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Fixed point theorem on functional intervals for sum of two operators and application in ODEs

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Abstract

In this paper, we present a generalization of the functional expansion-compression fixed point theorem developed by Avery et al. in [5] to the case of a k-set contraction perturbed by an operator T, where $I - T$ is Lipschitz invertible. The arguments are based upon recent fixed point index theory in cones of Banach spaces. Next, we apply the obtained result to discuss the existence of a nontrivial positive solution to a nonautonomous second order boundary value problem.

Keywords: Fixed point, sum of operators, positive solution, fixed point index, cones
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1 Introduction

This paper is a part of generalization of some results in fixed point theory on cones of Banach spaces for the sum of two operators. More precisely, we are interested in the theorems of functional types and their applications in the study of boundary value problems. Note that, Leggett and Williams [21] were the originators of this class of fixed point theorems. Since then, the literature has had a significant number of functional fixed point theorems developed promptly in different directions, especially those due to Avery et al. [1, 2, 3, 4, 6]. From a mathematical point of view, when functionals are used in applications instead of norms, we get more freedom and flexibility.

Recently, in [14] a new direction of research in the theory of fixed point in ordered Banach spaces for the sum of two operators was opened. Then, several fixed point theorems, including Leggett-Williams theorems type in cones, have been developed (see [9, 10, 16, 17, 18, 20]). These theorems have been applied to discuss the existence of positive solutions for various types of boundary and/or initial value problems (see [15, 16, 18]).

In [5], Avery et al. have developed an extension of the compression-expansion fixed point theorem of functional type by generalizing the underlaying set using functional-type interval which are sets of the form

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \; a < \alpha(x) \text{ and } \beta(x) < b\},$$

where $\mathcal{A}$ is an open subset of a cone $\mathcal{P}$. Motivated by this work, our purpose in this paper is to improve this result by considering the sum of two operators $T + F$, where $I - T$ is Lipschitz invertible and F a k-set contraction. The

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main tool used is a recent fixed point index theory in cones for this class of mappings developed by Mebarki et al. in [14, 18].

The paper is organized as follows: In Section 2, we provide some background material. In Section 3, we present our main contribution. In Section 4, we give an application to demonstrate how our theoretical result can improve conditions in applications to boundary value problems. We conclude the paper by giving in Section 5 an example with numerical computations.

2 Preliminaries

Definition 2.1. A closed convex set $\mathcal{P}$ in a Banach space E is said to be a cone if

1. $\lambda x \in \mathcal{P}$ for any $\lambda \geq 0$ and for any $x \in \mathcal{P}$,
2. $x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.

Definition 2.2. A subset D of a topological space is said to be relatively compact if its closure is compact, i.e. $\overline{D}$ is compact.

The interval $[a, b]$ is compact in $\mathbb{R}$ by Heine-Borel and (a, b) is relatively compact as $\overline{(a, b)} = [a, b]$ is compact. Note that (a, b) is not compact and every compact subset of a metric space is closed, hence, relatively compact. Let E be a real Banach space.

Definition 2.3. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The concept of set contraction is related to Kuratowski's measure of noncompactness χ.

Definition 2.4. [19] Let E be a real Banach space and Ω_E be the family of all nonempty and bounded subsets of E. The Kuratowski measure of noncompactness $\chi : \Omega_E \to [0, \infty)$ is defined by:

$$\chi(A) = \inf\{\delta > 0 : \; A = \bigcup_{i=1}^{n} A_i, \; A_i \subset E \text{ and } \; diam(A_i) \leq \delta, \text{ for all } i = 1, ..., n\} \tag{2.1}$$

where $diam(A_i) = \sup\{\|x - y\|_E, \; x, y \in A_i\}$ is the diameter of A_i.

For the main properties of measure of noncompactness and some of its applications we refer the reader to [7, 8, 11, 12, 13].

Proposition 2.5. Let E be a Banach space, $\lambda \in \mathbb{R}$ and $M, N \subset \Omega_E$. Then the Kuratowski measure of noncompactness χ has the following properties:

(i) $\chi(M \cup N) = \max\{\chi(M), \chi(N)\}$;

(ii) $\chi(M + N) \leq \chi(M) + \chi(N)$;

(iii) $\chi(\lambda M) = |\lambda| \chi(M)$;

(iv) $\chi(M) \leq \chi(N)$ for $M \subset N$;

(v) $\chi(convM) = \chi(M)$, $convM$ denotes the convex closure of M.

(vi) $\chi(M) = 0$ if and only if M is precompact.

Definition 2.6. A mapping $F : E \to E$ is said to be a k-set contraction if it is continuous, bounded and there exists a constant $k \geq 0$ such that

$$\chi(F(D)) \leq k\chi(D),$$

for any bounded set $D \subset E$. The mapping F is said to be a strict set contraction if $k < 1$.

Obviously, if $F : E \to E$ is a completely continuous mapping, then F is 0-set contraction. In fact,

$$\begin{aligned} F \text{ is completely continuous} \quad &\Rightarrow \quad \overline{F(D)} \text{ is compact, for any bounded set } D \subset E, \\ &\Rightarrow \quad \chi(F(D)) = \chi(\overline{F(D)}) = 0, \\ &\Rightarrow \quad F \text{ is } 0-\text{set contraction.} \end{aligned}$$

Definition 2.7. Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. An operator $T : D \to X$ is said to be γ-Lipschitz invertible on D if it is invertible and its inverse is Lipschitzian on $T(D)$ with constant γ.

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$ and U is a bounded open subset of $\mathcal{P}$ and we will denote $\mathcal{P} \setminus \{0\}$ by $\mathcal{P}^*$.
The fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P})$ defined by

$$i_* \left(T + F, U \cap \Omega, \mathcal{P}\right) = \begin{cases} i\left((I - T)^{-1}F, U, \mathcal{P}\right), & \text{if } U \cap \Omega \neq \emptyset \\ 0, & \text{if } U \cap \Omega = \emptyset, \end{cases} \tag{2.2}$$

is well defined whenever $T : \Omega \to E$ be a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ be a k-set contraction with $0 \leq k < \gamma^{-1}$ and $F(\overline{U}) \subset (I - T)(\Omega)$.
The proof of different results on the fixed point index i_* invokes the following main properties of this index (see [14, 18]):

(a) (Normalization). If $Fx = y_0$, for all $x \in \overline{U}$, where $(I - T)^{-1}y_0 \in U \cap \Omega$, then

$$i_* \left(T + F, U \cap \Omega, \mathcal{P}\right) = 1.$$

(b) (Additivity). For any pair of disjoint open subsets $U_1, U_2 \subset U$ such that $T + F$ has no fixed point on $(\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega$, we have

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i_*(T + F, U_1 \cap \Omega, \mathcal{P}) + i_*(T + F, U_2 \cap \Omega, \mathcal{P}).$$

(c) (Homotopy invariance). The fixed point index $i_*(T + H(t, .), U \cap \Omega, \mathcal{P})$ does not depend on the parameter $t \in [0, 1]$, where

- **(i)** $H : [0, 1] \times \overline{U} \to E$ is continuous and $H(t, x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,
- **(ii)** $H([0, 1] \times \overline{U}) \subset (I - T)(\Omega)$,
- **(iii)** $H(t, .) : \overline{U} \to E$ is a ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ for all $t \in [0, 1]$,
- **(iv)** $Tx + H(t, x) \neq x$ for all $t \in [0, 1]$ and $x \in \partial U \cap \Omega$.

(d) (Solvability). If $i_*(T + F, U \cap \Omega, \mathcal{P}) \neq 0$, then $T + F$ has a fixed point in $U \cap \Omega$.

Definition 2.8. A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y),$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$. A map β is said to be a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y),$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$.

Definition 2.9. Let X be a topological space, $D \subset X$ any subset. A set U is relatively open in D if there is an open set Ω in X such that $U = \Omega \cap D$.

Note that set $U \subset D$ can be relatively open to D without being an open set of X. For example:

1. $D = [0,1]$, the half-open interval $]a,1]$ is open in D for every $0 \le a < 1$, since $]a,1] = [0,1] \cap]a,3[$. It is clear that $]a,1]$ is not open in $\mathbb{R}$.
2. $X = \mathbb{R}^2$, $D = \mathbb{R} \times \{0\}$ and $U =]0,1[\times \{0\}$.

Definition 2.10. Let $\mathcal{A}$ be a relatively open subset of a cone $\mathcal{P}$, α be a nonnegative continuous concave functional on $\mathcal{P}$; β be a nonnegative continuous convex functional on $\mathcal{P}$ and let a, b be two positive real numbers, then the set

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \ a < \alpha(x) \text{ and } \beta(x) < b\},$$

is an interval of functional type.

Note that $\mathcal{A}(\beta, b, \alpha, a)$ is a subset of $\mathcal{P}(\beta, b, \alpha, a)$ defined by:

$$\mathcal{P}(\beta, b, \alpha, a) = \{x \in \mathcal{P} : \ a < \alpha(x) \text{ and } \beta(x) < b\}.$$

3 Main results

In the sequel, we will establish an extension of [5, Theorem 3.1] which guarantees the existence of at least one nontrivial positive solution to some equations of the form $Tx + Fx = x$ posed on cones of a Banach space.

Theorem 3.1. Let E be a Banach space; $\mathcal{P} \subset E$ be a cone, $\mathcal{A}$ be a relatively open subset of $\mathcal{P}$, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$ and β and θ are nonnegative continuous convex functionals on $\mathcal{P}$.

Let $T : \Omega \subset \mathcal{P} \to E$ be a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \mathcal{P} \to E$ be a k-set contraction mapping with $0 \le k < \gamma^{-1}$. Assume that there exist four nonnegative numbers a, b, c and d, and $\omega_0 \in \mathcal{A}(\beta, b, \alpha, a) \cap \mathcal{A}(\theta, c, \psi, d)$ such that

$$(I - T)^{-1}\omega_0 \in \mathcal{A}(\beta, b, \alpha, a), \tag{3.1}$$

$$\lambda F(\mathcal{A}(\beta, b, \alpha, a)) + (1 - \lambda)\omega_0 \subset (I - T)(\Omega), \text{ for all } \lambda \in [0,1], \tag{3.2}$$

$(\mathcal{H}_1)$ $\mathcal{A}(\beta, b, \alpha, a)$ is bounded, and $\partial\mathcal{A} \cap \overline{\mathcal{A}(\beta, b, \alpha, a)} = \emptyset$;

$(\mathcal{H}_2)$ if $x \in \mathcal{P}$ with $\alpha(x) = a$ and either $\theta(x) \le c$ or $\theta(Tx + Fx) > c$, then $\alpha(Tx + Fx) > a$;

$(\mathcal{H}_3)$ if $x \in \mathcal{P}$ with $\beta(x) = b$ and either $\psi(Tx + Fx) < d$ or $\psi(x) \ge d$, then $\beta(Tx + Fx) < b$;

$(\mathcal{H}_4)$ if $x \in \mathcal{P}$ with $\alpha(x) = a$, then $\alpha(Tx + \omega_0) > a$ and $\theta(Tx + \omega_0) \le c$;

$(\mathcal{H}_5)$ if $x \in \mathcal{P}$ with $\beta(x) = b$, then $\beta(Tx + \omega_0) < b$ and $\psi(Tx + \omega_0) \ge d$;

then $T + F$ has at least one fixed point $x^* \in \overline{\mathcal{A}(\beta, b, \alpha, a)}$.

Proof .

Claim 1: $Tx + Fx \ne x$ for all $x \in \partial\mathcal{A}(\beta, b, \alpha, a)$.

The functional interval $\mathcal{A}(\beta, b, \alpha, a) = \mathcal{A} \cap \mathcal{P}(\beta, b, \alpha, a)$, then by the condition $(\mathcal{H}_1)$,

$$\partial\mathcal{A}(\beta, b, \alpha, a) = \overline{\mathcal{A}} \cap \partial\mathcal{P}(\beta, b, \alpha, a).$$

Suppose that there exist $z_0 \in \partial\mathcal{A}(\beta, b, \alpha, a)$ such that $Tz_0 + Fz_0 = z_0$. Since $z_0 \in \partial\mathcal{P}(\beta, b, \alpha, a)$ so either $\beta(z_0) = b$ or $\alpha(z_0) = a$.

Case 1: $\beta(z_0) = b$.

If $\psi(Tz_0 + Fz_0) < d$ or $\psi(z_0) = \psi(Tz_0 + Fz_0) \ge d$, then by the condition $(\mathcal{H}_3)$,

$$\beta(Tz_0 + Fz_0) < b.$$

Hence we have that $Tz_0 + Fz_0 \ne z_0$.

Case 2: $\alpha(z_0) = a$.

If $\theta(z_0) = \theta(Tz_0 + Fz_0) \le c$ or $\theta(Tz_0 + Fz_0) > c$, then by the condition $(\mathcal{H}_2)$,

$$\alpha(Tz_0 + Fz_0) > a.$$

Hence we have that $Tz_0 + Fz_0 \neq z_0$.

Therefore, $T + F$ does not have any fixed points on $\partial\mathcal{A}(\beta, b, \alpha, a)$.

Claim 2: Let $H : [0,1] \times \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ be defined by

$$H(t, x) = tFx + (1 - t)\omega_0.$$

We have

(i) $H : [0,1] \times \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ is continuous and $H(t, x)$ is uniformly continuous in t with respect to $x \in \overline{\mathcal{A}(\beta, b, \alpha, a)}$,

(ii) $H([0,1] \times (\mathcal{A}(\beta, b, \alpha, a))) \subset (I - T)(\Omega)$,

(iii) $H(t, .) : \overline{\mathcal{A}(\beta, b, \alpha, a)} \to E$ is a k-set contraction with $0 \le k < \gamma^{-1}$ for all $t \in [0,1]$,

(iv) $Tx + H(t, x) \neq x$, for all $t \in [0,1]$ and $x \in \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$.

Properties $(i), (ii)$ and (iii) follow directly from the definition of H and the conditions on F and T. We only check (iv). Suppose the contrary, that is, there would exists $(t_1, x_1) \in [0,1] \times \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$ such that $Tx_1 + H(t_1, x_1) = x_1$. Since $x_1 \in \partial\mathcal{A}(\beta, b, \alpha, a) = \overline{\mathcal{A}} \cap \partial\mathcal{P}(\beta, b, \alpha, a)$, so $x_1 \in \partial\mathcal{P}(\beta, b, \alpha, a)$ we have that $\beta(x_1) = b$ or $\alpha(x_1) = a$.

(1) $\beta(x_1) = b$. Either $\psi(Tx_1 + Fx_1) < d$ or $\psi(Tx_1 + Fx_1) \ge d$.

If $\psi(Tx_1 + Fx_1) < d$, by the condition $(\mathcal{H}_3)$ we have that $\beta(Tx_1 + Fx_1) < b$, thus from the convexity of β and the condition $(\mathcal{H}_5)$ it follows that

$$\begin{aligned} b = \beta(x_1) &= \beta(Tx_1 + H(t_1, x_1)) \\ &= \beta(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \beta(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\ &\le t_1\beta(Tx_1 + Fx_1) + (1 - t_1)\beta(Tx_1 + \omega_0) \\ &< t_1 b + (1 - t_1)b \\ &= b, \end{aligned}$$

which is a contradiction.

If $\psi(Tx_1 + Fx_1) \ge d$, we have that $\psi(x_1) \ge d$ since

$$\begin{aligned} \psi(x_1) &= \psi(Tx_1 + H(t_1, x_1)) \\ &= \psi(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \psi(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\ &\ge t_1\psi(Tx_1 + Fx_1) + (1 - t_1)\psi(Tx_1 + \omega_0) \\ &\ge d, \end{aligned}$$

and thus by the condition $(\mathcal{H}_3)$ we have that $\beta(Tx_1 + Fx_1) < b$, which is the same contradiction we arrived at in the previous subcase.

(2) $\alpha(x_1) = a$. Either $\theta(Tx_1 + Fx_1) \le c$ or $\theta(Tx_1 + Fx_1) > c$.

If $\theta(Tx_1 + Fx_1) > c$, by the condition $(\mathcal{H}_2)$ we have that $\alpha(Tx_1 + Fx_1) > a$, thus from the concavity of α and the condition $(\mathcal{H}_4)$, it follows that

$$\begin{aligned} a = \alpha(x_1) &= \alpha(Tx_1 + H(t_1, x_1)) \\ &= \alpha(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\ &= \alpha(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\ &\ge t_1\alpha(Tx_1 + Fx_1) + (1 - t_1)\alpha(Tx_1 + \omega_0) \\ &> a, \end{aligned}$$

which is a contradiction.

If $\theta(Tx_1 + Fx_1) \leq c$, we have that $\theta(x_1) \leq c$ since

$$\begin{aligned}
\theta(x_1) &= \theta(Tx_1 + H(t_1, x_1)) \\
&= \theta(Tx_1 + t_1 Fx_1 + (1 - t_1)\omega_0) \\
&= \theta(t_1(Tx_1 + Fx_1) + (1 - t_1)(Tx_1 + \omega_0)) \\
&\leq t_1\theta(Tx_1 + Fx_1) + (1 - t_1)\theta(Tx_1 + \omega_0) \\
&\leq c,
\end{aligned}$$

and thus by the condition $(\mathcal{H}_2)$ we have that $\alpha(Tx_1 + Fx_1) > a$, which is the same contradiction we arrived at in the previous subcase.

Therefore, $Tx + H(t, x) \neq x$, for all $t \in [0, 1]$ and $x \in \partial\mathcal{A}(\beta, b, \alpha, a) \cap \Omega$. By the homotopy invariance property and the normality property of the fixed point index i_*

$$i_*(T + F, \mathcal{A}(\beta, b, \alpha, a), \mathcal{P}) = i_*(T + \omega_0, \mathcal{A}(\beta, b, \alpha, a), \mathcal{P}) = 1.$$

Then $T + F$ has at least one fixed point in $\mathcal{A}(\beta, b, \alpha, a)$. $\square$

4 Application

In this section, we will investigate the equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \tag{4.1}$$

with boundary conditions

$$x(0) = x'(1) = 0, \tag{4.2}$$

where $f : (0, 1) \times [0, \infty) \to [0, \infty)$ is a continuous function. The Green's function $G(t, s)$ for $-x''(t) = 0,\ t \in (0, 1)$ satisfying (4.2) is

$$\begin{aligned}
G(t, s) &= \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1, \end{cases} \\
&= \min\{t, s\},\ (t, s) \in [0, 1] \times [0, 1].
\end{aligned}$$

Note that, if x is a fixed point of the operator S defined by

$$Sx(t) = \int_0^1 G(t, s) f(s, x(s))\, ds,$$

where then x is a solution of the boundary value problem (4.1)-(4.2). Suppose that

(H$_1$) The functions f satisfy $\tilde{A} \leq f(t, x(t)) \leq a_1(t) + a_2(t)|x(t)|^p$, for $t \in [0, 1]$ and $a_1, a_2 \in \mathcal{C}([0, 1])$, $0 < a_1(t), a_2(t) \leq A$ for $t \in [0, 1]$, where p, A and $\tilde{A}$ are nonnegative constants.

(H$_2$) There exist positive constants $\varepsilon, \eta, a, b, c, d, C_1, C_2, C_3, \rho$ and R such that

$$\begin{gathered}
\varepsilon \in (0, 1), \quad \eta \in (0, 1), \quad 2b \leq \min(R, \rho), \\
A(1 + b^p) < b, \\
\max\left(\frac{a}{2}, \frac{d}{C_2}\right) < C_1 < \min\left(b, \frac{c}{C_3}\right), \\
0 \leq C_3\left((1 - \varepsilon)\frac{1}{\Lambda}(a - C_1) + \varepsilon C_1\right) \leq c, \quad C_2\left((1 - \varepsilon)\Lambda b + \varepsilon C_1\right) \geq d,
\end{gathered} \tag{4.3}$$

where $\Lambda = \frac{\min(\frac{\eta}{8}\tilde{A}, C_1)}{\rho}$.

Remark 4.1. As in [20, Remark 4.1], we can discuss the validity of the inequality (4.3).

4.1 Main result

Theorem 4.2. If $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold then the problem (4.1)-(4.2) has at least one positive solution $x \in \mathcal{C}([0,1])$ such that $a < \min_{t\in[0,1]} x(t) + 2C_1$ and $\max_{t\in[0,1]} x(t) < b$.

Proof .

Define the Banach space $E = \mathcal{C}([0,1])$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and

$$\mathcal{P} = \{x \in E : \ x(t) \geq 0, \, t \in [0,1], \ \min_{t\in[\frac{\eta}{8},\eta]} x(t) \geq \Lambda\|x\|\}, \quad \Omega = \{x \in \mathcal{P} : \ \|x\| \leq \rho\}.$$

For $x \in \mathcal{P}$ define the convex functionals

$$\beta(x) = \max_{t\in[0,1]} |x(t)|, \qquad \theta(x) = C_3 \max_{t\in[0,1]} |x(t)|.$$

For $x \in \mathcal{P}$ define the concave functionals

$$\alpha(x) = \min_{t\in[0,1]} x(t) + 2C_1, \qquad \psi(x) = C_2 \min_{t\in[\frac{\eta}{8},\eta]} x(t).$$

For $x \in \mathcal{P}$ define the operators

$$Tx(t) = (1-\varepsilon)x(t) + (\varepsilon - 1)C_1$$

$$Fx(t) = \varepsilon \int_0^1 G(t,s) f(s, x(s))\, ds + (1-\varepsilon)C_1, \ \ t \in [0,1].$$

Note that if $x \in \mathcal{P}$ is a fixed point of the operator $T + F$ then it is a positive solution to the problem (4.1)-(4.2). We set

$$\mathcal{A} = \mathcal{P} \cap B(0,R) = \{x \in \mathcal{P} : \ \|x\| < R\},$$

$$\mathcal{A}(\beta, b, \alpha, a) = \{x \in \mathcal{A} : \ a < \alpha(x) \text{ and } \beta(x) < b\}.$$

1. For $x, y \in \mathcal{P}$, we have

$$|(I-T)x(t) - (I-T)y(t)| = \varepsilon|x(t) - y(t)|, \ t \in [0,1].$$

Hence

$$\|(I-T)x - (I-T)y\| = \varepsilon\|x - y\|.$$

Thus, $I - T : \mathcal{P} \to E$ is Lipschitz invertible with a constant $\gamma = \frac{1}{\varepsilon}$.

2. Since f is continuous, F is continuous. Also, for $x \in \mathcal{P}$, we have

$$\begin{aligned} |Fx(t)| &= |\varepsilon \int_0^1 G(t,s) f(s,x(s))\, ds + C_1 - \varepsilon C_1| \\ &\leq \varepsilon(a_1(t) + a_2(t)|x(t)|^p) + C_1 \\ &\leq \varepsilon\, A(1 + |x(t)|^p) + C_1 \\ &\leq \varepsilon\, A(1 + \|x\|^p) + C_1 < \infty, \quad t \in [0,1] \end{aligned}$$

and

$$\begin{aligned} |(Fx)'(t)| &= |\varepsilon \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,x(s))\, ds + C_1 - \varepsilon C_1| \\ &\leq |\varepsilon(a_1(t) + a_2(t)|x(t)|^p) + C_1| \\ &\leq \varepsilon\, A(1 + |x(t)|^p) + C_1 \\ &\leq \varepsilon\, A(1 + \|x\|^p) + C_1 < \infty, \quad t \in [0,1]. \end{aligned}$$

Consequently, by Ascoli-Arzelà compactness criteria, $F : \mathcal{P} \to E$ is a completely continuous operator. Then $F : \mathcal{P} \to E$ is a 0-set contraction.

3. Let $\omega_0 \equiv C_1 \in \mathcal{A}(\beta,b,\alpha,a) \cap \mathcal{A}(\theta,c,\psi,d)$ be a constant, we have

$$(I-T)x = \varepsilon x + (1-\varepsilon)C_1$$
$$(I-T)^{-1}x = \frac{x-C_1}{\varepsilon} + C_1,\ x \in \mathcal{P}.$$

Hence,

$$\begin{aligned}
\left|(I-T)^{-1}\omega_0(t)\right| = \left|(I-T)^{-1}C_1\right| &= \left|\frac{C_1-C_1}{\varepsilon} + C_1\right| \\
&= C_1 \\
&< R, \quad t \in [0,1],
\end{aligned}$$

and

$$\min_{t\in[\frac{\eta}{8},\eta]}((I-T)^{-1}\omega_0(t)) = \frac{C_1-C_1}{\varepsilon} + C_1 = C_1 \geq \Lambda C_1.$$

Also,

$$\begin{aligned}
\alpha((I-T)^{-1}\omega_0) = \min_{t\in[0,1]}((I-T)^{-1}\omega_0(t)) + 2C_1 &= \min_{t\in[0,1]}((I-T)^{-1}C_1) + 2C_1 \\
&> \frac{C_1-C_1}{\varepsilon} + 3C_1 \\
&> a,
\end{aligned}$$

and

$$\begin{aligned}
\beta((I-T)^{-1}\omega_0) = \max_{t\in[0,1]}|(I-T)^{-1}\omega_0(t)| &= \max_{t\in[0,1]}\left|\frac{C_1-C_1}{\varepsilon} + C_1\right| \\
&= C_1 \\
&< b.
\end{aligned}$$

Then

$$(I-T)^{-1}\omega_0 \in \mathcal{A}(\beta,b,\alpha,a).$$

4. We have $\mathcal{A}(\beta,b,\alpha,a) \subset \mathcal{P}\cap B(0,b)$, then $\mathcal{A}(\beta,b,\alpha,a)$ is bounded and by construction of $\mathcal{A}$, $\partial\mathcal{A}\cap\overline{\mathcal{A}(\beta,b,\alpha,a)} = \emptyset$. So, the condition $(\mathcal{H}_1)$ holds.
5. Let $x \in \mathcal{P}$ with $\alpha(x) = a$ and either $\theta(x) \leq c$ or $\theta(Tx+Fx) > c$. Then

$$\begin{aligned}
\alpha(Tx+Fx) &= \min_{t\in[0,1]}(Tx(t)+Fx(t)) + 2C_1 \\
&> a.
\end{aligned}$$

Consequently, the condition $(\mathcal{H}_2)$ holds.
6. Let $x \in \mathcal{P}$ with $\beta(x) = b$ and either $\psi(Tx+Fx) < d$ or $\psi(x) \geq d$. Then

$$\begin{aligned}
\beta(Tx+Fx) &= \max_{t\in[0,1]}|Tx(t)+Fx(t)| \\
&= \max_{t\in[0,1]}\left|(1-\varepsilon)x(t) + \varepsilon\int_0^1 G(t,s)f(s,x(s))\,ds\right| \\
&\leq (1-\varepsilon)\max_{t\in[0,1]}|x(t)| + \varepsilon\max_{t\in[0,1]}\int_0^1 |G(t,s)||f(s,x(s))|\,ds \\
&\leq (1-\varepsilon)b + \varepsilon A(1+\|x\|^p) \\
&\leq (1-\varepsilon)b + \varepsilon A(1+b^p) \\
&< b.
\end{aligned}$$

Consequently, the condition $(\mathcal{H}_3)$ holds.
7. Let $x \in \mathcal{P}$ with $\alpha(x) = a$. Then

$$\begin{aligned}
\alpha(Tx+\omega_0) &= \min_{t\in[0,1]}(Tx(t)+C_1) + 2C_1 \\
&= \min_{t\in[0,1]}((1-\varepsilon)x(t) + (\varepsilon-1)C_1 + C_1) + 2C_1 \\
&\geq (1-\varepsilon)\min_{t\in[0,1]}x(t) + \varepsilon C_1 + 2C_1 \\
&= (1-\varepsilon)(a-2C_1) + (\varepsilon+2)C_1 \\
&> a,
\end{aligned}$$

and

$$\begin{aligned}
\theta(Tx+\omega_0) &= C_3 \max_{t\in[0,1]} |Tx(t)+C_1| \\
&= C_3 \max_{t\in[0,1]} |(1-\varepsilon)x(t)+\varepsilon C_1| \\
&\le C_3\left((1-\varepsilon)\max_{t\in[0,1]}|x(t)|+\varepsilon C_1\right) \\
&\le C_3\left((1-\varepsilon)\frac{1}{\Lambda}\min_{t\in[\frac{\eta}{8},\eta]} x(t)+\varepsilon C_1\right) \\
&\le C_3\left((1-\varepsilon)\frac{1}{\Lambda}(a-C_1)+\varepsilon C_1\right) \\
&\le c.
\end{aligned}$$

Consequently, the condition($\mathcal{H}_4$) holds.

8. Let $x\in\mathcal{P}$ with $\beta(x)=b$. Then

$$\begin{aligned}
\beta(Tx+\omega_0) &= \max_{t\in[0,1]} |Tx(t)+C_1| \\
&= \max_{t\in[0,1]} |(1-\varepsilon)x(t)+\varepsilon C_1| \\
&\le (1-\varepsilon)\max_{t\in[0,1]}|x(t)|+\varepsilon C_1 \\
&\le (1-\varepsilon)b+\varepsilon C_1 \\
&< b,
\end{aligned}$$

and

$$\begin{aligned}
\psi(Tx+\omega_0) &= C_2 \min_{t\in[\frac{\eta}{8},\eta]} (Tx(t)+C_1) \\
&= C_2\left(\min_{t\in[\frac{\eta}{8},\eta]}(1-\varepsilon)x(t)+\varepsilon C_1\right) \\
&\ge C_2\left((1-\varepsilon)\min_{t\in[\frac{\eta}{8},\eta]} x(t)+\varepsilon C_1\right) \\
&\ge C_2\left((1-\varepsilon)\Lambda\|x\|+\varepsilon C_1\right) \\
&\ge C_2\left((1-\varepsilon)\Lambda b+\varepsilon C_1\right) \\
&\ge d.
\end{aligned}$$

Consequently, the condition ($\mathcal{H}_5$) holds.

9. Let $\lambda\in[0,1]$ is fixed and $\tilde{x}\in\mathcal{A}(\beta,b,\alpha,a)$ is arbitrary chosen. Set

$$w(t)=\lambda\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds+(1-\lambda)C_1.$$

We have that $w(t)\ge 0$, $t\in[0,1]$, and

$$w(t)\le A(1+|\tilde{x}(t)|^p)+(1-\lambda)C_1 < A\left(1+b^p\right)+b, \quad t\in[0,1].$$

So,

$$\|w\| < A\left(1+b^p\right)+b\le\rho,$$

and

$$\begin{aligned}
\min_{t\in[\frac{\eta}{8},\eta]} w(t) &= \min_{t\in[\frac{\eta}{8},\eta]}\left(\lambda\int_0^1 G(t,s)f(s,\tilde{x}(s))\,ds+(1-\lambda)C_1\right) \\
&\ge \lambda\frac{\eta}{8}\tilde{A}+(1-\lambda)C_1 \\
&\ge \frac{\min(\frac{\eta}{8}\tilde{A},C_1)}{\rho}\rho \\
&\ge \Lambda\|w\|.
\end{aligned}$$

Therefore $\omega \in \Omega$. Also, we have

$$\begin{aligned}
\lambda F\tilde{x}(t) + (1-\lambda)\omega_0 &= \lambda \left[\varepsilon \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds + (1-\varepsilon)C_1\right] + (1-\lambda)C_1 \\
&= \lambda\varepsilon \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds + \lambda(1-\varepsilon)C_1 + (1-\lambda)C_1 \\
&= \lambda\varepsilon \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds - \lambda\varepsilon C_1 + C_1 \\
&= \lambda\varepsilon \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds - \lambda\varepsilon C_1 + C_1 - \varepsilon C_1 + \varepsilon C_1 \\
&= \varepsilon \left(\lambda \int_0^1 G(t,s) f(s,\tilde{x}(s))\, ds + (1-\lambda)C_1\right) + (1-\varepsilon)C_1 \\
&= \varepsilon w(t) + (1-\varepsilon)C_1 \\
&= (I-T)w(t), \quad t \in [0,1].
\end{aligned}$$

Then

$$\lambda F(\mathcal{A}(\beta, b, \alpha, a)) + (1-\lambda)\omega_0 \subset (I-T)(\Omega), \text{ for all } \lambda \in [0,1].$$

Hence, all the conditions of Theorem 3.1 are satisfied, and it follows that the operator $T+F$ has at least one fixed point $x^* \in \mathcal{A}(\beta, b, \alpha, a)$, which is a positive solution of the problem (4.1)-(4.2). This completes the proof. $\square$

5 Example

Let,

$$\varepsilon = \eta = \frac{1}{2}, \quad a = 8, \quad b = 10, \quad c = 12, \quad d = \frac{1}{3}, \quad C_1 = 8, \quad C_2 = C_3 = 1,$$

$$A = 2, \quad \tilde{A} = \frac{1}{4},$$

$$p = \frac{1}{2}, \quad R = 25, \quad \rho = 25.$$

So,

$$2b = 20 \leq \min(\rho, R) = 25, \quad A(1+b^p) = 2(1+10^{\frac{1}{2}}) = 8.3246 < b = 10,$$

$$\max\left(\frac{a}{2}, \frac{d}{C_2}\right) = \max\left(4, \frac{1}{3}\right) = 4 < C_1 = 8 < \min\left(b, \frac{c}{C_3}\right) = \min(10,12) = 10,$$

$$\Lambda = \frac{\min(\frac{\eta}{8}\tilde{A}, C_1)}{\rho} = 6.25 \times 10^{-4},$$

$$C_3\left((1-\varepsilon)\frac{1}{\Lambda}(a-C_1) + \varepsilon C_1\right) = 4 \leq c = 12,$$

$$C_2\left((1-\varepsilon)\Lambda b + \varepsilon C_1\right) = 4.0031 \geq d = \frac{1}{3}.$$

Thus, $(\mathbf{H}_2)$ holds. Now, by our main result, it follows that the boundary value problem:

$$\begin{cases} x''(t) = \frac{t+1}{4} + 2\frac{\sqrt{x(t)}}{1+x(t)^4}, \quad t \in (0,1), \\ x(0) = x'(1) = 0, \end{cases}$$

has at least one positive solution.

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