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# On coefficients of a new Ma-Minda type class connected to binomial distribution

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**Abstract**

In this paper, we define a new Ma-Minda type class based on binomial distribution series. Our investigation will be focused on the coefficients of the function $f$ belonging to that class.

Keywords: Coefficients, Convolution or Hadamard product, Ma-Minda type class, Subordination, Binomial distribution
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## 1 Introduction

Throughout the paper, we denote by $\mathbb{D}$ the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Denote also $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. The family of all analytic and normalized functions in $\mathbb{D}$ will be denoted by $\mathcal{A}$. It is well-known that each function $f$ in the class $\mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, and $f$ be defined by (1.1). Then, their Hadamard product is defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ is said to be univalent in $\mathbb{D}$, if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$. We denote by $\mathcal{S}$ the class of all univalent functions in $\mathbb{D}$. For two functions $f$ and $g$ belong to $\mathcal{A}$, we say that $f$ is subordinate to $g$, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w : \mathbb{D} \to \overline{\mathbb{D}}$, $w(z) = w_1 z + w_2 z^2 + \cdots$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. If $g$ is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

A function $f \in \mathcal{A}$ is starlike in $\mathbb{D}$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

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We denote by $\mathcal{S}^*$ the class of all starlike functions in $\mathbb{D}$. Also, a function $f \in \mathcal{A}$ is convex in the unit disk $\mathbb{D}$, if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0, \quad (z\in\mathbb{D}).$$

The class of all convex functions in $\mathbb{D}$ is denoted by $\mathcal{K}$. It is well-known that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$, see [3]. There is an other way to define the class of starlike and convex functions. Indeed, applying the subordination relation gives

$$f\in\mathcal{S}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad (z\in\mathbb{D}),$$

and

$$f\in\mathcal{K} \Leftrightarrow 1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \quad (z\in\mathbb{D}).$$

It is easy to see that $(1+z)/(1-z)$ is a univalent function in $\mathbb{D}$ and maps $\mathbb{D}$ onto the right-half plane. Also, it is clear that it maps 0 to 1. During the past few years, several researchers have defined new subclasses of analytic functions by replacing some special functions instead of $(1+z)/(1-z)$, see for instance [4, 5, 6, 7, 8, 9, 11, 16, 18, 19, 20]. These kinds of classes are known as Ma-Minda type classes. Motivated by mentioned works, we are aiming to introduce a new subclass of analytic functions. It should be noted that our new class is related to binomial distribution. At first, we introduce a special function as follows:

**Definition 1.1.** For $\alpha \in (0,1]$, let $\psi_\alpha$ be defined as

$$\psi_\alpha(z)=\frac{e^{\alpha z}-1}{\alpha(1-\alpha z)}, \quad (z\in\mathbb{D}). \tag{1.2}$$

It easy to check that $\psi_\alpha$ has the following expansion

$$\psi_\alpha(z)=z+\frac{3\alpha}{2}z^2+\frac{5\alpha^2}{3}z^3+\frac{41\alpha^3}{24}z^4+\cdots, \quad (z\in\mathbb{D}). \tag{1.3}$$

In order to obtain our results, it is important to obtain the radius of convexity of the function $\psi_\alpha$. In the next lemma, we find it.

**Lemma 1.2.** The function $\psi_\alpha$ is convex univalent in the unit disk $\mathbb{D}$ for all $\alpha \in (0, 0.32]$.

**Proof .** Suppose that $\alpha \in (0,1]$. It follows from (1.2), after some calculations that

$$1+\frac{z\psi_\alpha''(z)}{\psi_\alpha'(z)}=1+\frac{\alpha z\left(e^{\alpha z}-\alpha z e^{\alpha z}\right)}{2e^{\alpha z}-\alpha z e^{\alpha z}-1}+\frac{2\alpha z}{1-\alpha z}, \quad (z\in\mathbb{D}). \tag{1.4}$$

Applying the triangle inequality and since $e^{-\alpha|z|} \le |e^{\alpha z}| \le e^{\alpha|z|}$ for all $z \in \mathbb{D}$, the last equality (1.4) implies that

$$\begin{aligned}
\Re\left\{1+\frac{z\psi_\alpha''(z)}{\psi_\alpha'(z)}\right\} &= \Re\left\{1+\frac{\alpha z e^{\alpha z}(1-\alpha z)}{2e^{\alpha z}-\alpha z e^{\alpha z}-1}+\frac{2\alpha z}{1-\alpha z}\right\} \\
&\ge 1-\left|\frac{\alpha z e^{\alpha z}(1-\alpha z)}{2e^{\alpha z}-\alpha z e^{\alpha z}-1}+\frac{2\alpha z}{1-\alpha z}\right| \\
&\ge 1-\frac{\alpha r e^{\alpha r}(1+\alpha r)}{2e^{-\alpha r}-\alpha r e^{\alpha r}-1}-\frac{2\alpha r}{1-\alpha r}, \quad (|z|=r) \\
&= \frac{2e^{-\alpha r}(1-3\alpha r)+\left(\alpha^3 r^3+3\alpha^2 r^2-2\alpha r\right)e^{\alpha r}+3\alpha r-1}{\left(2e^{-\alpha r}-\alpha r e^{\alpha r}-1\right)(1-\alpha r)} =: h(r,\alpha),
\end{aligned}$$

where $r = |z| < 1$. It is easy to check that $h(r,\alpha) > 0$ if and only if $g(r,\alpha) := 2e^{-\alpha r} - \alpha r e^{\alpha r} - 1 > 0$. Computer experiment (`Mathematica`) shows that $g(r,\alpha) > 0$ for all $r \in (0,1)$ and $\alpha \in (0, 0.32]$, see Figure 1 for more details. The proof is now complete. □

Figure 2 shows the image of the unit disk $\mathbb{D}$ under the function $\psi_\alpha(z)$, where $\alpha = 0.32$ and $\alpha = 0.5$. In 2014, Porwal [14] introduced a Poisson distribution series and obtained necessary and sufficient conditions for this series belonging to some certain subclasses of analytic functions. Following, we introduce a power series whose coefficients are probabilities of the binomial distribution. Before, we recall the binomial distribution.

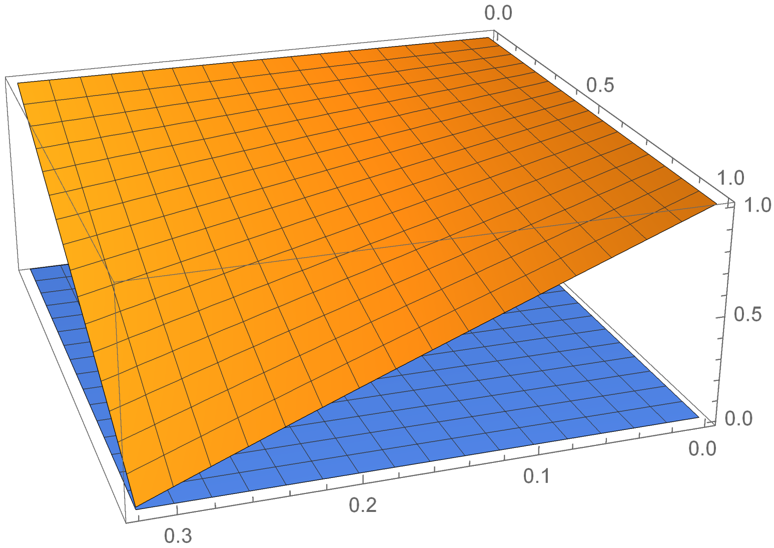


Figure 1: The 3D plot of $g(r, \alpha)$ for $r \in (0, 1)$ and $\alpha \in (0, 0.32]$.

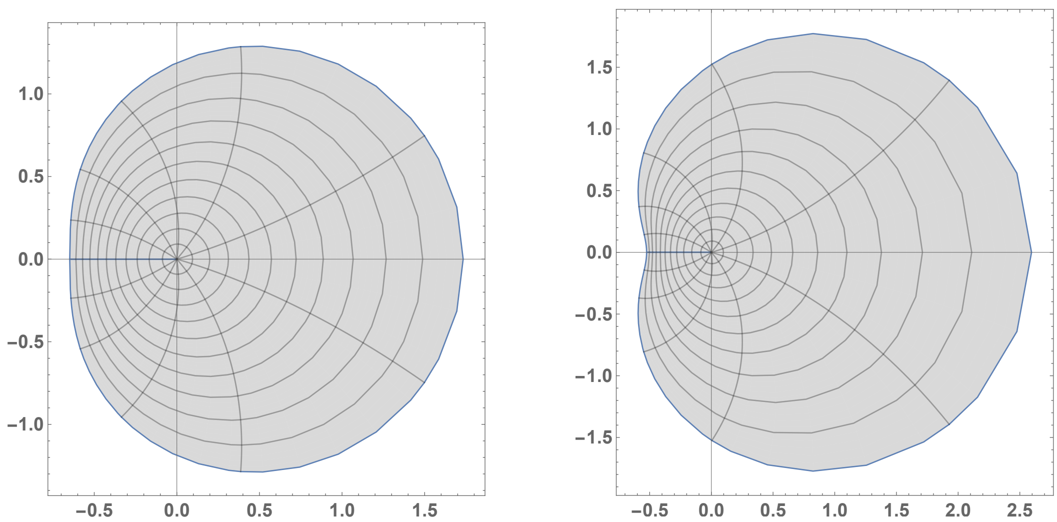


Figure 2: (a): The image of $\mathbb{D}$ under $\psi_{0.32}(z)$ (convex) (b): The image of $\mathbb{D}$ under $\psi_{0.5}(z)$ (non-convex).

### 1.1 Binomial Distribution

A binomial distribution is used when a trial has exactly two mutually exclusive outcomes. A successful outcome will be tagged "success" and a failure will be tagged "failure". With a binomial distribution, $x$ successes are calculated in $m$ trials, with $p$ representing the probability of success on a single trial. For all trials, $p$ is assumed to be fixed in a binomial distribution. A binomial probability mass function can be expressed as follows:

$$P(x; p, n) = \binom{m}{x} p^x q^{m-x}, \quad (m \geq 1, x = 0, 1, 2, \dots, m), \tag{1.5}$$

where $q = 1 - p$, and $\binom{m}{x} = m!/x!(m-x)!$. Therefore, binomial cumulative probability can be calculated using the following formula:

$$F(x; p, m) = \sum_{i=0}^{x} \binom{m}{i} p^i q^{m-i}. \tag{1.6}$$

By using (1.5) we define a new analytic and normalized function whose coefficients are binomial distribution probabilities as follows:

$$K(m,p,q;z) := z + \sum_{n=2}^{\infty} \binom{m}{n-2} p^{n-2} q^{m-n+2} z^n \tag{1.7}$$
$$= z + q^m z^2 + mpq^{m-1} z^3 + \cdots .$$

It is easy to see that by ratio test, the radius of convergence of the above series $K(m,p,q;z)$ is infinity. Applying the Hadamard product for $K(m,p,q;z)$ and $f \in \mathcal{A}$, we define a new analytic function as follows:

$$F(z) := F(m,p,q;z) = K(m,p,q;z) * f(z) \tag{1.8}$$
$$= \left( z + \sum_{n=2}^{\infty} \binom{m}{n-2} p^{n-2} q^{m-n+2} z^n \right) * \left( z + \sum_{n=2}^{\infty} a_n z^n \right)$$
$$= z + \sum_{n=2}^{\infty} C(m,n) z^n,$$

where

$$C_n := C(m,n,p,q) = \binom{m}{n-2} p^{n-2} q^{m-n+2} a_n. \tag{1.9}$$

In particular,

$$C_2 = q^m a_2, \; C_3 = mpq^{m-1} a_3, \quad \text{and} \quad C_4 = \binom{m}{2} p^2 q^{m-2} a_4. \tag{1.10}$$

In this paper by $F(z)$, $\psi_\alpha(z)$, and the subordination relation we are going to introduce a new Ma-Minda type class.

**Definition 1.3.** Let $\psi_\alpha(z)$ be defined by (1.2), $\alpha \in (0,1]$, $0 \le p \le 1$, $q = 1-p$, and $m \ge 1$. We say that a function $f(z) \in \mathcal{A}$ belongs to the class $B_\psi(\alpha,m,p,q)$, if it satisfies the following subordination relation:

$$\left( \frac{z(K(m,p,q;z) * f(z))'(z)}{(K(m,p,q;z) * f(z))(z)} - 1 \right) \prec \psi_\alpha(z), \quad (z \in \mathbb{D}), \tag{1.11}$$

where $K(z)$ is defined as in (1.7).

In order to estimate the coefficients of the function $f \in B_\psi(\alpha,m,p,q)$, the following lemmas will be useful.

**Lemma 1.4.** [12, p. 172] Assume that $w$ is a Schwarz function, so that $w(z) = \sum_{n=1}^{\infty} w_n z^n$. Then

$$|w_1| \le 1 \quad \text{and} \quad |w_n| \le 1 - |w_1|^2, \quad (n = 2,3,\ldots).$$

**Lemma 1.5.** (Prokhorov and Szynal [15]) If $w$ is a Schwarz function of the form $w(z) = \sum_{n=1}^{\infty} w_n z^n$, then for any complex numbers $\rho$ and $\tau$ the following sharp estimate holds:

$$|c_3 + \rho c_1 c_2 + \tau c_1^3| \le 1,$$

where $(\rho,\tau) \in \Omega_1 \cup \Omega_2$ with

$$\Omega_1 = \left\{ (\rho,\tau) : |\rho| \le \frac{1}{2}, |\tau| \le 1 \right\},$$

and

$$\Omega_2 = \left\{ (\rho,\tau) : \frac{1}{2} \le |\rho| \le 2, \frac{4}{27}(|\rho|+1)^3 - (|\rho|+1) \le \tau \le 1 \right\}.$$

**Lemma 1.6.** [2, Lemma 1] If $w(z) = w_1 z + w_2 z^2 + \cdots$ is a Schwarz function, then

$$|w_2 - t w_1^2| \le \begin{cases} -t, & t \le -1; \\ 1 & -1 \le t \le 1; \\ t, & t \ge 1. \end{cases}$$

All inequalities are sharp.

**Lemma 1.7.** [10] Let the function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{D}$ and $\Re\{p(z)\} > 0$ for all $z \in \mathbb{D}$. Then for any complex number $\mu$

$$|p_2 - \mu p_1^2| \leq \begin{cases} -4\mu + 2, & \mu \leq 0; \\ 2, & 0 \leq \mu \leq 1; \\ 4\mu - 2, & \mu \geq 1. \end{cases}$$

The result is sharp.

In the present paper, we estimate the initial coefficients of the function $f \in B_\psi(\alpha, m, p, q)$ and obtain some coefficients inequalities.

## 2 Main Results

We begin with the following result.

**Theorem 2.1.** Let $\alpha \in (0, 1]$, $0 \leq p \leq 1$, $q = 1 - p$, and $m \geq 1$. If a function $f(z)$ is of the form (1.1) belongs to the class $B_\psi(\alpha, m, p, q)$, then

$$|a_2| \leq \frac{1}{q^m} \quad \textit{and} \quad |a_3| \leq \frac{3\alpha + 2}{4mpq^{m-1}}. \tag{2.1}$$

If $\alpha \in (0, 0.625]$, then

$$|a_4| \leq \frac{4 + (9\alpha + 6)}{6m(m-1)p^2 q^{m-2}}. \tag{2.2}$$

All inequalities are sharp.

**Proof .** Let the function $f \in \mathcal{A}$ belong to the class $B_\psi(\alpha, m, p, q)$, where $\alpha \in (0, 1]$, $0 \leq p \leq 1$, $q = 1 - p$, and $m \geq 1$. Then there exists a Schwarz function $w(z)$ such that

$$\frac{z(K(m, p, q; z) * f(z))'(z)}{(K(m, p, q; z) * f(z))(z)} - 1 = \psi_\alpha(w(z)), \quad (z \in \mathbb{D}), \tag{2.3}$$

where $K(m, p, q; z)$ and $\psi_\alpha$ are defined as in (1.7) and (1.2), respectively. A simple calculation gives

$$\frac{z(K(m, p, q; z) * f(z))'(z)}{(K(m, p, q; z) * f(z))(z)} - 1 = C_2 z + (2C_3 - C_2^2)z^2 + (3C_4 - 3C_2C_3 + C_2^3)z^3 + \cdots, \tag{2.4}$$

where $C_n$ defined in (1.9). On the other hand, if $w(z) = w_1 z + w_2 z^2 + \cdots$ is a Schwarz function, then by use of (1.3) we obtain

$$\psi_\alpha(w(z)) = w_1 z + \left(w_2 + \frac{3\alpha}{2} w_1^2\right) z^2 + \left(w_3 + 3\alpha w_1 w_2 + \frac{5\alpha^2}{3} w_1^3\right) z^3 + \cdots. \tag{2.5}$$

Equating the corresponding coefficients in (2.4) and (2.5) gives us the following

$$q^m a_2 = w_1, \quad 2mpq^{m-1} a_3 - q^{2m} a_2^2 = w_2 + \frac{3\alpha}{2} w_1^2 \tag{2.6}$$

and

$$\frac{3}{2} m(m-1)p^2 q^{m-2} a_4 - 3mpq^{2m-1} a_2 a_3 + q^{3m} a_2^3 = w_3 + 3\alpha w_1 w_2 + \frac{5\alpha^2}{3} w_1^3. \tag{2.7}$$

It follows from Lemma 1.4 that $q^m a_2 = w_1$ which implies the first inequality of (2.1). In order to estimate $a_3$ we obtain

$$2mpq^{m-1} a_3 = w_2 - \left(-\frac{3\alpha}{2} - 1\right) w_1^2. \tag{2.8}$$

As an application of Lemma 1.6 we get the second inequality of (2.1). By (2.6) and (2.7) we have

$$\frac{3}{2} m(m-1)p^2 q^{m-2} a_4 = w_3 + 3\alpha w_1 w_2 + \frac{5}{3}\alpha^2 w_1^3 + \frac{3}{2} w_1 \left(w_2 + \left(\frac{3\alpha}{2} + 1\right) w_1^2\right).$$

Therefore, by Lemma 1.6 and Lemma 1.5 we get

$$\frac{3}{2}m(m-1)p^2q^{m-2}|a_4| \le \left|w_3 + 3\alpha w_1 w_2 + \frac{5}{3}\alpha^2 w_1^3\right| + \frac{3}{2}|w_1|\left|w_2 + \left(\frac{3\alpha}{2}+1\right)w_1^2\right|$$
$$\le 1 + \frac{3}{2}\left(\frac{3\alpha}{2}+1\right).$$

It is enough to show that $\left|w_3 + 3\alpha w_1 w_2 + 5\alpha^2 w_1^3/3\right| \le 1$. We consider two cases for $\alpha$.

**Case 1:** $0 < \alpha \le 1/6$. Let $\rho = 3\alpha$ and $\tau = 5\alpha^2/3$. Then it is easy to see that $0 < \rho \le 1/2$ and $0 < \tau < 1$. Thus $(\rho,\tau) \in \Omega_1$, where $\Omega_1$ is defined in Lemma 1.5, which means that $\left|w_3 + 3\alpha w_1 w_2 + 5\alpha^2 w_1^3/3\right| \le 1$.

**Case 2:** $1/6 \le \alpha \le 0.625$. In this case, we see that $1/2 \le \rho \le 1.86$ and because $5\alpha^2/3 < 1$, and

$$h(\alpha) := \frac{4}{27}(3\alpha+1)^3 - (3\alpha+1) - \frac{5\alpha^2}{3} \le 0$$

for all $\alpha \in [1/6, 0.625]$ (see Figure 3), therefore, $(\rho,\tau) \in \Omega_2$, where $\Omega_2$ is defined in Lemma 1.5. In this case also we have

$$\left|w_3 + 3\alpha w_1 w_2 + 5\alpha^3 w_1^3/3\right| \le 1.$$

All inequalities are sharp, when $f$ is a solution of the equation (2.3) with $w(z) = z$. The proof is now complete. □

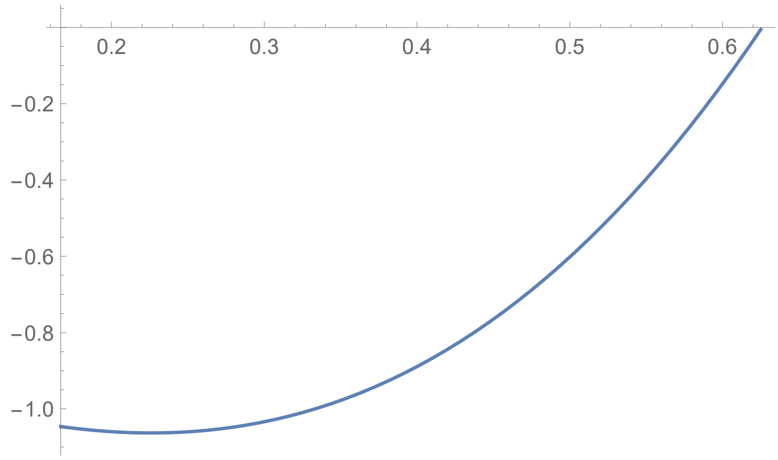


Figure 3: The graph of $h(\alpha)$ for $1/6 \le \alpha \le 0.625$.

Complex function theory relies heavily on the $k$th root transform in many different ways. Based on subordination, Ali et al. [1] investigated Fekete-Szegö coefficient functionals for the $k$th root transform of several classes of analytic functions. Here, we recall that for a univalent function $f$ of the form (1.1), the $k$-th ($k \ge 1$) root transform is defined by

$$F_k(z) := (f(z^k))^{1/k} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}, \quad (z \in \mathbb{D}). \tag{2.9}$$

Therefore, for $f \in \mathcal{A}$ we have,

$$(f(z^k))^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right) z^{2k+1} + \cdots. \tag{2.10}$$

Following, we estimate $|b_{2k+1} - \mu b_{k+1}|$ which is known as Fekete-Szegö problem of the $k$th root transform of $f$.

**Theorem 2.2.** Let the function $f \in \mathcal{A}$ belong to the class $B_\psi(\alpha, m, p, q)$, where $\alpha \in (0, 1]$, $0 \le p \le 1$, $q = 1 - p$, and

$m \geq 1$. Then for all $\mu \in \mathbb{C}$ we have

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \begin{cases} \frac{1}{4kmpq^{m-1}}\left(\frac{2(1-k)mp}{kq^{m+1}} - \frac{4mp\mu}{kq^{m+1}} + 3\alpha + 2\right), & \mu \leq \delta_1; \\ \frac{1}{2kmpq^{m-1}}, & \delta_1 \leq \mu \leq \delta_2; \\ \frac{1}{4kmpq^{m-1}}\left(\frac{4mp\mu}{kq^{m+1}} - \frac{2(1-k)mp}{kq^{m+1}} - 3\alpha - 2\right), & \mu \geq \delta_2, \end{cases}$$

where

$$\delta_1 := \frac{3\alpha k q^{m+1}}{4mp} - \frac{1}{2}(k-1), \quad (k \geq 1), \tag{2.11}$$

and

$$\delta_2 := \frac{kq^{m+1}}{mp} + \frac{3\alpha k q^{m+1}}{4mp} - \frac{1}{2}(k-1), \quad (k \geq 1). \tag{2.12}$$

The result is sharp.

**Proof .** Let $\alpha \in (0,1]$, $0 \leq p \leq 1$, $q = 1-p$, and $m \geq 1$. If a function $f \in \mathcal{A}$ belongs to the class $B_\psi(\alpha, m, p, q)$, then there exists a Schwarz function $w(z)$ such that (2.3) holds true. Define

$$\frac{1+w(z)}{1-w(z)} =: p(z) = 1 + p_1 z + p_2 z^2 + \cdots . \tag{2.13}$$

It clear that $p(0) = 1$ and $\Re\, p(z) > 0$, where $z \in \mathbb{D}$. It follows from (2.13) that

$$w(z) = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \cdots . \tag{2.14}$$

From (1.3) and (2.14), we get

$$1 + \psi_\alpha(w(z)) = 1 + \frac{1}{2}p_1 z + \left(\frac{3\alpha}{8}p_1^2 + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)\right) z^2 + \cdots . \tag{2.15}$$

Now by (2.3), (2.4), and (2.15), we obtain

$$\frac{1}{2}p_1 = C_1, \quad \text{and} \quad \frac{3\alpha}{8}p_1^2 + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) = 2C_3 - C_2^2, \tag{2.16}$$

where $C_2$ and $C_3$ are defined as in (1.10). The first identity of (2.16) gives

$$a_2 = \frac{p_1}{2q^m}, \tag{2.17}$$

while the second identity of (2.16) gives

$$a_3 = \frac{1}{4mpq^{m-1}}\left(p_2 + \frac{3\alpha}{4}p_1^2\right). \tag{2.18}$$

Equating the corresponding coefficients of (2.9) and (2.10) give

$$b_{k+1} = \frac{1}{k}a_2, \tag{2.19}$$

and

$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{2.20}$$

It follows from (2.17) and (2.19) that

$$b_{k+1} = \frac{p_1}{2kq^m}. \tag{2.21}$$

Also, by (2.18) and (2.20) we obtain

$$b_{2k+1} = \frac{1}{4kmpq^{m-1}}\left(p_2 + \frac{3\alpha}{4}p_1^2\right) - \frac{1}{2}\frac{k-1}{k^2}\frac{p_1^2}{4q^{2m}}. \tag{2.22}$$

Now by (2.21) and (2.22) we get

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{4kmpq^{m-1}}\left(p_2 - \frac{1}{2}\left[\frac{k-1}{k}\frac{mp}{q^{m+1}} + \frac{2mp\mu}{kq^{m+1}} - \frac{3\alpha}{2}\right]p_1^2\right), \tag{2.23}$$

where $\mu \in \mathbb{C}$. If we let

$$\mu' = \frac{1}{2}\left[\frac{k-1}{k}\frac{mp}{q^{m+1}} + \frac{2mp\mu}{kq^{m+1}} - \frac{3\alpha}{2}\right],$$

and then apply Lemma 1.7, we get the desired result. The result is sharp for a solution of the equation (2.3) with $w(z) = z$. $\square$

Pommerenke [13] was the first to study the Hankel determinant $H_{q,n}(f)$ of a function $f$ given by (1.1). The Hankel determinant $H_{q,n}(f)$ is given as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Based on different values of $q$ and $n$, Hankel determinants for various orders can be derived. Here is how the above-defined determinant looks when $n$ is equal to 1 and $q$ is equal to 2

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left|a_3 - a_2^2\right|, \quad (a_1 = 1).$$

In addition, if $q = n = 2$, then we have the second Hankel determinant

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left|a_2 a_4 - a_3^2\right|.$$

The upper bound of $H_{2,2}(f)$ for different subclasses of analytical functions has been studied and investigated by many authors in recent years, see for instance [1-2-3]. Motivated by aforementioned works, we estimate $H_{2,1}(f)$ and $H_{2,2}(f)$, where $f \in B_\psi(\alpha, m, p, q)$.

**Theorem 2.3.** Let $0 < p \le 1$, $0 < q \le 1$, $m \ge 1$, and $\alpha \in (0, 1]$. Also, let the function $f$ be of the form (1.1) belong to $B_\psi(\alpha, m, p, q)$. Then

$$H_{2,1}(f) \le \frac{1}{2mpq^{m-1}}\left(\frac{3\alpha}{2} + 1\right) + \frac{1}{q^{2m}}.$$

The result is sharp.

**Proof .** If the function $f \in \mathcal{A}$ belongs to the class $B_\psi(\alpha, m, p, q)$, then by (2.6) we get

$$a_3 - a_2^2 = \frac{1}{2mpq^{m-1}}\left[w_2 - \left(-\frac{3\alpha}{2} - 1\right)w_1^2\right] - \frac{1}{q^{2m}}w_1^2.$$

Therefore,

$$\begin{aligned} \left|a_3 - a_2^2\right| &= \left|\frac{1}{2mpq^{m-1}}\left[w_2 - \left(-\frac{3\alpha}{2} - 1\right)w_1^2\right] - \frac{1}{q^{2m}}w_1^2\right| \\ &\le \frac{1}{2mpq^{m-1}}\left|w_2 - \left(-\frac{3\alpha}{2} - 1\right)w_1^2\right| + \frac{1}{q^{2m}}\left|w_1\right|^2 \\ &\le \frac{1}{2mpq^{m-1}}\left(\frac{3\alpha}{2} + 1\right) + \frac{1}{q^{2m}}, \end{aligned}$$

concluded the result by Lemma 1.6. The result is sharp for a solution of the equation (2.3) with $w(z) = z$. $\square$

**Theorem 2.4.** Let $0 < p \leq 1$, $0 < q \leq 1$, $m \geq 1$, and $\alpha \in (0,1]$. If a function $f$ is of the form (1.1) belongs to $B_\psi(\alpha, m, p, q)$, then

$$H_{2,2}(f) \leq \frac{2}{3m(m-1)p^2q^{2(m-1)}}\left(1+\frac{3}{2}\left(\frac{3\alpha}{2}+1\right)\right)+\frac{1}{4m^2p^2q^{2(m-1)}}\left(\frac{3\alpha}{2}+1\right)^2.$$

The result is sharp.

**Proof .** By (2.6) and (2.7), and a simple calculation, we see that

$$\begin{aligned} a_2a_4-a_3^2 &= \frac{2}{3m(m-1)p^2q^{2(m-1)}}\left\{w_3+3\alpha w_1w_2+\frac{5\alpha^2}{3}w_1^3+\frac{3}{2}\left[w_2-\left(-\frac{3\alpha}{2}-1\right)w_1^2\right]+w_1^3\right\}w_1 \\ &\quad -\frac{1}{4m^2p^2q^{2(m-1)}}\left[w_2-\left(-\frac{3\alpha}{2}-1\right)w_1^2\right]^2. \end{aligned}$$

Now, by triangle inequality, the proof of Theorem 2.1 and Lemma 1.6 we obtain

$$\begin{aligned} H_{2,2}(f) &\leq \frac{2}{3m(m-1)p^2q^{2(m-1)}}\left\{\left|w_3+3\alpha w_1w_2+\frac{5\alpha^2}{3}w_1^3\right|+\frac{3}{2}\left|w_2-\left(-\frac{3\alpha}{2}-1\right)w_1^2\right|+|w_1|^3\right\}|w_1| \\ &\quad +\frac{1}{4m^2p^2q^{2(m-1)}}\left|w_2-\left(-\frac{3\alpha}{2}-1\right)w_1^2\right|^2 \\ &\leq \frac{2}{3m(m-1)p^2q^{2(m-1)}}\left(1+\frac{3}{2}\left(\frac{3\alpha}{2}+1\right)\right)+\frac{1}{4m^2p^2q^{2(m-1)}}\left(\frac{3\alpha}{2}+1\right)^2. \end{aligned}$$

The result is sharp for a solution of the equation (2.3) with $w(z) = z$. The proof now is complete. □

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