsilon \in \;\mathbb{R}^n_+$;
$\omega > \upsilon \;\Leftrightarrow\; \omega_j > \upsilon_j,\; j = 1, 2, 3, ..., n \;\Leftrightarrow\; \omega - \upsilon \in \; int\mathbb{R}^n_+$;
$\omega \geq \upsilon \;\Leftrightarrow\; \omega_j \geq \upsilon_j,\; j = 1, 2, 3, ..., n$, but $\omega \neq \upsilon \;\Leftrightarrow\; \omega - \upsilon \in \; int\mathbb{R}^n_+ \setminus \{0\}$.

First of all, we recall some definitions.

Definition 2.1. Suppose $\Gamma : D \to \overline{\mathbb{R}}$ is an extended real valued function, $\omega \in D$ and $\Gamma(\omega)$ is finite. Then the *lower and upper Dini derivatives* of Γ at $\omega \in D$ in the direction $\upsilon \in \mathbb{R}^n$, are denoted and defined as follows:

$$\Gamma^-(\omega, \upsilon) = \liminf_{\lambda \to 0} \frac{\Gamma(\omega + \lambda\upsilon) - \Gamma(\omega)}{\lambda},$$

$$\Gamma^+(\omega, \upsilon) = \limsup_{\lambda \to 0} \frac{\Gamma(\omega + \lambda\upsilon) - \Gamma(\omega)}{\lambda}.$$

Definition 2.2. [9] Suppose $\Gamma : D \to \overline{\mathbb{R}}$ is an extended real valued function, $\omega \in D$ and $\Gamma(\omega)$ is finite. Then Γ is said to be:

(i) *an upper convexificator* $\partial^*\Gamma(\omega) \subseteq \mathbb{R}^n$ at $\omega \in D$, if $\partial^*\Gamma(\omega)$ is closed and for every $\upsilon \in \mathbb{R}^n$, we have

$$\Gamma^-(\omega, \upsilon) \leq \sup_{\zeta \in \partial^*\Gamma(\omega)} \langle \zeta, \upsilon \rangle\,;$$

(ii) *a lower convexificator* $\partial_*\Gamma(\omega) \subseteq \mathbb{R}^n$ at $\omega \in D$, if $\partial_*\Gamma(\omega)$ is closed and for every $\upsilon \in \mathbb{R}^n$, we have

$$\Gamma^+(\omega, \upsilon) \geq \inf_{\zeta \in \partial_*\Gamma(\omega)} \langle \zeta, \upsilon \rangle\,;$$

(iii) *a convexificator* $\partial^*_*\Gamma(\omega) \subseteq \mathbb{R}^n$ at $\omega \in D$, if $\partial^*_*\Gamma(\omega)$ is both upper and lower convexificators of Γ at ω. That is, for every $\upsilon \in \mathbb{R}^n$, we have

$$\Gamma^-(\omega, \upsilon) \leq \sup_{\zeta \in \partial^*_*\Gamma(\omega)} \langle \zeta, \upsilon \rangle\,, \quad \Gamma^+(\omega, \upsilon) \geq \inf_{\zeta \in \partial^*_*\Gamma(\omega)} \langle \zeta, \upsilon \rangle\,.$$

We are capable of extending the definitions and characteristics discussed above to a locally Lipschitz vector-valued mapping $\Gamma : D \to \mathbb{R}^p$. We designate the components of Γ by $\Gamma_j,\; j \in K = \{1, 2, 3, ..., p\}$. The convexificator of Γ at $\omega \in D$ is the set

$$\partial^*_*\Gamma(\omega) = \partial^*_*\Gamma_1(\omega) \times \partial^*_*\Gamma_2(\omega) \times \partial^*_*\Gamma_3(\omega) \times ... \times \partial^*_*\Gamma_p(\omega).$$

The concepts of generalized approximate convexity were introduced by Bhatia et al. [1] in 2013. We define the ∂^*_*-approximate convex function, ∂^*_*-approximate pseudoconvex function of type I and II and ∂^*_*-approximate quasiconvex function of type I and II in terms of convexificator as follows:

Definition 2.3. Suppose $\Gamma : D \to \mathbb{R}$ is a locally Lipschitz function at $\omega_0 \in D$ and admits a bounded convexificator $\partial_*^*\Gamma(\omega_0)$ at ω_0. Then Γ is said to be:

(i) [11]∂_*^*-*approximate convex* at $\omega_0 \in D$, if for every $e > 0$, there is $\varrho > 0$ such that

$$\Gamma(\upsilon) - \Gamma(\omega) \geqq \langle \zeta, \upsilon - \omega \rangle - e\|\upsilon - \omega\|, \ \forall\, \zeta \in \partial_*^*\Gamma(\omega), \ \forall\, \omega, \upsilon \in \mathbb{B}(\omega_0, \varrho);$$

(ii) ∂_*^*-*approximate pseudoconvex of type I* at $\omega_0 \in D$, if for every $e > 0$, there is $\varrho > 0$ such that, whenever $\omega, \upsilon \in \mathbb{B}(\omega_0, \varrho)$ and if

$$\langle \zeta, \upsilon - \omega \rangle \geqq 0, \ \text{for some } \zeta \in \partial_*^*\Gamma(\omega),$$

then

$$\Gamma(\upsilon) - \Gamma(\omega) \geqq -e\|\upsilon - \omega\|;$$

(iii) ∂_*^*-*approximate pseudoconvex of type II* (∂_*^*-*strictly approximate pseudoconvex of type II*) at $\omega_0 \in D$, if for every $e > 0$, there is $\varrho > 0$ such that, whenever $\omega, \upsilon \in \mathbb{B}(\omega_0, \varrho)$ and if

$$\langle \zeta, \upsilon - \omega \rangle + e\|\upsilon - \omega\| \geqq 0, \ \text{for some } \zeta \in \partial_*^*\Gamma(\omega),$$

then

$$\Gamma(\upsilon) - \Gamma(\omega) \geqq (>)0;$$

(iv) ∂_*^*-*approximate quasiconvex of type I* at $\omega_0 \in D$, if for every $e > 0$, there is $\varrho > 0$ such that, whenever $\omega, \upsilon \in \mathbb{B}(\omega_0, \varrho)$ and if

$$\Gamma(\upsilon) - \Gamma(\omega) \leqq 0,$$

then

$$\langle \zeta, \upsilon - \omega \rangle - e\|\upsilon - \omega\| \leqq 0, \ \text{for every } \zeta \in \partial_*^*\Gamma(\omega);$$

(v) ∂_*^*-*approximate quasiconvex of type II* (∂_*^*-*strictly approximate quasiconvex of type II*) at $\omega_0 \in D$, if for every $e > 0$, there is $\varrho > 0$ such that, whenever $\omega, \upsilon \in \mathbb{B}(\omega_0, \varrho)$ and if

$$\Gamma(\upsilon) - \Gamma(\omega) \leqq (<)e\|\upsilon - \omega\|,$$

then

$$\langle \zeta, \upsilon - \omega \rangle \leqq 0, \ \text{for every } \zeta \in \partial_*^*\Gamma(\omega).$$

Remark 2.4. If we assume $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)$ and $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p}$, $\epsilon > 0$ the concepts of generalized approximate convexity might be generalized to the vector case.

3 Approximate Minty and Stampacchia vector variational Inequality

Consider the following nonsmooth vector optimization problem (for short, $NVOP$)

$$\text{Min } \Gamma(\omega) = (\Gamma_1(\omega), \Gamma_2(\omega), ..., \Gamma_p(\omega)) \text{ so that } \omega \in D,$$

where $\Gamma_j : D \to \mathbb{R}, \ j = 1, 2, 3, ..., p$ are non-differentiable locally Lipschitz functions on D.

The following notions of approximation efficient solutions were presented by Mishra et al. [19]. When an efficient solution cannot be demonstrated, these notions are useful.

Definition 3.1. Let $\Gamma : D \to \mathbb{R}^p$ be a function. A vector $\omega_0 \in D$ is said to be:

(i) AES of the $NVOP$ of type I (in short, $(AES)_1$), if for $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p}$, $\epsilon > 0$ as small as possible, there is no $\varrho > 0$ such that

$$\Gamma(\omega) - \Gamma(\omega_0) \leq e\|\omega - \omega_0\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho) \setminus \{\omega_0\};$$

(ii) AES of the $NVOP$ of type II (in short, $(AES)_2$), if for $\epsilon > 0$ as small as possible, there is $\varrho > 0$ such that

$$\Gamma(\omega) - \Gamma(\omega_0) \nleq e\|\omega - \omega_0\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho);$$

(iii) AES of the $NVOP$ of type III (in short, $(AES)_3$), if for $\epsilon > 0$ as small as possible, there is no $\varrho > 0$ such that

$$\Gamma(\omega) - \Gamma(\omega_0) \nleq -e\|\omega - \omega_0\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho).$$

The following VVI problems of Minty type in terms of convexificators have been utilized in the sequel to describe an AES of the $NVOP$ in the next section.

$(AMVVI)_1$: Find $\omega_0 \in D$ so that, for any $\epsilon > 0$ as small as possible, there is no $\varrho > 0$ such that

$$\langle \zeta, \omega - \omega_0) \rangle \leq e\|\omega - \omega_0)\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho) \setminus \{\omega_0\}, \ \zeta \in \partial^*_*\Gamma(\omega);$$

$(AMVVI)_2$: Find $\omega_0 \in D$ so that, for any $\epsilon > 0$ as small as possible, there is $\varrho > 0$ such that

$$\langle \zeta, \omega - \omega_0) \rangle \nleq e\|\omega - \omega_0)\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho), \ \zeta \in \partial^*_*\Gamma(\omega);$$

$(AMVVI)_3$: Find $\omega_0 \in D$ so that, for any $\epsilon > 0$, there is $\varrho > 0$ such that

$$\langle \zeta, \omega - \omega_0) \rangle \nleq -e\|\omega - \omega_0)\|, \ \forall\, \omega \in \mathbb{B}(\omega_0, \varrho), \ \zeta \in \partial^*_*\Gamma(\omega).$$

The following theorem gives the conditions under which an AES of the $NVOP$ is a solution of $AMVVI$.

Theorem 3.2. Let $\Gamma : D \to \mathbb{R}^p$ be a locally Lipschitz function on D, which permits a bounded convexificator $\partial^*_*\Gamma(\omega_0)$ at $\omega_0 \in D$. Then

(i) if Γ is ∂^*_*-approximate pseudoconvex of type II at $\omega_0 \in D$ and ω_0 is an $(AES)_1$ of the $NVOP$, then ω_0 is also a solution of the $(AMVVI)_1$;

(ii) if Γ is ∂^*_*-approximate pseudoconvex of type II at $\omega_0 \in D$ and ω_0 is an $(AES)_2$ of the $NVOP$, then ω_0 is also a solution of the $(AMVVI)_2$;

(iii) if Γ is ∂^*_*-strictly approximate pseudoconvex of type II at $\omega_0 \in D$ and ω_0 is an $(AES)_3$ of the $NVOP$, then ω_0 is also a solution of the $(AMVVI)_3$.

Proof .(i) Suppose that ω_0 is not a solution of the $(AMVVI)_1$. Then, for some $\epsilon > 0$ as small as possible, there is $\tilde{\varrho} > 0$ such that

$$\langle \zeta, \omega - \omega_0 \rangle \leq e\|\omega - \omega_0\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \tilde{\varrho}) \text{ and } \zeta \in \partial^*_*\Gamma(\omega),$$

where $e = \underbrace{(\epsilon, \epsilon, \dots, \epsilon)}_{p} \in \mathrm{int}\mathbb{R}^p_+$. It can be expressed as follows:

$$\langle \zeta, \omega_0 - \omega \rangle + e\|\omega - \omega_0\| \geq 0. \tag{3.1}$$

Since Γ is a ∂^*_*-approximate pseudoconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\langle \zeta, \omega_0 - \omega \rangle + e\|\omega - \omega_0\| \geqq 0, \text{ for some } \zeta \in \partial^*_*\Gamma(\omega),$$

then

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq 0.$$

Using (3.1) and the hypothesis of ∂^*_*-approximate pseudoconvex function of type II, and taking $\hat{\varrho} = \min\{\tilde{\varrho}, \breve{\varrho}\}$, we have

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq 0 < e\|\omega - \omega_0)\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}) \text{ and } \zeta \in \partial^*_*\Gamma(\omega),$$

which is a contradiction that ω_0 is an $(AES)_1$ of $NVOP$.

(ii) Suppose that ω_0 is not a solution of the $(AMVVI)_2$. Then, for some $\epsilon > 0$ as small as possible and for every $\tilde{\varrho} > 0$, there is $\omega \in \mathbb{B}(\omega_0, \tilde{\varrho})$ and $\zeta \in \partial^*_*\Gamma(\omega)$ such that

$$\langle \zeta, \omega - \omega_0)\rangle \leq e\|\omega - \omega_0)\|,$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$. It can be expressed as follows:

$$\langle \zeta, \omega_0 - \omega\rangle + e\|\omega - \omega_0\| \geq 0. \tag{3.2}$$

Since Γ is a ∂^*_*-approximate pseudoconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\langle \zeta, \omega_0 - \omega\rangle + e\|\omega - \omega_0\| \geqq 0, \text{ for some } \zeta \in \partial^*_*\Gamma(\omega),$$

then

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq 0.$$

Using (3.2) and the hypothesis of ∂^*_*-approximate pseudoconvex function of type II, and taking $\hat{\varrho} = \min\{\tilde{\varrho}, \breve{\varrho}\}$, we have

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq 0 < e\|\omega - \omega_0)\|, \text{ for some } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}) \text{ and } \zeta \in \partial^*_*\Gamma(\omega),$$

which is a contradiction that ω_0 is an $(AES)_2$ of $NVOP$.

(iii) Suppose that ω_0 is not a solution of the $(AMVVI)_3$. Then, for some $\epsilon > 0$ and for every $\tilde{\varrho} > 0$, we have

$$\langle \zeta, \omega - \omega_0)\rangle \leq -e\|\omega - \omega_0)\| < e\|\omega - \omega_0)\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \tilde{\varrho}) \text{ and } \zeta \in \partial^*_*\Gamma(\omega),$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$. It can be expressed as follows:

$$\langle \zeta, \omega_0 - \omega\rangle + e\|\omega - \omega_0\| \geq 0. \tag{3.3}$$

Since Γ is a ∂^*_*-strictly approximate pseudoconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\langle \zeta, \omega_0 - \omega\rangle + e\|\omega - \omega_0\| \geq 0, \text{ for some } \zeta \in \partial^*_*\Gamma(\omega),$$

then

$$\Gamma(\omega) - \Gamma(\omega_0) < 0.$$

Using (3.3) and the hypothesis of ∂^*_*-strictly approximate pseudoconvex function of type II, and taking $\hat{\varrho} = \min\{\tilde{\varrho}, \breve{\varrho}\}$, we have

$$\Gamma(\omega) - \Gamma(\omega_0) < 0, \text{ for every } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}) \text{ and } \zeta \in \partial^*_*\Gamma(\omega).$$

This implies that there is $\epsilon > 0$ as small as possible such that

$$\Gamma(\omega) - \Gamma(\omega_0) \leq -e\|\omega - \omega_0)\|,$$

which is a contradiction that ω_0 is $(AES)_3$ of $NVOP$. $\Box$

Now, we consider approximation of the Stampacchia VVI problems by expressing them about convexificators.

$(ASVVI)_1$ Find $\omega_0 \in D$ so that, for any $\epsilon > 0$ as small as possible, there are $\omega \in D \setminus \{\omega_0\}$ and $\zeta_0 \in \partial^*_*\Gamma(\omega_0)$ such that

$$\langle \zeta_0, \omega - \omega_0)\rangle \nleq e\|\omega - \omega_0)\|;$$

$(ASVVI)_2$ Find $\omega_0 \in D$ so that, for any $\epsilon > 0$ as small as possible, for every $\omega \in D$ and $\zeta_0 \in \partial^*_*\Gamma(\omega_0)$ satisfying

$$\langle \zeta_0, \omega - \omega_0)\rangle \nleq e\|\omega - \omega_0)\|;$$

$(ASVVI)_3$ Find $\omega_0 \in D$ so that, for any $\epsilon > 0$, there is $\varrho > 0$ such that

$$\langle \zeta_0, \omega - \omega_0)\rangle \nleq -e\|\omega - \omega_0)\| \text{ for every } \omega \in \mathbb{B}(\omega_0, \varrho),\ \zeta_0 \in \partial^*_*\Gamma(\omega_0).$$

Theorem 3.3. Let $f : D \to \mathbb{R}^p$ be a locally Lipschitz function on D, which permits a bounded convexificator $\partial_*^*\Gamma(\omega_0)$ at $\omega_0 \in D$. Then

(i) if Γ is ∂_*^*-approximate quasiconvex of type II at $\omega_0 \in D$ and ω_0 is a solution of $(ASVVI)_1$, then ω_0 is also $(AES)_1$ of the $NVOP$;

(ii) if Γ is ∂_*^*-approximate quasiconvex of type II at $\omega_0 \in D$ and ω_0 is a solution of $(ASVVI)_2$, then ω_0 is also $(AES)_2$ of the $NVOP$;

(iii) if Γ is ∂_*^*-approximate pseudoconvex of type II at $\omega_0 \in D$ and ω_0 is a solution of $(ASVVI)_3$, then ω_0 is also $(AES)_3$ of the $NVOP$.

Proof .(i) Suppose that ω_0 is not an $(AES)_1$ of the $NVOP$. Then, for some $\epsilon > 0$ as small as possible, there is $\tilde{\varrho} > 0$ such that

$$\Gamma(\omega) - \Gamma(\omega_0) \leq e\|\omega - \omega_0)\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \tilde{\varrho}), \ \omega \neq \omega_0, \tag{3.4}$$

where $e = \underbrace{(\epsilon, \epsilon, \dots, \epsilon)}_{p} \in \text{int } \mathbb{R}^p_+$.

Since Γ is a ∂_*^*-approximate quasiconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq e\|\omega - \omega_0\|,$$

then

$$\langle \zeta_0, \omega - \omega_0) \rangle \leqq 0.$$

Using (3.4) and the hypothesis of ∂_*^*-approximate quasiconvex function of type II, and taking $\hat{\varrho} = \min\{\tilde{\varrho}, \breve{\varrho}\}$, we have

$$\langle \zeta_0, \omega - \omega_0) \rangle \leqq 0, \text{ for every } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}), \ \omega \neq \omega_0 \text{ and } \zeta_0 \in \partial_*^*\Gamma(\omega_0).$$

This implies, for $\epsilon > 0$,

$$\langle \zeta_0, \omega - \omega_0) \rangle \leqq 0 < e\|\omega - \omega_0)\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}), \ \omega \neq \omega_0 \text{ and } \zeta_0 \in \partial_*^*\Gamma(\omega_0),$$

which is a contradiction that ω_0 is a solution of $(ASVVI)_1$.

(ii) Suppose that ω_0 is a solution of the $(ASVVI)_2$. Then, for any $\epsilon > 0$ as small as possible, for every $\omega \in D$ and $\zeta_0 \in \partial_*^*\Gamma(\omega_0)$, we have

$$\langle \zeta_0, \omega - \omega_0) \rangle \nleq e\|\omega - \omega_0)\|,$$

which implies

$$\langle \zeta_0, \omega - \omega_0) \rangle \nleq 0. \tag{3.5}$$

Since Γ is a ∂_*^*-approximate quasiconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq e\|\omega - \omega_0)\|,$$

then

$$\langle \zeta_0, \omega - \omega_0) \rangle \leqq 0.$$

Using (3.5) and the hypothesis of ∂_*^*-approximate quasiconvex function of type II, it means for ω sufficiently close to ω_0, we have

$$\Gamma(\omega) - \Gamma(\omega_0) \nleq e\|\omega - \omega_0)\|, \text{ for every } \omega \in \mathbb{B}(\omega_0, \breve{\varrho}) \text{ and } \omega \neq \omega_0.$$

Hence ω_0 is an $(AES)_2$ of the $NVOP$.

(iii) Suppose that ω_0 is not an $(AES)_3$ of the $NVOP$. Then, for some $\epsilon > 0$, and for every $\tilde{\varrho} > 0$, there is $\omega \in \mathbb{B}(\omega_0, \tilde{\varrho})$ such that

$$\Gamma(\omega) - \Gamma(\omega_0) \leq -e\|\omega - \omega_0)\| < 0, \tag{3.6}$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int } \mathbb{R}^p_+$.

Since Γ is a ∂^*_*-approximate pseudoconvex function of type II at $\omega_0 \in D$, it means for every $\epsilon > 0$, there is $\breve{\varrho} > 0$ such that, whenever $\omega, \omega_0 \in \mathbb{B}(\omega_0, \breve{\varrho})$ and if

$$\langle\zeta_0, \omega - \omega_0)\rangle + e\|\omega - \omega_0)\| \geqq 0, \text{ for some } \zeta_0 \in \partial^*_*\Gamma(\omega_0),$$

then

$$\Gamma(\omega) - \Gamma(\omega_0) \leqq 0.$$

Using (3.6) and the hypothesis of ∂^*_*-approximate pseudoconvexity of type II, and taking $\hat{\varrho} = \min\{\tilde{\varrho}, \breve{\varrho}\}$, we have

$$\langle\zeta_0, \omega - \omega_0)\rangle < -e\|\omega - \omega_0)\|, \text{ for some } \omega \in \mathbb{B}(\omega_0, \hat{\varrho}) \text{ and for every } \zeta_0 \in \partial^*_*\Gamma(\omega_0),$$

which is a contradiction that ω_0 is a solution of $(ASVVI)_3$.

□

4 Numerical Example

Following is an illustration that demonstrates the applicability of the main results:

Example 4.1. Consider the $NVOP$ as follows:

$$\min \Gamma(\omega) = (\Gamma_1(\omega), \Gamma_2(\omega)), \text{ subject to } \omega \in \mathbb{R},$$

where

$$\Gamma_1(\omega) = \begin{cases} 3\omega + 1, & \text{if } \omega \geqq 0; \\ 2\omega - e^{\omega}, & \text{if } \omega < 0; \end{cases}$$

and

$$\Gamma_2(\omega) = \begin{cases} 2\omega^3 + \omega, & \text{if } \omega \geqq 0; \\ 2\omega, & \text{if } \omega < 0. \end{cases}$$

The convexificators of Γ_1 and Γ_2 at ω are defined as follows:

$$\partial^*_*\Gamma_1(\omega) = \begin{cases} 3, & \text{if } \omega > 0; \\ [1, 3], & \text{if } \omega = 0; \\ 2 - e^{\omega}, & \text{if } \omega < 0; \end{cases}$$

and

$$\partial^*_*\Gamma_2(\omega) = \begin{cases} 6\omega^2 + 1, & \text{if } \omega > 0; \\ [1, 2], & \text{if } \omega = 0; \\ 2, & \text{if } \omega < 0. \end{cases}$$

Suppose $e = (\epsilon, \epsilon)$, for $\epsilon > 0$ and take $\varrho = min(1, \frac{\epsilon}{3})$ so that for every $\omega, \upsilon \in \mathbb{B}(0, \varrho)$, $\zeta_1 \in \partial^*_*\Gamma_1(\omega)$, $\zeta_1 \in \partial^*_*\Gamma_2(\omega)$, we have

$$\langle\zeta_1, \upsilon - \omega\rangle + \epsilon\|\upsilon - \omega\| = \begin{cases} 3(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| > 0, & \text{if } \omega > 0, \upsilon > 0, \upsilon - \omega > 0; \\ 3(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| < 0, & \text{if } \omega > 0, \upsilon > 0, \upsilon - \omega < 0; \\ 3(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| < 0, & \text{if } \omega < 0, \upsilon \leqq 0; \\ (2 - e^{\omega})(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| > 0, & \text{if } \omega > 0, \upsilon \geqq 0; \\ (2 - e^{\omega})(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| > 0, & \text{if } \omega < 0, \upsilon < 0, \upsilon - \omega > 0; \\ (2 - e^{\omega})(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| < 0, & \text{if } \omega < 0, \upsilon < 0, \upsilon - \omega < 0; \\ r_1(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| > 0, & \text{if } \upsilon = 0, \omega > 0, r_1 \in [1, 3]; \\ r_1(\upsilon - \omega) + \epsilon\|\upsilon - \omega\| < 0, & \text{if } \upsilon = 0, \omega < 0, r_1 \in [1, 3]; \end{cases}$$

and

$$\langle \zeta_2, \upsilon-\omega\rangle + \epsilon\|\upsilon-\omega\| = \begin{cases} (6\omega^2+1)(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| > 0, & \text{if } \omega>0, \upsilon>0, \upsilon-\omega>0;\\ (6\omega^2+1)(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| < 0, & \text{if } \omega>0, \upsilon>0, \upsilon-\omega<0;\\ (6\omega^2+1)(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| < 0, & \text{if } \omega<0, \upsilon\leqq 0;\\ 2(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| > 0, & \text{if } \omega>0, \upsilon\geqq 0;\\ 2(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| > 0, & \text{if } \omega<0, \upsilon<0, \upsilon-\omega>0;\\ 2(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| < 0, & \text{if } \omega<0, \upsilon<0, \upsilon-\omega<0;\\ r_2(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| > 0, & \text{if } \upsilon=0, \omega>0, r_2\in[1,2];\\ r_2(\upsilon-\omega)+\epsilon\|\upsilon-\omega\| < 0, & \text{if } \upsilon=0, \omega<0, r_2\in[1,2]. \end{cases}$$

Also,

$$\begin{aligned} \Gamma_1(\upsilon)-\Gamma_1(\omega) &= \begin{cases} 3(\upsilon-\omega), & \text{if } \omega>0, \upsilon>0, \upsilon-\omega>0;\\ 3\upsilon-2\omega+1+e^{\omega}, & \text{if } \omega<0, \upsilon\geqq 0;\\ 2(\upsilon-\omega)+e^{\omega}-e^{\upsilon}, & \text{if } \omega<0, \upsilon<0, \upsilon-\omega>0;\\ 3\upsilon+1, & \text{if } \omega=0, \upsilon>0; \end{cases}\\ &> 0, \end{aligned}$$

and

$$\begin{aligned} \Gamma_2(\upsilon)-\Gamma_2(\omega) &= \begin{cases} (\upsilon-\omega)(2\upsilon^2+2\omega^2+2\omega\upsilon+1), & \text{if } \omega>0, \upsilon>0, \upsilon-\omega>0;\\ 2\upsilon^3+\upsilon-2\omega, & \text{if } \omega<0, \upsilon\geqq 0;\\ 2(\upsilon-\omega), & \text{if } \omega<0, \upsilon<0, \upsilon-\omega>0;\\ 2\upsilon^3+\upsilon, & \text{if } \omega=0, \upsilon>0; \end{cases}\\ &\geqq 0. \end{aligned}$$

Hence $\Gamma=(\Gamma_1,\Gamma_2)$ is ∂^*_*-approximate pseudoconvex of type II at $\omega_0=0$.

So, for any $\omega\in\mathbb{B}(\omega_0,\varrho)$, if $\omega>0$,

$$\langle \zeta_{0_1}, \omega-\omega_0)\rangle + \epsilon\|\omega-\omega_0)\| = r_1\omega+\epsilon\|\omega\| > 0,\ r_1\in[1,3],$$

and

$$\langle \zeta_{0_1}, \omega-\omega_0)\rangle + \epsilon\|\omega-\omega_0)\| = r_2\omega+\epsilon\|\omega\| > 0,\ r_2\in[1,2].$$

That is, $\langle \zeta_0, \omega-\omega_0)\rangle + e\|\omega-\omega_0)\| \nleq 0$. Hence $\omega_0=0$ is a solution of $(ASVVI)_3$. Thus, for any $\epsilon>0$, there is $\varrho>0$, such that for all $\omega>0$, $\omega\in\mathbb{B}(\omega_0,\varrho)$, we have

$$\Gamma_1(\omega)-\Gamma_1(\omega_0)+\epsilon\|\omega-\omega_0)\| = 3\omega+1+\epsilon\|\omega\| > 0,$$

and

$$\Gamma_2(\omega)-\Gamma_2(\omega_0)+\epsilon\|\omega-\omega_0)\| = 2\omega^3+\omega+\epsilon\|\omega\| > 0.$$

That is, $\Gamma(\omega)-\Gamma(\omega_0)+e\|\omega-\omega_0)\| \nleq 0$. Hence $\omega_0=0$ is an $(AES)_3$ of the $NVOP$. Thus, Theorem 3.3 is verified. So, for any $\omega\in\mathbb{B}(\omega_0,\varrho)$, if $\omega>0$,

$$\langle \zeta_1, \omega-\omega_0)\rangle + \epsilon\|\omega-\omega_0)\| = 3\omega+\epsilon\|\omega\| > 0,$$

and

$$\langle \zeta_2, \omega_0-\omega\rangle + \epsilon\|\omega-\omega_0)\| = 6\omega^3+\omega+\epsilon\|\omega\| > 0.$$

That is, $\langle \zeta, \omega-\omega_0)\rangle + e\|\omega-\omega_0)\| \nleq 0$. Hence $\omega_0=0$ is a solution of $(AMVVI)_3$. Thus, Theorem 3.2 is verified.

Remark 4.2. In the above mentioned example

(i) Γ_1,Γ_2 are ∂^*_*-approximate pseudoconvex of type I and II at $\omega_0=0$ (approximate quasiconvex of type I and type II at $\omega_0=0$) for $e=(\epsilon,\epsilon)$, $\epsilon>0$, but if we assume $\upsilon<0$ and $\omega=0$, then the inequality of ∂^*_*-approximate convexity does not hold. Thus generalized approximate convex functions are now valuable as a consequence of this research.

(ii) the convexificators of Γ_1 and Γ_2 are rigidly confined in the Clarke or Michel-Penot subdifferentials. Convexity with convexificators is easier than that of other subdifferentials. Thus our results are easy to use.

5 Conclusion

In this paper, we defined the AES, $AMVVI$ and $ASVVI$ in terms of convexificators. Furthermore by utilizing ∂_*^*-approximate pseudoconvex function of type II and ∂_*^*-approximate quasiconvex function of type II, we have established the relationships between the AES of the Minty and Stampacchia VVI and the $NVOP$.

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