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Complex dynamics of a predator-prey model with harvesting effects on both predator and prey

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Abstract

A discrete predator-prey model with harvesting effects on both predator and prey is examined to reveal its chaotic dynamics. The model's existence and local stability analysis are investigated. It is demonstrated that the model experiences period-doubling bifurcation and Neimark-Sacker bifurcation by using bifurcation theory. Moreover, numerical examples are used to demonstrate the consistency of analytical conclusions as well as the model's complexity owing to harvesting effects. It is shown that changing the harvesting parameters affects not only the number of fixed points in the model, but also the occurrence of different bifurcations.

Keywords: Predator-prey model, harvesting, stability, period-doubling bifurcation, Neimark-Sacker bifurcation
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1 Introduction

One of the most important aspects of forecasting ecological systems is to understand the relationship between the predator and the prey. Much of recent research has focused on understanding the interaction between biological species in general and predator-prey species in particular, as well as predicting the survival (or not) of the living organisms investigated. A variety of effective mathematical biology models have been presented to predict these interactions and the survival of diverse species. Many existing models have as their primary goal the investigation of interactions between diverse biological models, as well as the prediction of interactions and the survival of understudied species.

We can analyze the behavior of populations through models created by difference or differential equations [20, 14, 25, 4, 13, 3]. A sufficient number of continuous-time prey-predator models have been introduced to explain the complex relationship of the species in the literature. But in ecology, populations evolve in discrete-time steps because there is no overlapping between successive generations for many species. For such population dynamics, it would make sense to use difference equations. To use discrete-time models of prey-predator interaction is another possible way to understand the behavior of these populations. The studies on discretization of prey-predator models governed by difference equations have received remarkable attention. The dynamic behaviors of discrete-time models are discussed

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by the authors [1, 7, 8, 9, 18]. It is biologically significant that a discrete-time models lead to unpredictable dynamic behaviors. Therefore, bifurcation theory is widely used in mathematical investigations of dynamical systems to predict this behavior of populations. This is a good way to highlight the prospect that the laws that govern ecological models are generally simple and easy to find. The rich dynamic behaviors revealed by such models have caused the discrete variant to become a significant study topic. There has been a substantial quantity of literature on bifurcation of discrete-time population models (we refer the readers to [5, 10, 11, 12, 21, 27]).

Analysis of prey-predator models with harvesting effect has an important place in dynamic systems, and is necessary to consider the harvesting of populations in some models. Harvesting has also an important role on control of populations. In recent times, the effect of harvesting in prey-predator models has been investigated by a lot of researchers [17, 19, 22, 28].

2 Model Formulation

In [16], the authors considered a Lotka-Volterra type predator-prey model

$$\begin{cases} \frac{dx}{dt} = r_0 x(1-\frac{x}{k}) - b_0 xy, \\ \frac{dy}{dt} = (-d_0 + cx)y, \end{cases} \tag{2.1}$$

where $x(t)$ and $y(t)$ represent prey and predator populations respectively, $b_0 x$ is the predator's functional response, which measures the number of prey individuals devoured by an individual predator per unit area per unit time, c denotes the efficiency with which prey are converted into predators, cxy denotes the predator's numerical response, and d_0 denotes the predator's mortality rate. After using the scaled variables, the model (2.1) was rewritten as [16]

$$\begin{cases} \frac{dx}{dt} = rx(1-x) - bxy, \\ \frac{dy}{dt} = (-d + bx)y, \end{cases} \tag{2.2}$$

where r, b, and d are all positive parameters, $r = r_0 k, b = ck^2$ and $d = d_0 k$. In this paper, we studied the model (2.2) by introducing the harvesting effect on both predator and prey populations. We consider the following model

$$\begin{cases} \frac{dx}{dt} = rx(1-x) - bxy - h_1 x, \\ \frac{dy}{dt} = (-d + bx)y - h_2 y, \end{cases} \tag{2.3}$$

where h_1 is the harvesting effect on prey population and h_2 is the harvesting effect on predator population. The model (2.3) is simplified to following discrete model after applying forward Euler method.

$$\begin{cases} x_{n+1} = x_n + h(rx_n(1-x_n) - bx_n y_n - h_1 x_n), \\ y_{n+1} = y_n + h((-d + bx_n)y_n - h_2 y_n). \end{cases} \tag{2.4}$$

where $h > 0$ denotes the step size. The paper is organized as follows: The existence and stability of fixed points of model (2.4) are discussed in section 3. In section 4, we discuss local bifurcation analysis at unique positive fixed point of model (2.4) by using center manifold theorem and bifurcation theory. Some numerical examples are offered in section 5 to validate our theoretical conclusions. Some final thoughts are included in the section 6.

3 Stability analysis of fixed points

The model (2.4) has following three fixed points

$$P_0(0,0),\ P_1\left(\frac{r-h_1}{r}, 0\right),\ P_2\left(\frac{d+h_2}{b}, \frac{br - dr - bh_1 - rh_2}{b^2}\right).$$

Note that for the existence of P_1, it is required that $r > h_1$. Moreover, P_2 is the unique fixed point of the model (2.4) if $br - dr - bh_1 - rh_2 > 0$. For biologically meaningful our aim is to describe the dynamics of the model (2.4) at the unique positive fixed point P_2. The variational matrix of the model (2.4) evaluated at any point $(\bar{x}, \bar{y})$ is

$$J(\bar{x}, \bar{y}) = \begin{bmatrix} 1 - h(h_1 - r + 2r\bar{x} + b\bar{y}) & -bh\bar{x} \\ bh\bar{y} & 1 - dh - hh_2 + bh\bar{x} \end{bmatrix}.$$

The variational matrix at the fixed point P_2 is

$$J(P_2) = \begin{bmatrix} 1 - \frac{h(d+h_2)r}{b} & -h(d+h_2) \\ -\frac{h(b(h_1-r)+(d+h_2)r)}{b} & 1 \end{bmatrix}.$$

The characteristic polynomial of $J(P_2)$ is

$$F(w) = w^2 + Sw + T, \tag{3.1}$$

where

$$S = -2 + A_1 h,\ T = 1 - A_1 h + A_2 h^2,$$

$$A_1 = \frac{r(d+h_2)}{b} > 0,\ A_2 = \frac{(d+h_2)(br - dr - bh_1 - rh_2)}{b} > 0.$$

We obtain the following after easy calculations

$$F(0) = 1 - A_1 h + A_2 h^2,\ F(1) = A_2 h^2,\ F(-1) = 4 - 2A_1 h + A_2 h^2.$$

The following results are used to study the stability of fixed points of model (2.4).

Lemma 3.1. [6] Let $F(w) = w^2 + Sw + T$ be the characteristic polynomial associated to the variational matrix at fixed point $(\bar{x}, \bar{y})$. If w_1, w_2 are two roots of $F(w) = 0$, then $(\bar{x}, \bar{y})$ is

(i) sink and therefore locally asymptotically stable if $|w_{1,2}| < 1$,

(ii) source and therefore unstable if $|w_{1,2}| > 1$,

(iii) saddle point if $|w_1| < 1$ and $|w_2| > 1$ (or $|w_1| > 1$ and $|w_2| < 1$),

(iv) non-hyperbolic if either $|w_1| = 1$ or $|w_2| = 1$.

Lemma 3.2. [6] Let $F(w) = w^2 + Sw + T$. Assume that $F(1) > 0$. If w_1, w_2 are two roots of $F(w) = 0$, then

(i) $|w_{1,2}| < 1$ iff $F(-1) > 0$ and $T < 1$,

(ii) $|w_1| < 1$ and $|w_2| > 1$ (or $|w_1| > 1$ and $|w_2| < 1$) iff $F(-1) < 0$,

(iii) $|w_1| > 1$ and $|w_2| > 1$ iff $F(-1) > 0$ and $T > 1$,

(iv) $w_1 = -1$ and $|w_2| \neq 1$ iff $F(-1) = 0$ and $S \neq 0, 2$,

(v) w_1 and w_2 are complex and $|w_{1,2}| = 1$ iff $S^2 - 4T < 0$ and $T = 1$.

Using lemma (3.2), we obtain the local dynamics of the fixed point P_2.

Proposition 3.3. Assume that $br - dr - bh_1 - rh_2 > 0$. The fixed point P_2 of the model (2.4) is

(i) a sink and therefore it is locally asymptotically stable if one of the following conditions holds

(a) $A_1^2 - 4A_2 < 0$ and $0 < h < \frac{A_1}{A_2}$,

(b) $A_1^2 - 4A_2 \geq 0$ and $0 < h < \frac{A_1 - \sqrt{A_1^2 - 4A_2}}{A_2}$,

(ii) a source and therefore it is unstable if one of the following conditions holds

(a) $A_1^2 - 4A_2 \leq 0$ and $h > \frac{A_1}{A_2}$,

(b) $A_1^2 - 4A_2 > 0$ and $h > \frac{A_1 + \sqrt{A_1^2 - 4A_2}}{A_2}$,

(iii) a saddle point if the following condition holds

$A_1^2 - 4A_2 > 0$ and $\frac{A_1 - \sqrt{A_1^2 - 4A_2}}{A_2} < h < \frac{A_1 + \sqrt{A_1^2 - 4A_2}}{A_2}$,

(iv) non-hyperbolic point if one of the following conditions holds

(a) $A_1^2 - 4A_2 > 0$ and $h = \frac{A_1 \pm \sqrt{A_1^2 - 4A_2}}{A_2}$,

(b) $A_1^2 - 4A_2 < 0$ and $h = \frac{A_1}{A_2}$.

It is clear that if $A_1^2 - 4A_2 < 0$ and $h = \frac{A_1}{A_2}$, then eigenvalues of $J(P_2)$ are complex with unit modulus. Therefore, model (2.4) experiences Neimark-Sacker bifurcation at fixed point P_2 when parameters vary in a small neighbourhood of the following set:

$$\left\{ h, r, b, d, h_1, h_2 \in \mathbb{R}^+ \middle| A_1^2 - 4A_2 < 0,\ h = \frac{A_1}{A_2} \right\}.$$

Moreover, if $A_1^2 - 4A_2 > 0$ and $h = \frac{A_1 \pm \sqrt{A_1^2 - 4A_2}}{A_2}$, then one of the eigenvalues of $J(P_2)$ is -1 and other eigenvalue λ satisfies $|\lambda| \neq 1$. Therefore a period-doubling bifurcation can occur if parameters vary in a small neighbourhood of either of the following sets:

$$\left\{ h, r, b, d, h_1, h_2 \in \mathbb{R}^+ \middle| A_1^2 - 4A_2 > 0,\ h = \frac{A_1 + \sqrt{A_1^2 - 4A_2}}{A_2} \right\},$$

$$\left\{ h, r, b, d, h_1, h_2 \in \mathbb{R}^+ \middle| A_1^2 - 4A_2 > 0,\ h = \frac{A_1 - \sqrt{A_1^2 - 4A_2}}{A_2} \right\}.$$

4 Local Bifurcation Analysis

Local bifurcation analysis is an effective approach for analyzing the qualitative behavior of dynamical systems around fixed points. It helps in identifying critical points at which the system's behavior changes qualitatively. These bifurcations may result in complicated behaviors like chaotic oscillations or the establishment of stable limit cycles, which has consequences for the stability and sustainability of predator-prey relationships in nature. Different bifurcation types are addressed in this section at unique positive fixed point P_2 of the model (2.4). For detailed bifurcation theory, we refer the readers to [15, 26, 2, 23, 29, 24].

4.1 Period-Doubling Bifurcation at $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$:

In this section, we discuss period-doubling bifurcation at fixed point $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ for the domain Ω_3. Similar arguments can be used for the domain Ω_2. Consider the domain

$$\Gamma_3 = \left\{ h, r, b, d, h_1, h_2 \in \mathbb{R}^+ \middle| A_1^2 - 4A_2 > 0,\ h = H_1 = \frac{A_1 - \sqrt{A_1^2 - 4A_2}}{A_2} \right\}.$$

Assuming that $(h, r, b, d, h_1, h_2) \in \Gamma_3$, and δ be small perturbation in H_1, we consider the following perturbation of the model (2.4):

$$\begin{cases} x_{n+1} = x_n + (H_1 + \delta)(r x_n(1 - x_n) - b x_n y_n - h_1 x_n), \\ y_{n+1} = y_n + (H_1 + \delta)((-d + b x_n) y_n - h_2 y_n), \end{cases} \tag{4.1}$$

where $\delta, |\delta| \ll 1$, is a small perturbation parameter. We define $a_n = x_n - \frac{d+h_2}{b},\ b_n = y_n - \frac{br-dr-bh_1-rh_2}{b^2}$, to translate fixed point $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ to origin. Under this translation map the model (4.1) becomes

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - A_1 H_1 & -\frac{b A_1 H_1}{r} \\ \frac{r A_2 H_1}{A_1 b} & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} F(a_n, b_n, \delta) \\ G(a_n, b_n, \delta) \end{bmatrix}, \tag{4.2}$$

where

$$\begin{aligned} F(a_n, b_n, \delta) &= -\frac{b A_1}{r} \delta b_n - A_1 \delta a_n - b H_1 a_n b_n - b \delta a_n b_n - H_1 r a_n^2 - r \delta a_n^2, \\ G(a_n, b_n, \delta) &= (r - h_1 - A_1) \delta a_n + b H_1 a_n b_n + b \delta a_n b_n. \end{aligned}$$

For $H_1 = \frac{A_1-\sqrt{A_1^2-4A_2}}{A_2}$, the eigenvalues of $J(P_2)$ are $\lambda_1 = -1$ and $\lambda_2 = 3 - A_1H_1$. Let

$$T = \begin{bmatrix} \frac{2A_1b}{r(-A_1+\sqrt{A_1^2-4A_2})} & \frac{A_1^3b-A_1^2b\sqrt{A_1^2-4A_2}-2A_1A_2b}{A_2r(-A_1+\sqrt{A_1^2-4A_2})} \\ 1 & 1 \end{bmatrix}.$$

Under the following transformation

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = T \begin{bmatrix} e_n \\ f_n \end{bmatrix}, \tag{4.3}$$

the model (4.2) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} F(e_n, f_n, \delta) \\ G(e_n, f_n, \delta) \end{bmatrix}, \tag{4.4}$$

where

$$\begin{aligned} \lambda_2 &= \frac{2A_1^4 - 2A_1^3\sqrt{A_1^2-4A_2} - 11A_1^2A_2 + 7A_1A_2\sqrt{A_1^2-4A_2} + 12A_2^2}{A_2(-A_1^2 + A_1\sqrt{A_1^2-4A_2} + 4A_2)}, \\ F(e_n, f_n, \delta) &= D_1e_n^2 + D_2e_nf_n + D_3f_n^2 + D_4e_n\delta + D_5e_n^2\delta + D_6f_n\delta + D_7e_nf_n\delta + O((|e_n| + |f_n| + |\delta|)^4), \\ G(e_n, f_n, \delta) &= E_1e_n^2 + E_2e_nf_n + E_3f_n^2 + E_4e_n\delta + E_5e_n^2\delta + E_6f_n\delta + E_7e_nf_n\delta + O((|e_n| + |f_n| + |\delta|)^4), \end{aligned}$$

where values of coefficients D_i and E_i are given in Appendix A. Next, we determine the center manifold $W^C(0,0,0)$ for (4.4), which can be represented as follows:

$$W^C(0,0,0) = \left\{ (e_n, f_n, \delta) \in \mathbb{R}^3 \middle| f_n = c_1e_n^2 + c_2e_n\delta + c_3\delta^2 + O((|e_n| + |\delta|)^3) \right\},$$

where

$$\begin{aligned} c_1 &= \frac{A_2b(A_1(2b+r) + r\sqrt{A_1^2-4A_2})}{r(A_1^4 - A_1^3\sqrt{A_1^2-4A_2} - 5A_1^2A_2 + 3A_1A_2\sqrt{A_1^2-4A_2} + 4A_2^2)}, \\ c_2 &= -\frac{2A_2^2(A_1^2b + A_1b(h_1 - r) + A_2r)}{r(A_1^2 - 4A_2)(-A_1 + \sqrt{A_1^2-4A_2})(-A_1^2 + A_1\sqrt{A_1^2-4A_2} + 2A_2)}, \ c_3 = 0. \end{aligned}$$

Thus the model (4.4) restricted to the center manifold is given by

$$\tilde{F} : e_{n+1} = -e_n + D_1e_n^2 + D_4\delta e_n - \frac{D_6E_4}{1+\lambda_2}\delta^2e_n + (D_5 - \frac{D_6E_1}{-1+\lambda_2} - \frac{D_2E_4}{1+\lambda_2})\delta e_n^2 + \frac{D_2E_1}{1-\lambda}e_n^3. \tag{4.5}$$

In order for map (4.5) to undergo period-doubling bifurcation it is required that following two quantities are non-zero.

$$l_1 = \tilde{F}_\delta\tilde{F}_{e_ne_n} + 2\tilde{F}_{e_n\delta}\Big|_{(0,0)}, \ l_2 = \frac{1}{2}(\tilde{F}_{e_ne_n})^2 + \frac{1}{3}\tilde{F}_{e_ne_ne_n}.$$

From simple computations, we obtain

$$l_1 = 2D_4, \ l_2 = 2D_1^2 + \frac{2D_2E_1}{1-\lambda_2}.$$

As a consequence of the preceding analysis, we have the following result.

Theorem 4.1. The model (2.4) experiences period-doubling bifurcation at the fixed point $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ if $l_2 \neq 0$ and h varies in a small neighbourhood of $H_1 = \frac{A_1-\sqrt{A_1^2-4A_2}}{A_2}$. Moreover, if $l_2 > 0$ (respectively $l_2 < 0$), then the period-2 orbits that bifurcate from $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ are stable (respectively, unstable).

4.2 Neimark-Sacker Bifurcation at $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$:

In this section, we discuss Neimark-Sacker bifurcation at fixed point $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ for the domain Ω_1. Consider the domain

$$\Gamma_1 = \left\{ h, r, b, d, h_1, h_2 \in \mathbb{R}^+ \middle| A_1^2 - 4A_2 < 0,\ h = H_2 = \frac{A_1}{A_2} \right\}.$$

Assuming that $(h, r, b, d, h_1, h_2) \in \Omega_1$, and δ be small perturbation in H_2, we consider the following perturbation of the model (2.4):

$$\begin{cases} x_{n+1} = x_n + (H_2+\delta)(rx_n(1-x_n) - bx_ny_n - h_1x_n), \\ y_{n+1} = y_n + (H_2+\delta)((-d+bx_n)y_n - h_2y_n), \end{cases} \tag{4.6}$$

We define $a_n = x_n - \frac{d+h_2}{b}$, $b_n = y_n - \frac{br-dr-bh_1-rh_2}{b^2}$, to translate fixed point $P_2(\frac{d+h_2}{b}, \frac{br-dr-bh_1-rh_2}{b^2})$ to origin. Under this translation map the model (4.6) becomes

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{-A_1r(d+h_2)+A_2(b-(d+h_2)r\delta)}{A_2b} & -(d+h_2)(\frac{A_1}{A_2}+\delta) \\ -\frac{(b(h_1-r)+(d+h_2)r)(A_1+A_2\delta)}{A_2b} & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} F(a_n, b_n) \\ G(a_n, b_n) \end{bmatrix}, \tag{4.7}$$

where

$$F(a_n, b_n) = -b(\frac{A_1}{A_2}+\delta)a_nb_n - r(\frac{A_1}{A_2}+\delta)a_n^2,$$
$$G(a_n, b_n) = b(\frac{A_1}{A_2}+\delta)a_nb_n.$$

At the fixed point $(0,0)$, the characteristic equation of the linearized part of the model (4.7) is

$$\lambda^2 - \alpha(\delta)\lambda + \beta(\delta) = 0, \tag{4.8}$$

where

$$\alpha(\delta) = 2 - \frac{A_1^2}{A_2} - A_1\delta,$$
$$\beta(\delta) = 1 + A_1\delta + A_2\delta^2.$$

The roots of the equation (4.8) are complex with the property $|\lambda_{1,2}| = 1$, which are given by

$$\lambda_{1,2} = \frac{\alpha(\delta) \pm i\sqrt{4\beta(\delta) - \alpha^2(\delta)}}{2}.$$

By computations, we obtain

$$|\lambda_1| = |\lambda_2| = \sqrt{\beta(\delta)}$$

and

$$\left(\frac{d|\lambda_1|}{d\delta}\right)_{\delta=0} = \left(\frac{d|\lambda_2|}{d\delta}\right)_{\delta=0} = \frac{A_1}{2} > 0.$$

Moreover, it is required that $\lambda_1^i, \lambda_2^i \neq 1$ for $i = 1,2,3,4$ at $\delta = 0$ which is equivalent to $\alpha(0) \neq \pm 2, 0, 1$. Since $A_1 > 0, A_2 > 0$, $A_1^2 - 4A_2 < 0$ and $\alpha(0) = 2 - \frac{A_1^2}{A_2}$, therefore $\alpha(0) \neq \pm 2$. We only need to require that $\alpha(0) \neq 0, 1$, which leads to $A_1^2 \neq 2A_2, A_2$. To convert the linear part of (4.7) into its canonical form at $\delta = 0$, we employ the aforementioned mapping:

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} -\frac{bA_1^2}{rA_2} & 0 \\ \frac{A_1^2}{2A_2} & -\frac{A_1\sqrt{4A_2-A_1^2}}{2A_2} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}. \tag{4.9}$$

Under the transformation (4.9), the model (4.7) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} F(e_n, f_n) \\ G(e_n, f_n) \end{bmatrix}, \tag{4.10}$$

where

$$\mu = 1 - \frac{A_1^2}{2A_2},\ \nu = \frac{A_1\sqrt{4A_2 - A_1^2}}{2A_2},$$

$$F(e_n, f_n) = \frac{A_1^3 b}{2A_2^2}e_n^2 + \frac{A_1^2 b\sqrt{4A_2 - A_1^2}}{2A_2^2}e_n f_n + O((|e_n| + |f_n| + |\delta|)^4),$$

$$G(e_n, f_n) = \frac{A_1^4 b(2b + r)}{2A_2^2 r\sqrt{4A_2 - A_1^2}}e_n^2 + \frac{A_1^3 b(-2b + r)}{2A_2^2 r}e_n f_n + O((|e_n| + |f_n| + |\delta|)^4),$$

In a model with NS bifurcation, the aforementioned value L specifies the direction in which the invariant curve occurs.

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\eta_{20}\eta_{11}\right) - \frac{1}{2}|\eta_{11}|^2 - |\eta_{02}|^2 + Re(\lambda_2\eta_{21})\right]\right)_{\delta=0},$$

where

$$\begin{aligned}
\eta_{20} &= \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} + 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} - 2F_{e_nf_n})\right],\\
\eta_{11} &= \frac{1}{4}\left[F_{e_ne_n} + F_{f_nf_n} + i(G_{e_ne_n} + G_{f_nf_n})\right],\\
\eta_{02} &= \frac{1}{8}\left[F_{e_ne_n} - F_{f_nf_n} - 2G_{e_nf_n} + i(G_{e_ne_n} - G_{f_nf_n} + 2F_{e_nf_n})\right],\\
\eta_{21} &= \frac{1}{16}\left[F_{e_ne_ne_n} + F_{e_nf_nf_n} + G_{e_ne_nf_n} + G_{f_nf_nf_n} + i(G_{e_ne_ne_n} + G_{e_nf_nf_n} - F_{e_ne_nf_n} - F_{f_nf_nf_n})\right].
\end{aligned}$$

The preceding computations lead to the aforementioned theorem for the presence and direction of NS bifurcation.

Theorem 4.2. Assume that $A_1^2 - 4A_2 < 0$ and $A_1^2 \neq 2A_2, A_2$. When the parameter h changes within a neighbourhood of $H_2 = \frac{A_1}{A_2}$, the model (2.4) undergoes NS bifurcation at the fixed point P_2 if $L \neq 0$. In addition, an attracting invariant closed curve bifurcates from the fixed point if $L < 0$, while a repelling invariant closed curve bifurcates from the fixed point if $L > 0$.

5 Numerical examples

In this section, we will provide some numerical simulations to back up our theoretical analysis of the model's multiple qualitative characteristics. We consider the following set of parameter values for bifurcation analysis.

Table 1: Parameter values

Cases	Fixed parameters and initial conditions	varying parameter
Case (i)	$r = 4.5,\ b = 2.26,\ d = 0.5,\ h_1 = 1.3,\ h_2 = 1.1,\ x_0 = 0.6,\ y_0 = 0.006$	$0.6 \leq h \leq 0.85$
Case (ii)	$r = 3.5,\ b = 4.6,\ d = 0.5, h_1 = 1.3,\ h_2 = 1.1,\ x_0 = 0.3,\ y_0 = 0.2$	$0.7 \leq h \leq 0.85$
Case (iii)	$r = 3.5,\ b = 4.6,\ d = 0.5, h = 0.774336,\ h_2 = 1.1,\ x_0 = 0.3,\ y_0 = 0.2$	$0.8 \leq h_1 \leq 1.5$
Case (iv)	$r = 3.5,\ b = 4.6,\ d = 0.5, h = 0.774336,\ h_1 = 1.3,\ x_0 = 0.3,\ y_0 = 0.2$	$0.4 \leq h_2 \leq 1.4$

Example 5.1. Period-Doubling bifurcation of the model (2.4) at P_2 with respect to bifurcation parameter h. We take parameters values as in case (i) of table (1). The positive fixed point of (2.4) for these parametric values is $P_2(0.707965, 0.00626517)$. The eigenvalues of $J(P_2)$ for $h = 0.629185$ are $\lambda_1 = -1, \lambda_2 = 0.995516$, indicating that the model (2.4) is experiencing period doubling bifurcation at $P_2(0.707965, 0.00626517)$ as the bifurcation parameter h crosses $h = H_1 = 0.629185$. Figures (1a, 1b) show bifurcation diagrams for both prey and predator populations, respectively, for $h \in [0.6, 0.85]$. The MLE is plotted in figure (1c).

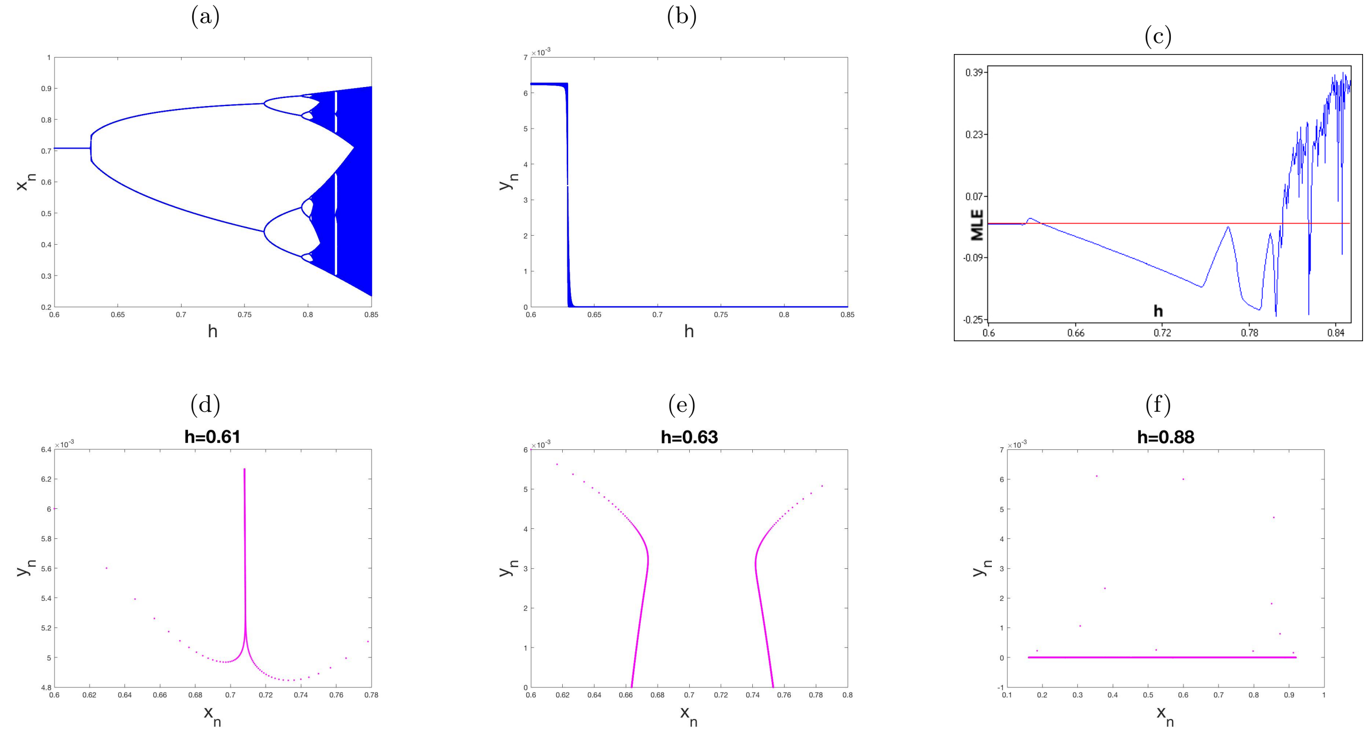

Figure 1: Bifurcation diagrams, MLE graph, phase portraits for some values of h for case (i) set of values of table (1).

The fixed point P_2 is locally asymptotically stable for these parametric values if and only if $0 < h < 0.629185$. Figures (1d,1e,1f) show phase portraits of the model (2.4) for various values of h. These figures express that fixed point $P_2(0.707965, 0.00626517)$ is locally asymptotically stable for $0 < h < 0.629185$, but loses its stability at $h = 0.629185$, where the model (2.4) undergoes period-doubling bifurcation.

Example 5.2. Neimark-Sacker bifurcation of the model (2.4) at P_2. We take parameters values as in case (ii) of table (1). The positive fixed point of (2.4) for these parametric values is $P_2(0.347826, 0.213611)$. The eigenvalues of $J(P_2)$ for $h = 0.774336$ are $\lambda_1 = 0.528665 - 0.848831i, \lambda_2 = 0.528665 + 0.848831i$ with $|\lambda_{1,2}| = 1$, indicating that the model (2.4) is experiencing Neimark-Sacker bifurcation at $P_2(0.347826, 0.213611)$ as the bifurcation parameter h crosses $h = H_2 = 0.774336$. Figures (2a), (2b) depict bifurcation diagrams for both prey and predator populations, respectively, for $h \in [0.7, 0.85]$. The MLE is plotted in figure (2c).

The fixed point P_2 is locally asymptotically stable for these parametric values if and only if $h < 0.774336$. Figures (2d,2e,2f) show phase portraits of the model (2.4) for various values of h. From the figures, it is evident that fixed point P_2 is locally asymptotically stable for $h < 0.774336$, but it loses its stability at $h = 0.774336$, where the model (2.4) undergoes Neimark-Sacker bifurcation. For $h \geq 0.774336$ a smooth invariant curve emerges and it increases its radius as h is increasing. By increasing the value of h, the invariant curve disappears suddenly and some periodic orbit appears and then again we have an invariant curve in place of a periodic orbit. We observe that large values of h lead to the appearance of a strange chaotic attractor as presented in figure (2f).

It is described in figures (3a,3b) that harvesting effect on prey population leads to complex dynamics by using the set of values described in case (iii) of table (1). Similarly, it is described in figures (3c, 3d) that harvesting effect on predator population leads to complex dynamics by using the set of values described in case (iv) of table (1).

6 Conclusion

In this paper, we examine stability and local bifurcation at unique positive fixed point of a discrete-time predator-prey model with harvesting effects on both predator and prey. Some numerical simulations are presented to justify our theoretical conclusions. Moreover, It is discussed that harvesting effects lead to complex dynamics in the model. For larger values of harvesting effects, the model shows stability but for smaller values of harvesting the model experiences

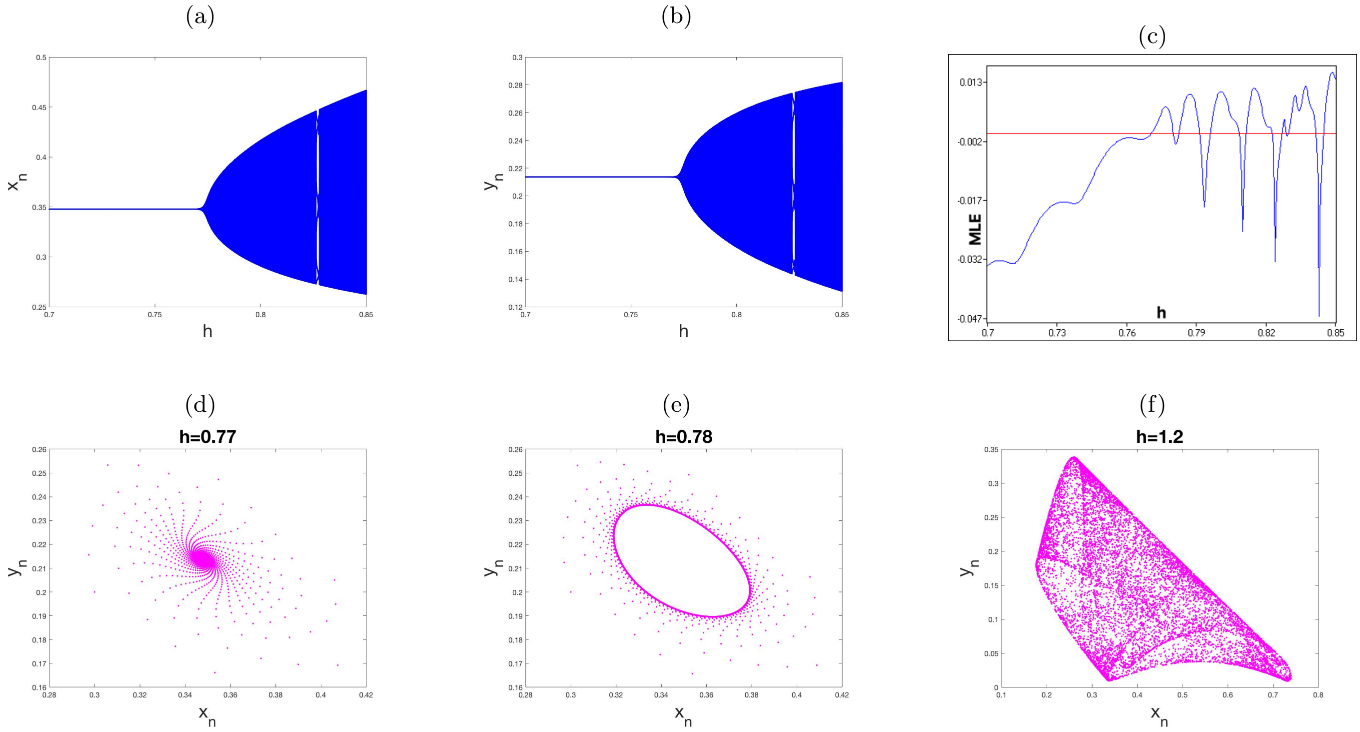

Figure 2: Bifurcation diagrams, MLE graph, phase portraits for some values of h for case (ii) set of values of table (1).

bifurcation. The harvesting effect on prey and predator can stabilize the model. The system (2.4) is shown to have three fixed points. The trivial fixed point P_0 always exists, but harvesting impacts the existence of the boundary and interior fixed points P_1 and P_2. By analyzing period-doubling and Neimrk-Sacker bifurcations, one may determine the significance of the harvesting rate parameters h_1 and h_2. In other words, when the harvesting rate parameters h_1 or h_2 vary, not only does the number of fixed points of the model (2.4) change, but also different types of bifurcations arise.

Appendix A

$$
\begin{aligned}
D_1 &= \frac{2b(A_1^3 b - A_1^2 b\sqrt{A_1^2 - 4A_2} + A_2 r\sqrt{A_1^2 - 4A_2} + A_1 A_2(-2b + r))}{A_2 r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)}, \\
D_2 &= \frac{2A_1 b(A_1^4 b - A_1^3 b\sqrt{A_1^2 - 4A_2} + A_1 A_2(b - r)\sqrt{A_1^2 - 4A_2} - 2A_2^2 r + A_1^2 A_2(-3b + r))}{A_2^2 r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)}, \\
D_3 &= -\Bigg(2b\Bigg(A_2^2 r\sqrt{A_1^2 - 4A_2} - A_1^5(b + r) + A_1^4(b + r)\sqrt{A_1^2 - 4A_2} \\
&\quad - A_1^2 A_2(2b + 3r)\sqrt{A_1^2 - 4A_2} - A_1 A_2^2(2b + 5r) + A_1^3 A_2(4b + 5r)\Bigg)\Bigg) \\
&\quad \Bigg/ \Bigg(A_2^2 r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)\Bigg), \\
D_4 &= -\Bigg(2\Bigg(-A_1^4 b + A_1^2 b(2A_2 + (h_1 - r)\sqrt{A_1^2 - 4A_2}) + 2A_1 A_2 b(h_1 - r) - 2A_2^2 r \\
&\quad + A_1^3 b(\sqrt{A_1^2 - 4A_2} - h_1 + r)\Bigg)\Bigg) \Bigg/ \Bigg((-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)r\Bigg),
\end{aligned}
$$

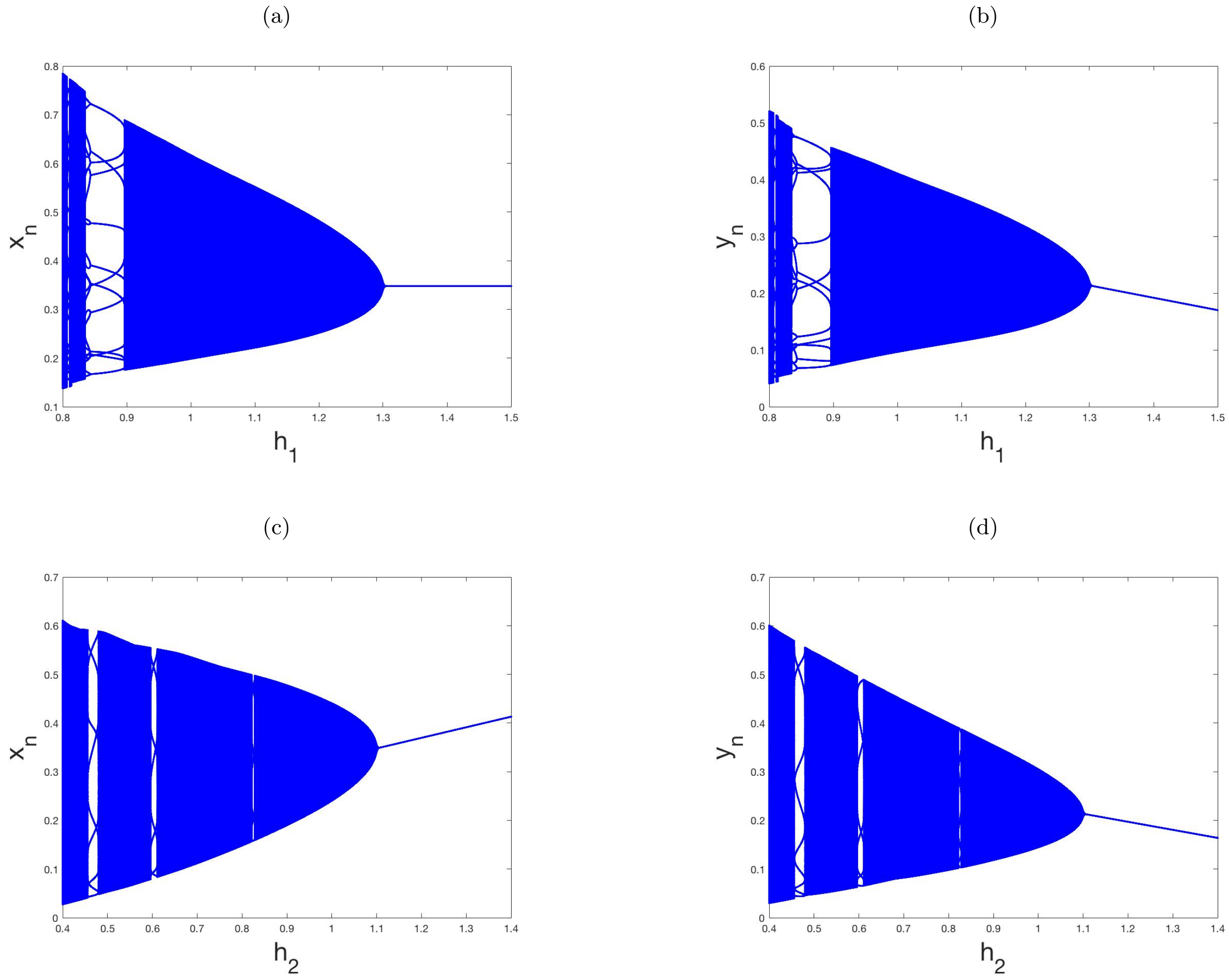

Figure 3: Bifurcation diagrams for cases (iii) and (iv) set of values of table (1).

$$D_5 = \frac{4b\Big(- A_1^3 b + A_1^2 b\sqrt{A_1^2 - 4A_2} + A_1 A_2(2b - r) - A_2 r\sqrt{A_1^2 - 4A_2}\Big)}{r(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},$$

$$D_6 = -\frac{2\Big(- A_1^4 + A_1^3\sqrt{A_1^2 - 4A_2} + 4A_1^2 A_2 - 2A_1 A_2\sqrt{A_1^2 - 4A_2} - 2A_2^2\Big)\Big(A_1^2 b + A_1 b(h_1 - r) + A_2 r\Big)}{rA_2(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},$$

$$D_7 = \frac{2A_1 b\Big(- A_1^4 b + A_1^3 b\sqrt{A_1^2 - 4A_2} + A_1^2 A_2(3b - r) + 2A_2^2 r + A_1 A_2(-b + r)\sqrt{A_1^2 - 4A_2}\Big)}{rA_2(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},$$

$$E_1 = -\frac{2b(r\sqrt{A_1^2 - 4A_2} + A_1(2b + r))}{r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},\ E_2 = \frac{2A_1 b(2A_2 r - A_1^2(b + r) + A_1(b + r)\sqrt{A_1^2 - 4A_2})}{A_2 r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},$$

$$E_3 = \Big(2b\Big(- A_1^3 A_2(b - 5r) + A_1 A_2^2(2b - 5r) + A_1^2 A_2(b - 3r)\sqrt{A_1^2 - 4A_2} - A_1^5 r + A_1^4 r\sqrt{A_1^2 - 4A_2} + A_2^2 r\sqrt{A_1^2 - 4A_2}\Big)\Big)\Big/\Big(A_2^2 r(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)\Big),$$

$$E_4 = -\frac{4A_2(A_1^2 b + A_1 b(h_1 - r) + A_2 r)}{r(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)}$$

$$E_5 = \frac{4A_2 b(r\sqrt{A_1^2 - 4A_2} + A_1(2b + r))}{r(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)},$$

$$E_6 = \Big(2\Big(- 2A_2^2 r - A_1^4(b + r) + 2A_1 A_2(b(h_1 - r) - r\sqrt{A_1^2 - 4A_2}) + A_1^3(r\sqrt{A_1^2 - 4A_2} + b(\sqrt{A_1^2 - 4A_2} - h_1 + r)) + A_1^2(b(h_1 - r)\sqrt{A_1^2 - 4A_2} + 2A_2(b + 2r))\Big)\Big) \Big/ \Big(r(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)\Big),$$

$$E_7 = -\frac{2A_1 b\Big(2A_2 r - A_1^2(b + r) + A_1(b + r)\sqrt{A_1^2 - 4A_2}\Big)}{r(-A_1 + \sqrt{A_1^2 - 4A_2})(-A_1^2 + A_1\sqrt{A_1^2 - 4A_2} + 4A_2)}.$$

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