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On a generalized Caputo for Langevin fractional differential equations in Banach spaces

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Abstract

In this research article, we study the existence, uniqueness and Ulam-Hyers stability of solutions in connection to the generalized Caputo Langevin fractional differential equations in Banach Space. The existence, uniqueness, and stability in the sense of Ulam are established for the proposed system. Our approach is based on the technique of measure of noncompactness combined with the Mönch fixed point theorem, the implementation Banach contraction principle fixed point theorem. Moreover, the Ulam–Hyers stability is discussed by utilizing the Urs's. Lastly, we deliver an example to check the efficiency and accuracy of the proposed methods.

Keywords: Fractional Langevin equation, Generalized Caputo, Ulam-Hyers stability, Kuratowski measures of noncompactness, fixed point theorems, Banach space
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1 Introduction

Probably, the fractional differential equation is better and more accurate than the integer-order differential equations regarding to there natures in describing some phenomena and processes of many scientific and engineer fields, one can get a good opinion in consulting various studies of fractional differential equations by the mean of published papers involving fractional derivatives, in particular Liouville and Caputo fractional derivatives. The detailed literature including the basic theory of fractional calculus can be found in [27, 31, 36, 37, 43]. More recently, in [10] Almeida proposed a new fractional derivative with respect to a kernel function named by ψ–Caputo fractional derivative and extended the work of several famous scientists [31, 33]. Since then several studies showed interest in the ψ–Caputo fractional derivative, we suggest some works [3, 10, 11, 12, 15, 39]. In the same context, theoretical results concerning existence, uniqueness, and Ulam–stability of solutions to fractional differential equations with various definitions of well known fractional derivatives can be found in the articles [1, 8, 16, 17, 18, 19, 20, 28, 29] and the references therein as well as to the books of several famous scientists [1, 7, 43].

On the other hand, the Langevin equation was formulated by Paul Langevin in 1908 to describe the evolution of physical phenomena in fluctuating environments such as Brownian motion [32]. After that, various generalizations of the Langevin equation were proposed and studied by many scholars we mention here some works [23]. Recently, many researchers have investigated sufficient conditions for the existence, uniqueness, and stability of solutions for the nonlinear fractional Langevin equations involving various types of fractional derivatives and by using different

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types of methods such as (standard fixed point theorems, Leray-Schauder theory, variational methods, etc.) For more details see [8, 19, 20, 21, 34, 41]. However, to the best of our knowledge, few results can be found on the existence and uniqueness of solutions for fractional Langevin equations with the ψ–Caputo fractional derivative except that of [39].

Motivated by novel developments in ψ-fractional calculus, in the present research, we investigate the existence and uniqueness of the solutions and UH-type stability of the nonlinear fractional Langevin differential equations (FLDEs) described by

$$\begin{cases} \mathbb{D}^{\beta;\psi}_{a^+}(\mathbb{D}^{\alpha;\psi}_{a^+}+\lambda)\omega(\varsigma)=\mathbb{F}(\varsigma,\omega(\varsigma)), \quad \varsigma\in \mathrm{J}=[a,b],\\ \omega(a)=0,\\ \omega(b)+\lambda\mathbb{I}^{\alpha;\psi}_{a^+}\omega(b)=0,\\ \mathbb{D}^{\alpha;\psi}_{a^+}\omega(\xi)+\lambda\omega(\xi)=0, \quad \xi\in]a,b]. \end{cases} \tag{1.1}$$

where $\mathbb{D}^{\varepsilon}_{a^+}$ denote the ψ–Caputo fractional derivatives of order $\varepsilon\in\{\alpha,\beta\}$ such that $\alpha\in(0,1]$, $\beta\in(1,2]$ and $\mathbb{I}^{\alpha;\psi}_{a^+}$ is the fractional ψ–integral of the Reimann-Liouville type. $\mathbb{F}:\mathrm{J}\times\mathbb{X}\to\mathbb{X}$ are continuous functions. λ are real constants.

Here is a brief outline of the paper. The section 2 provides the definitions and preliminary results that we will need to prove our main results. In Section 3, we establish existence and uniqueness and stability in the sense of Ulam for system (1.1).

2 Preliminaries and lemmas

We start this section by introducing some necessary definitions and basic results required for further developments. Consider the space of real and continuous functions $\mathbb{C}=C(\mathrm{J},\mathbb{X})$ space with the norm

$$\|\omega\|_\infty=\sup\{\|\omega(\varsigma)\|: \quad \varsigma\in\mathrm{J}\}.$$

And $\mathfrak{M}_{\mathcal{U}}$ represents the class of all bounded mappings in $\mathbb{C}$. Let $L^1(\mathrm{J},\mathbb{X})$ be the Banach space of measurable functions $\omega:\mathrm{J}\to\mathbb{X}$ which are Bochner integrable, equipped with the norm

$$\|\omega\|_{L^1}=\int_{\mathrm{J}}|\omega(\varsigma)|\,dt.$$

Let $\psi\in\mathcal{C}^1=C^1(\mathrm{J},\mathbb{R})$ be an increasing differentiable function such that $\psi'(\varsigma)\neq0$, for all $\varsigma\in\mathrm{J}$. Now, we start by defining ψ-FODs as follows:

Definition 2.1. [31] The ψ–Riemann–Liouville fractional integral of order $\alpha>0$ for an integrable function $\omega\colon\mathrm{J}\longrightarrow\mathbb{R}$ is given by

$$\mathbb{I}^{\alpha;\psi}_{a^+}\omega(\varsigma)=\frac{1}{\Gamma(\alpha)}\int_a^\varsigma\psi'(s)(\psi(\varsigma)-\psi(s))^{\alpha-1}\omega(s)\mathrm{ds}, \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2. [31] Let $n-1<\alpha<n$ ($n\in\mathbb{N}$), $\omega:\mathrm{J}\to\mathbb{R}$ is an integrable function, and $\psi\in C^n(\mathrm{J},\mathbb{R})$, the ψ–Riemann–Liouville FOD of a function ω of order α is given by

$$\mathbb{D}^{\alpha;\psi}_{a^+}\omega(\varsigma)=\left(\frac{D_\varsigma}{\psi'(\varsigma)}\right)^n\ \mathbb{I}^{n-\alpha;\psi}_{a^+}\omega(\varsigma),$$

where $n=[\alpha]+1$ and $D_\varsigma=\frac{d}{dt}$.

Definition 2.3. [10] For $n-1<\alpha<n$ ($n\in\mathbb{N}$) and $\omega,\psi\in C^n(\mathrm{J},\mathbb{R})$, the ψ–Caputo FOD of a function ω of order α is given by

$${}^c\mathbb{D}^{\alpha;\psi}_{a^+}\omega(\varsigma)=\ \mathbb{I}^{n-\alpha;\psi}_{a^+}\omega^{[n]}_\psi(\varsigma),$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$, and $\omega^{[n]}_\psi(\varsigma)=\left(\frac{D_\varsigma}{\psi'(\varsigma)}\right)^n\omega(\varsigma)$.

From the above definition, we can express ψ–Caputo FOD by formula

$${}^c\mathbb{D}^{\alpha;\psi}_{a^+}\omega(\varsigma)=\begin{cases}\int_a^\varsigma\frac{\psi'(s)(\psi(\varsigma)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\omega^{[n]}_\psi(s)\mathrm{ds} & ,\quad \text{if } \alpha\notin\mathbb{N},\\ \omega^{[n]}_\psi(\varsigma) & ,\quad \text{if } \alpha\in\mathbb{N}.\end{cases} \tag{2.2}$$

Also, the ψ–Caputo FOD of order α of ω is defined as

$$^{c}\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\varsigma) = \mathbb{D}_{a^+}^{\alpha;\psi}\left[\omega(\varsigma) - \sum_{k=0}^{n-1}\frac{\omega_{\psi}^{[k]}(a)}{k!}(\psi(\varsigma)-\psi(a))^k\right].$$

For more details see [10, Theorem 3].

Lemma 2.4. [31] For $\alpha, \beta > 0$, and $\omega \in C(\mathrm{J}, \mathbb{R})$, we have

$$\mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\beta;\psi}\omega(\varsigma) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi}\omega(\varsigma), \ a.e. \ \varsigma \in \mathrm{J}.$$

Lemma 2.5. [11] Let $\alpha > 0$. If $\omega \in C(\mathrm{J}, \mathbb{R})$, then

$$^{c}\mathbb{D}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\alpha;\psi}\omega(\varsigma) = \omega(\varsigma), \ \varsigma \in \mathrm{J},$$

and if $\omega \in C^{n-1}(\mathrm{J}, \mathbb{R})$, then

$$\mathbb{I}_{a^+}^{\alpha;\psi}\ {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\varsigma) = \omega(\varsigma) - \sum_{k=0}^{n-1}\frac{\omega_{\psi}^{[k]}(a)}{k!}\left[\psi(\varsigma)-\psi(a)\right]^k, \quad \varsigma \in \mathrm{J}.$$

Lemma 2.6. [10, 31] For $\varsigma > a$, $\alpha \geq 0$, $\beta > 0$, and let $\chi(\varsigma) = \psi(\varsigma) - \psi(a)$. Then

- $\mathbb{I}_{a^+}^{\alpha;\psi}(\chi(\varsigma))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\chi(\varsigma))^{\beta+\alpha-1}$,
- ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}(\chi(\varsigma))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\chi(\varsigma))^{\beta-\alpha-1}$,
- ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}(\chi(\varsigma))^{k} = 0$, for all $k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.7. [9, 13] The mapping $\kappa : \mathfrak{M}_{\mathcal{U}} \to [0, \infty)$ for Kuratowski measure of non-compactness is defined as:

$$\kappa(B) = \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter} \leq \varepsilon\Big\}.$$

Proposition 2.8. The Kuratowski measure of noncompactness satisfies some properties [9, 13].

1. $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$,
2. $\kappa(A) = 0$ if and only if A is relatively compact,
3. $\kappa(A) = \kappa(\overline{A}) = \kappa(conv(A))$, where $\overline{A}$ and conv(A) represent the closure and the convex hull of A respectively,
4. $\kappa(A + B) \leq \kappa(A) + \kappa(B)$,
5. $\kappa(\lambda A) = |\lambda|\kappa(A), \lambda \in \mathbb{R}$.

Definition 2.9. A map $\mathbb{F} : \mathrm{J} \times \mathbb{X} \to \mathbb{X}$ is said to be Caratheodory if
(i) $\varsigma \mapsto \mathbb{F}(\varsigma, \omega)$ is measurable for each $\omega \in \mathbb{X}$;
(ii) $\omega \mapsto F(\varsigma, \omega)$ is continuous for almost all $\varsigma \in \mathrm{J}$.

For a given set V of functions $\omega : \mathrm{J} \to \mathbb{X}$, let us denote by

$$V(\varsigma) = \{\omega(\varsigma) : \omega \in V\}, \varsigma \in \mathrm{J},$$

and

$$V(\mathrm{J}) = \{\omega(\varsigma) : \omega \in V, \varsigma \in \mathrm{J}\}.$$

To conclude this section, we show the following fixed point theorems.

Theorem 2.10. (Banach's fixed point theorem [24]). Let ($\mathbb{X}$, d) be a nonempty complete metric space with a contraction mapping $\mathcal{G} : \mathbb{X} \to \mathbb{X}$ i.e., $\mathrm{d}(\mathcal{G}\omega, \mathcal{G}\varpi) \leq k\ \mathrm{d}(\omega, \varpi)$ for all $\omega, \varpi \in \mathbb{X}$, where $k \in (0, 1)$ is a constant. Then $\mathcal{G}$ has a unique fixed point.

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.11. ([6, 35, 40]) Let $\mathcal{D}$ be a bounded, closed and convex subset of a Banach space such that $0 \in \mathcal{D}$, and let N be a continuous mapping of $\mathcal{D}$ into itself. If the implication

$$V = \overline{conv} N(V) \ \text{ or } \ V = N(V) \cup \{0\} \Rightarrow \kappa(V) = 0 \tag{2.3}$$

holds for every subset V of $\mathbb{D}$, then N has a fixed point.

Lemma 2.12. ([40]) Let $\mathcal{D}$ be a bounded, closed and convex subset of the Banach space $\mathbb{C}$, G a continuous function on $\mathrm{J} \times \mathrm{J}$ and $\mathbb{F}$ a function from $\mathrm{J} \times \mathbb{X} \longrightarrow \mathbb{X}$ which satisfies the Caratheodory conditions, and suppose there exists $\mathcal{P} \in L^1(\mathrm{J}, \mathbb{R}^+)$ such that, for each $\varsigma \in \mathrm{J}$ and each bounded set $\mathcal{B} \subset \mathbb{X}$, we have

$$\lim_{h\to 0^+} \kappa(\mathbb{F}(\mathrm{J}_{\varsigma,h} \times \mathcal{B})) \leq \mathcal{P}(\varsigma)\kappa(\mathcal{B}); \text{ here } \mathrm{J}_{\varsigma,h} = [\varsigma - h, \varsigma] \cap \mathrm{J}.$$

If V is an equicontinuous subset of $\mathcal{D}$, then

$$\kappa\left(\left\{\int_{\mathrm{J}} G(s,\varsigma)\mathbb{F}(s,\omega(s))ds : \omega \in V\right\}\right) \leq \int_{\mathrm{J}} \|G(\varsigma,s)\|\mathcal{P}(s)\kappa(V(s))ds.$$

3 Main Results

In this section, we are concerned with the existence, uniqueness and Ulam stability of solutions of problem (1.1). Let us start by defining what we meant by a solution of the problem (1.1).

Definition 3.1. By a solution of the system (1.1) we mean a measurable functions $\omega \in \mathbb{C}$ such that $\omega(0) = 0, \omega(b) + \lambda\mathbb{I}_{a^+}^{\alpha;\psi}\omega(b) = 0$ and $\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\xi) + \lambda\omega(\xi) = 0$, and the equations $\mathbb{D}_{a^+}^{\beta;\psi}(\mathbb{D}_{a^+}^{\alpha;\psi} + \lambda)\omega(\varsigma) = \mathbb{F}(\varsigma, \omega(\varsigma))$ are satisfied on J.

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 3.2. Let $\alpha \in (0, 1]$, $\beta \in (1, 2]$. Then the boundary value problem

$$\begin{cases} \mathbb{D}_{a^+}^{\beta;\psi}(\mathbb{D}_{a^+}^{\alpha;\psi} + \lambda)\omega(\varsigma) = \sigma(\varsigma), \quad \varsigma \in (a, b), \\ \omega(a) = 0 \\ \omega(b) + \lambda\mathbb{I}_{a^+}^{\alpha;\psi}\omega(b) = 0 \\ \mathbb{D}_{a^+}^{\alpha;\psi}\omega(\xi) + \lambda\omega(\xi) = 0, \quad \xi \in]a, b]. \end{cases} \tag{3.1}$$

has a unique solution defined by

$$\omega(\varsigma) + \lambda\mathbb{I}_{a^+}^{\alpha;\psi}\omega(\varsigma) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(\varsigma) + \mu(\varsigma)\mathbb{I}_{a^+}^{\beta;\psi}\sigma(\xi) + \nu(\varsigma)\mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(b). \tag{3.2}$$

where

$$\mu(\varsigma) = \frac{(\psi(\varsigma) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{(\psi(\varsigma) - \psi(0))\,|\omega_3|}{(\alpha+1)|\Theta|} - \frac{1}{|\Theta|}\right] \tag{3.3}$$

and

$$\nu(\varsigma) = \frac{(\psi(\varsigma) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{|\omega_1|}{|\Theta|} - \frac{(\psi(\varsigma) - \psi(0))\,|\omega_2|}{(\alpha+1)|\Theta|}\right] \tag{3.4}$$

with

$$\omega_1 = \frac{(\psi(\varsigma) - \psi(0))^{\alpha+1}}{\Gamma(\alpha+2)}, \quad \omega_2 = \frac{(\psi(\varsigma) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)}, \quad \omega_3 = (\psi(\varsigma) - \psi(0)), \tag{3.5}$$

and

$$\Theta = \omega_2 - \omega_1\omega_3 \neq 0. \tag{3.6}$$

Proof . Applying the integrator operator $\mathbb{I}_{a^+}^{\beta;\psi}$ to (3.1) and using the Lemma 2.6 we get

$$(\mathbb{D}_{a^+}^{\alpha}+\lambda)\,\omega(\varsigma)=c_1+c_2\left(\psi(\varsigma)-\psi(0)\right)+\mathbb{I}_{a^+}^{\beta;\psi}\sigma(\varsigma),\varsigma\in(0,b]. \tag{3.7}$$

We apply again the operator $\mathbb{I}_{a^+}^{\alpha;\psi}$ and use the results of Lemmas 2.6 to get the general solution representation of problem (3.1)

$$\omega(\varsigma)=\mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(\varsigma)-\lambda\mathbb{I}_{a^+}^{\alpha}\omega(\varsigma)+c_0+c_1\frac{(\psi(\varsigma)-\psi(0))^{\alpha+1}}{\Gamma(\alpha+2)}+c_2\frac{(\psi(\varsigma)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \tag{3.8}$$

where $c_0,c_1,c_2\in\mathbb{R}$. By using the boundary conditions in problem (3.1) and the above equation, we observe that $c_0=0$, and

$$c_1\frac{(\psi(\varsigma)-\psi(0))^{\alpha+1}}{\Gamma(\alpha+2)}+c_2\frac{(\psi(\varsigma)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}+\mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(b)=0. \tag{3.9}$$

Moreover, we obtain

$$c_1+c_2\left(\psi(\xi)-\psi(0)\right)+\mathbb{I}_{a^+}^{\beta}\sigma(\xi)=0. \tag{3.10}$$

Also, by using (3.20), equations (3.9) and (3.10) can be written as

$$c_1\omega_1+c_2\omega_2 = 0 \tag{3.11}$$
$$c_1+c_2\omega_3 = 0. \tag{3.12}$$

Solving the last two equations in c_1 and c_2, we end up with

$$c_1 = \frac{\omega_3}{\Theta}\mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(b)-\frac{\omega_2}{\Theta}\mathbb{I}_{a^+}^{\beta}\sigma(\xi) \tag{3.13}$$
$$c_2 = \frac{\omega_1}{\Theta}\mathbb{I}_{a^+}^{\beta}\sigma(\xi)-\frac{1}{\Theta}\mathbb{I}_{a^+}^{\alpha+\beta;\psi}\sigma(b). \tag{3.14}$$

Substituting c_1 and c_2 in (3.8), we get the desired solution representation (3.2). Besides and by the help of the results in Lemmas 2.6 one can easily figure out that Eq. (3.2) solves problem (3.1). This finishes the proof. □

We will need the following properties for the functions μ and ν defined in next Lemma

Lemma 3.3. The functions μ and ν are continuous functions on J and satisfy the following properties:
(1) $\mu^*=\max_{0\le\varsigma\le b}|\mu(\varsigma)|$,
(2) $\nu^*=\max_{0<\varsigma<b}|\nu(\varsigma)|$,

In what follows, we present the solution representation associated with problem (1.1).

Lemma 3.4. Assume that $\mathbb{F}:\mathrm{J}\times\mathbb{X}\to\mathbb{X}$ is continuous. A function $\omega(\varsigma)$ solves the system (1.1) if and only if it is a fixed-point of the operator $\mathcal{G}:\mathbb{C}\to\mathbb{C}$ defined by

$$\begin{aligned}\mathcal{G}\omega(\varsigma)=&\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{ds}+\lambda\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)\omega(s)\mathrm{ds}+\mu(\varsigma)\int_a^{\xi}\mathcal{G}_{\psi}^{\beta}(\xi,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{ds}\\&+\nu(\varsigma)\int_a^{b}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{ds}\end{aligned} \tag{3.15}$$

3.1 Uniqueness Result via Banach contraction

We introduce the following conditions:
(H1) The function $\mathbb{F}:[a,b]\times\mathbb{X}\to\mathbb{X}$ is continuous.
(H2) There exists a constant $\mathcal{L}>0$ such that,

$$|\mathbb{F}(\varsigma,\omega)-\mathbb{F}(\varsigma,\varpi)|\le\mathcal{L}|\omega-\varpi|,\quad \varsigma\in\mathrm{J},\omega,\varpi\in\mathbb{R}.$$

For simplicity, we denote

$$\Delta := \left(\mathcal{L}\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(\mathcal{L}\nu^{*}\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \mathcal{L}\mu^{*}\frac{(\psi(\xi)-\psi(a))^{\alpha}}{\Gamma(\beta+1)}\right) \tag{3.16}$$

$$\mathcal{G}_{\psi}^{\chi}(\varsigma,s) = \frac{\psi'(s)(\psi(\varsigma)-\psi(s))^{\chi-1}}{\Gamma(\chi)}, \quad \chi>0 \tag{3.17}$$

Now, we are ready to present our main results. The first existence result is based on Banach's fixed point theorem.

Theorem 3.5. Assume that (H1) and (H2) are satisfied. Then the problem (1.1) has a unique solution on J. Provided that $\Delta<1$.

Proof . In view of Lemma 3.2 we introduce an operator $\mathcal{G}:\mathbb{C}\to\mathbb{C}$ associated with the problem (1.1) as follows:

$$\begin{aligned}\mathcal{G}\omega(\varsigma) &= \int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{d}s + \lambda\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)\omega(s)\mathrm{d}s\\ &\quad+\mu(\varsigma)\int_a^{\xi}\mathcal{G}_{\psi}^{\beta}(\xi,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{d}s + \nu(\varsigma)\int_a^{b}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)\mathbb{F}\left(s,\omega(s)\right)\mathrm{d}s\end{aligned} \tag{3.18}$$

The operator $\mathcal{G}$ is well defined as $\mathbb{F}$ is continuous function. Then the fixed point of operator $\mathcal{G}$ coincides with the solution of the problem (1.1). Next, the Banach contraction principle will be used to prove that $\mathcal{G}$ has a fixed point. For this reason, we shall show that $\mathcal{G}$ is a contraction. Indeed, let $\omega,\varpi\in\mathbb{C}$. Then, for every $\varsigma\in$ J, using (H2), we can get

$$\begin{aligned}|\mathcal{G}\varpi(\varsigma)-\mathcal{G}\omega(\varsigma)| &= \int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)|\mathbb{F}(s,\varpi(s))-\mathbb{F}(s,\omega(s))|\mathbf{ds} + |\lambda|\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)|\varpi(s)-\omega(s)|\mathbf{ds}\\ &\quad+\nu^{*}\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)|\mathbb{F}(s,\varpi(s))-\mathbb{F}(s,\omega(s))|\mathbf{ds} + \mu^{*}\int_a^{\varsigma}\mathcal{G}_{\psi}^{\beta}(\xi,s)|\mathbb{F}(s,\varpi(s))-\mathbb{F}(s,\omega(s))|\mathbf{ds}\\ &\le \int_a^{\varsigma}\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)|\varpi(s)-\omega(s)|\mathbf{ds} + |\lambda|\int_a^{\varsigma}\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)|\varpi(s)-\omega(s)|\mathbf{ds}\\ &\quad+\nu^{*}\int_a^{\varsigma}\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)|\varpi(s)-\omega(s)|\mathbf{ds} + \mu^{*}\int_a^{\varsigma}\mathcal{L}\mathcal{G}_{\psi}^{\beta}(\xi,s)|\varpi(s)-\omega(s)|\mathbf{ds}\\ &= \int_a^{\varsigma}\left(\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)+|\lambda|\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)+\nu^{*}\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)+\mu^{*}\mathcal{L}\mathcal{G}_{\psi}^{\beta}(\xi,s)\right)|\varpi(s)-\omega(s)|\mathbf{ds}\\ &\le\|\varpi-\omega\|_{\infty}\int_a^{\varsigma}\left(\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)+|\lambda|\mathcal{G}_{\psi}^{\alpha}(\varsigma,s)+\nu^{*}\mathcal{L}\mathcal{G}_{\psi}^{\alpha+\beta}(b,s)+\mu^{*}\mathcal{L}\mathcal{G}_{\psi}^{\beta}(\xi,s)\right)\mathrm{ds}\end{aligned} \tag{3.19}$$

Also note that

$$\int_a^{\varsigma}\mathcal{G}_{\psi}^{\chi}(\varsigma,s)\mathrm{d}s \le \frac{(\psi(b)-\psi(a))^{\chi}}{\Gamma(\chi+1)}, \quad \chi>0,$$

Using the above arguments, we get

$$\begin{aligned}\|\mathcal{G}\varpi-\mathcal{G}\omega\|_{\infty} &\le \left(\mathcal{L}\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right)\|\varpi-\omega\|\\ &\quad+\left(\mathcal{L}\nu^{*}\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \mathcal{L}\mu^{*}\frac{(\psi(\xi)-\psi(a))^{\alpha}}{\Gamma(\beta+1)}\right)\|\varpi-\omega\|\\ &= \Delta\|\varpi-\omega\|\end{aligned}$$

As $\Delta<1$, we deduce that $\mathcal{G}$ is a contraction. Hence, by the Banach fixed point theorem, $\mathcal{G}$ has a unique fixed point which is a unique solution of the initial value problem (1.1). This completes the proof. □

In the following subsections, we establish the existence of solutions for the system (1.1) by applying Mönch fixed point theorems.

3.2 Existance result via Mönch Fixed Point Theorem

We further will use the following hypotheses.
(A1) For any $\mathbb{F} : \mathrm{J} \times \mathbb{X} \to \mathbb{X}$ satisfies the Caratheodory conditions;
(A2) There exists $\mathcal{P} \in L^1(\mathrm{J}, \mathbb{R}^+) \cap C(\mathrm{J}, \mathbb{R}^+)$, such that,

$$\|\mathbb{F}(\varsigma, \omega)\| \leq \mathcal{P}(\varsigma)\|\omega\|, \text{ for } \varsigma \in \mathrm{J} \text{ and each } \omega \in \mathbb{X},$$

(A3) For any $\varsigma \in \mathrm{J}$ and each bounded measurable sets $\mathcal{B} \subset \mathbb{X}$, we have $\lim_{\mathbb{H} \to 0^+} \kappa(\mathbb{F}(\mathrm{J}_{\varsigma,h} \times \mathcal{B}) \leq \mathcal{P}(\varsigma)\kappa(\mathcal{B})$, where κ is the Kuratowski measure of compactness and $\mathrm{J}_{\varsigma,h} = [\varsigma - h, \varsigma] \cap \mathrm{J}$. Set

$$\mathcal{P}^* = \sup_{\varsigma \in \mathrm{J}} \mathcal{P}(\varsigma). \tag{3.20}$$

Theorem 3.6. Assume that conditions (A1)-(A3) hold. If

$$\Lambda := (\mathcal{M}\mathcal{P}^* + \mathcal{N}) < 1 \tag{3.21}$$

where

$$\mathcal{M} = \left\{ \frac{(\psi(b) - \psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (\nu^* + 1) + \mu^* \frac{(\psi(\xi) - \psi(a))^{\beta}}{\Gamma(\beta+1)} \right\}, \quad \mathcal{N} \quad = |\lambda| \frac{(\psi(b) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.22}$$

then the system (1.1) has at least one solution on J.

Proof . We consider the operator $\mathcal{G} : \mathbb{C} \to \mathbb{C}$ defined by the formula (3.15).

Clearly, the fixed points of the operator $\mathcal{G}$ are solutions of the systlem (1.1). Let we take

$$\mathcal{D}_{\mathcal{R}} = \{\omega \in \mathbb{C} : \|\omega\| \leq \mathcal{R}\}.$$

Clearly, the subset $\mathcal{D}_{\mathcal{R}}$ is closed, bounded and convex. We shall show that $\mathcal{G}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps. □

Step 1: First we show that $\mathcal{G}$ is sequentially continuous:

Let $\{\omega_n\}_n$ be a sequence such that $\omega_n \to \omega$ in $\mathbb{C}$. Then for any $\varsigma \in \mathrm{J}$,

$$\begin{aligned}
\|(\mathcal{G}\omega_n)(\varsigma) - (\mathcal{G}\omega)(\varsigma)\| &\leq \int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma, s)|\mathbb{F}_n(s, \omega_n(s)) - \mathbb{F}(s, \omega(s))|\mathrm{ds} - \lambda \int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha}(\varsigma, s)\, |\omega_n - \omega|\, \mathrm{ds} \\
&+ \mu^* \int_a^{\xi} \mathcal{G}_{\psi}^{\alpha+\beta}(\xi, s)|\mathbb{F}_n(s, \omega_n(s)) - \mathbb{F}(s, \omega(s))|\mathrm{ds} + \nu^* \int_a^{b} \mathcal{G}_{\psi}^{\alpha+\beta}(b, s)|\mathbb{F}_n(s, \omega_n(s)) - \mathbb{F}(s, \omega(s))|\mathrm{ds} \\
&\leq \left\{ \frac{(\psi(b) - \psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (\nu^* + 1) + \mu^* \frac{(\psi(\xi) - \psi(a))^{\beta}}{\Gamma(\beta+1)} \right\} \times \|\mathbb{F}_n(s, \omega_n(s)) - \mathbb{F}(s, \omega(s))\| + \lambda \frac{(\psi(b) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)}\, \|\omega_n - \omega\|
\end{aligned}$$

Since for any $i = 1, 2$, the function $\mathbb{F}$ satisfies assumptions (A1), we have $\mathbb{F}(\varsigma, \omega_n(\varsigma))$ converge uniformly to $\mathbb{F}(\varsigma, \omega(\varsigma))$. Hence, the Lebesgue dominated convergence theorem implies that $(\mathcal{G}\omega_n)(\varsigma)$ converges uniformly to $(\mathcal{G}(\omega))(\varsigma)$ Thus $(\mathcal{G}\omega_n) \to (\mathcal{G}(\omega))$. Hence $\mathcal{G} : \mathcal{D}_{\mathcal{R}} \to \mathcal{D}_{\mathcal{R}}$ is sequentially continuous.

Step 2: Second we show that $\mathcal{G}$ maps $\mathbb{D}_{\mathcal{R}}$ into itself:

Take $\omega \in \mathcal{D}_{\mathcal{R}}$, by (A2), we have, for each $\varsigma \in \mathrm{J}$ and assume that $(\mathcal{G}(\omega))(\varsigma) \neq 0$.

$$\begin{aligned}
|\mathcal{G}u(\varsigma)| \leq & \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma,s)\left|\mathbb{F}(s,\omega(s))\right| \mathrm{ds} + \lambda \int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma,s)\left|\omega(s)\right| \mathrm{ds} + \mu^* \int_a^\varsigma \mathcal{G}_\psi^{\beta}(\xi,s)\left|\mathbb{F}(s,\omega(s))\right| \mathrm{ds} \\
& + \nu^* \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(b,s)\left|\mathbb{F}(s,\omega(s))\right| \mathrm{ds} \\
\leq & \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma,s)\mathcal{P}(\varsigma)\|\omega\|\mathrm{ds} + \lambda \int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma,s)\left|\omega(s)\right| \mathrm{ds} + \mu^* \int_a^\varsigma \mathcal{G}_\psi^{\beta}(\xi,s)\mathcal{P}(\varsigma)\|\omega\|\mathrm{ds} + \nu^* \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(b,s)\mathcal{P}(\varsigma)\|\omega\|\mathrm{ds} \\
\leq & \mathcal{P}^*\mathcal{R} \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma,s)(1)\mathrm{ds} + \lambda\mathcal{R} \int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma,s)(1)\mathrm{ds} + \mu^*\mathcal{P}^*\mathcal{R} \int_a^\varsigma \mathcal{G}_\psi^{\beta}(\xi,s)(1)\mathrm{ds} + \nu^*\mathcal{P}^*\mathcal{R} \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(b,s)(1)\mathrm{ds} \\
\leq & \mathcal{P}^*\mathcal{R}\left\{\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(\nu^*+1) + \mu^*\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right\} + \lambda\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\mathcal{R} \\
= & \mathcal{R}\left(\mathcal{M}\mathcal{P}^* + \mathcal{N}\right).
\end{aligned}$$

Hence we get

$$\|\mathcal{G}(\omega)\|_{\mathbb{C}} \leq \mathcal{R}\left(\mathcal{M}\mathcal{P}^* + \mathcal{N}\right) \leq \mathcal{R}. \tag{3.23}$$

Step 3: $\mathcal{G}(\mathcal{D}_{\mathcal{R}})$ is equicontinuous. By Step 2, it is obvious that $\mathcal{G}(\mathcal{D}_{\mathcal{R}}) \subset \mathbb{C}$ is bounded. For the equicontinuity of $\mathcal{G}(\mathcal{D}_{\mathcal{R}})$. Let $\varsigma_1, \varsigma_2 \in \mathrm{J}$, $\varsigma_1 < \varsigma_2$ and $\omega \in \mathcal{D}_{\mathcal{R}}$, so $\mathcal{G}\omega(\varsigma_2) - \mathcal{G}\omega(\varsigma_1) \neq 0$. Then

$$\begin{aligned}
\|\mathcal{G}\omega(\varsigma_2) - \mathcal{G}\omega(\varsigma_1)\| \leq & \int_a^{\varsigma_1} \left[\mathcal{G}_\psi^{\alpha+\beta}(\varsigma_2,s) - \mathcal{G}_\psi^{\alpha+\beta}(\varsigma_1,s)\right]\left|\mathbb{F}(s,\omega(s))\right| \mathrm{ds} + \int_{\varsigma_1}^{\varsigma_2} \mathcal{G}_\psi^{\alpha+\beta}(\varsigma_2,s)\left|\mathbb{F}(s,\omega(s))\right| \mathrm{ds} \\
& + \lambda \int_a^{\varsigma_1} \left[\mathcal{G}_\psi^{\alpha+\beta}(\varsigma_2,s) - \mathcal{G}_\psi^{\alpha+\beta}(\varsigma_1,s)\right]\left|\omega(s)\right| \mathrm{ds} + \lambda \int_{\varsigma_1}^{\varsigma_2} \mathcal{G}_\psi^{\alpha+\beta}(\varsigma_2,s)\left|\omega(s)\right| \mathrm{ds} \\
& + \left|\mu(\varsigma_2) - \mu(\varsigma_1)\right| \int_a^\varsigma \mathcal{G}_\psi^{\beta}(\xi,s)\left|\mathbb{F}(s,\omega(s))(\xi)\right| \mathrm{ds} + \left|\nu(\varsigma_2) - \nu(\varsigma_1)\right| \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(b,s)\left|\mathbb{F}(s,\omega(s))(b)\right| \mathrm{ds}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\mathcal{G}\omega(\varsigma_2) - \mathcal{G}\omega(\varsigma_1)\| \leq & \frac{2\mathcal{P}^*\mathcal{R}}{\Gamma(\alpha+\beta+1)}(\psi(\varsigma_2)-\psi(\varsigma_1))^{\alpha+\beta} + \frac{2\lambda\mathcal{R}}{\Gamma(\alpha+1)}(\psi(\varsigma_2)-\psi(\varsigma_1))^{\alpha} \\
& + \left|\nu(\varsigma_2) - \nu(\varsigma_1)\right| \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \left|\mu(\varsigma_2) - \mu(\varsigma_1)\right| \frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}.
\end{aligned}$$

As $\varsigma_1 \to \varsigma_2$, the right hand side of the above inequality tends to zero. This means that $\mathcal{G}(\mathcal{D}_{\mathcal{R}}) \subset \mathcal{D}_{\mathcal{R}}$.

Finally we show that the implication (2.3) holds. Let $V \subset \mathcal{D}_{\mathcal{R}}$ such that $V = \overline{conv}(\mathcal{G}(V) \cup \{0\})$. Since V is bounded and equicontinuous, and therefore the function $\omega \to \omega(\varsigma) = \kappa(V(\varsigma))$ is continuous on J. By hypothesis (A2), and the properties of the measure κ, for any $\varsigma \in \mathrm{J}$ we get.

$$\begin{aligned}
\omega(\varsigma) \leq & \kappa(\mathcal{G}(V)(\varsigma) \cup \{(0,0)\})) \leq \kappa((\mathcal{G}V)(\varsigma)) \leq \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds} \\
& + \lambda \int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma,s)\kappa(V(s))\mathrm{ds} + \mu^* \int_a^\xi \mathcal{G}_\psi^{\beta}(\xi,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds} + \nu^* \int_a^b \mathcal{G}_\psi^{\alpha+\beta}(b,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds} \\
\leq & \mathcal{P}^*\|v\|\left\{\int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds} + \mu^* \int_a^\xi \mathcal{G}_\psi^{\beta}(\xi,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds} + \nu^* \int_a^b \mathcal{G}_\psi^{\alpha+\beta}(b,s)\mathcal{P}(s)\kappa(V(s))\mathrm{ds}\right\} \\
& + \lambda\|v\| \int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma,s)\kappa(V(s))\mathrm{ds} \\
\leq & \mathcal{P}^*\|v\|\left\{\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(\nu^*+1) + \mu^*\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right\} + \lambda\|v\|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
= & \|v\|\left(\mathcal{M}\mathcal{P}^* + \mathcal{N}\right).
\end{aligned}$$

This means that $\|v\|\left(1 - \mathcal{M}\mathcal{P}^* - \mathcal{N}\right) \leq 0$. By (3.21) it follows that $\|v\| = 0$, that is $v(\varsigma) = 0$ for each $\varsigma \in \mathrm{J}$ and then $V(\varsigma)$ is relatively compact in $\mathbb{X}$. In view of the Ascoli-Arzela theorem, V is relatively compact in $\mathcal{D}_{\mathcal{R}}$. Applying now Theorem 2.12, we conclude that $\mathcal{G}$ has a fixed point which is a solution of the problem (1.1).

4 Ulam-Hyers stability analysis

In this section, we study the Ulam-Hyers stability, and Ulam-Hyers-Rassias stability of the problem (1.1). Let $\varepsilon > 0$ and $\Phi : [a, b] \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities:

$$\left|\mathbb{D}_{a^+}^{\beta;\psi}\left(\mathbb{D}_{a^+}^{a;\phi} - \mu^*\right)\bar{\omega}(\varsigma) - \mathbb{F}(\varsigma, \widetilde{\omega}(\varsigma))\right| \leq \varepsilon, \quad \varsigma \in [a, b]. \tag{4.1}$$

Definition 4.1. Equation (1.1) is Ulam-Hyers stable if there exists a real number $c_{\mathbb{F}} > 0$ such that, for each $\varepsilon > 0$ and for each solution $\bar{\omega} \in \mathbb{C}$ of inequalities (4.1) there exists a solution $\omega \in \mathbb{C}$ of (1.1) with

$$|\widetilde{\omega}(\varsigma) - \omega(\varsigma)| \leq \varepsilon c_{\mathbb{F}}, \quad \varsigma \in [a, b].$$

Definition 4.2. Equation (1.1) is generalized Ulam-Hyers stable if there exists $c_{\mathbb{F}} : \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ with $c_{\mathbb{F}}(0) = 0$ such that, for each $\varepsilon > 0$ and for each solution $\widetilde{\omega} \in \mathbb{C}$ of inequalities (4.1), there exists a solution $\omega \in \mathbb{C}$ of (1.1) with

$$|\widetilde{\omega}(\varsigma) - \omega(\varsigma)| \leq c_{\mathbb{F}}(\varepsilon), \quad \varsigma \in [a, b].$$

Remark 4.3. A function $\widetilde{\omega} \in \mathbb{C}$ is a solution of inequality (4.1) if and only if there exists a function $\mathbb{H} \in \mathbb{C}$ (which depends on solution $\widetilde{\omega}$) such that

(1) $|\mathbb{H}(\varsigma)| \leq \varepsilon, \varsigma \in [a, b]$.

(2) $\mathbb{D}_{a^+}^{\beta;\psi}(\mathbb{D}_{a^+}^{a,\psi} - \lambda)\widetilde{\omega}(\varsigma) = \mathbb{F}(\varsigma, \widetilde{\omega}(\varsigma)) + \mathbb{H}(\varsigma), \quad \varsigma \in [a, b]$.

Now, we discuss the Ulam-Hyers stability of solution to the problem (1.1).

Theorem 4.4. If all the assumptions (H1)-(H2) are fulfilled, then the problem (1.1) is Ulam-Hyers stable on $[a, b]$ and consequently generalized Ulam-Hyers stable provided that $\Delta < 1$.

Proof . Let $\varepsilon > 0$ and let $\widetilde{\omega} \in \mathbb{C}$ be a function which satisfies the inequality (4.1) and let $\omega \in \mathbb{C}$ the unique solution of the following problem

$$\begin{cases} \mathbb{D}_{a^+}^{\beta;\psi}(\mathbb{D}_{a^+}^{\alpha;\psi} + \lambda)\omega(\varsigma) = \mathbb{F}(\varsigma, \omega(\varsigma)), \quad \varsigma \in \mathrm{J} = [a, b], \\ \omega(a) = 0, \\ \omega(b) + \lambda \mathbb{I}_{a^+}^{\alpha;\psi}\omega(b) = 0, \\ \mathbb{D}_{a^+}^{\alpha;\psi}\omega(\xi) + \lambda\omega(\xi) = 0, \quad \xi \in]a, b]. \end{cases} \tag{4.2}$$

By Lemma 3.2, we have

$$\begin{aligned} \mathcal{G}\omega(\varsigma) = &\int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma, s)\mathbb{F}(s, \omega(s))\,\mathrm{d}s + \lambda\int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma, s)\omega(s)\mathrm{d}s \\ &+ \mu(\varsigma)\int_a^\xi \mathcal{G}_\psi^{\beta}(\xi, s)\mathbb{F}(s, \omega(s))\,\mathrm{d}s + \nu(\varsigma)\int_a^b \mathcal{G}_\psi^{\alpha+\beta}(b, s)\mathbb{F}(s, \omega(s))\,\mathrm{d}s. \end{aligned} \tag{4.3}$$

Since we have assumed that ω is a solution of (4.1), hence we have by Remark 4.3.

$$\begin{cases} \mathbb{D}_{a^+}^{\beta;\psi}(\mathbb{D}_{a^+}^{\alpha;\psi} + \lambda)\widetilde{\omega}(\varsigma) = \mathbb{F}(\varsigma, \widetilde{\omega}(\varsigma)) + \mathbb{H}(\varsigma), \quad \varsigma \in \mathrm{J} = [a, b], \\ \widetilde{\omega}(a) = 0 \\ \widetilde{\omega}(b) + \lambda \mathbb{I}_{a^+}^{\alpha;\psi}\widetilde{\omega}(b) = 0 \\ \mathbb{D}_{a^+}^{\alpha;\psi}\widetilde{\omega}(\xi) + \lambda\widetilde{\omega}(\xi) = 0, \quad \xi \in]a, b]. \end{cases} \tag{4.4}$$

Again by Lemma 3.2, we have

$$\begin{aligned} \widetilde{\omega}(\varsigma) = &\int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma, s)\mathbb{F}(s, \widetilde{\omega}(s))\,\mathrm{d}s + \lambda\int_a^\varsigma \mathcal{G}_\psi^{\alpha}(\varsigma, s)\widetilde{\omega}(s)\mathrm{d}s + \mu(\varsigma)\int_a^\xi \mathcal{G}_\psi^{\beta}(\xi, s)\mathbb{F}(s, \widetilde{\omega}(s))\,\mathrm{d}s \\ &+ \nu(\varsigma)\int_a^b \mathcal{G}_\psi^{\alpha+\beta}(b, s)\mathbb{F}(s, \widetilde{\omega}(s))\,\mathrm{d}s + \int_a^\varsigma \mathcal{G}_\psi^{\alpha+\beta}(\varsigma, s)|\mathbb{H}(s)|\mathrm{d}s + \nu^*\int_a^b \mathcal{G}_\psi^{\alpha+\beta}(b, s)|\mathbb{H}(s)|\mathrm{d}s + \mu^*\int_a^\xi \mathcal{G}_\psi^{\beta}(\xi, s)|\mathbb{H}(s)|\mathrm{d}s \end{aligned} \tag{4.5}$$

On the other hand, we have, for each $\varsigma \in [a,b]$

$$
\begin{aligned}
|\widetilde{\omega}(\varsigma)-\omega(\varsigma)| \leq & \int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)|\mathbb{H}(s)|\mathrm{d}s+\nu^*\int_a^{b} \mathcal{G}_{\psi}^{\alpha+\beta}(b,s)|\mathbb{H}(s)|\mathrm{d}s+\mu^*\int_a^{\xi} \mathcal{G}_{\psi}^{\beta}(\xi,s)|\mathbb{H}(s)|\mathrm{d}s \\
& +\int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha+\beta}(\varsigma,s)|\mathbb{F}(s,\widetilde{\omega}(s))-\mathbb{F}(s,\omega(s))|\mathrm{d}\mathbf{s}+|\lambda|\int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha}(\varsigma,s)|\widetilde{\omega}(s)-\omega(s)|\mathrm{d}\mathbf{s} \\
& +\nu^*\int_a^{\varsigma} \mathcal{G}_{\psi}^{\alpha+\beta}(b,s)|\mathbb{F}(s,\widetilde{\omega}(s))-\mathbb{F}(s,\omega(s))|\mathrm{d}\mathbf{s}+\mu^*\int_a^{\varsigma} \mathcal{G}_{\psi}^{\beta}(\xi,s)|\mathbb{F}(s,\widetilde{\omega}(s))-\mathbb{F}(s,\omega(s))|\mathrm{d}\mathbf{s}
\end{aligned}
$$

Hence using part (1) of Remark 4.3 and (H2) we can get

$$
|\widetilde{\omega}(\varsigma)-\omega(\varsigma)| \leq \left(\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}[\nu^*+1]+\mu^*\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right)\varepsilon+\Delta\|\widetilde{\omega}-\omega\|
$$

where Δ is defined in (3.16). In consequence, it follows that

$$
\|\widetilde{\omega}-\omega\|_{\infty} \leq \left(\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\frac{[\nu^*+1]}{(1-\Delta)}+\frac{\mu^*}{(1-\Delta)}\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right)\varepsilon
$$

If we let $c_{\mathbb{F}}=\left(\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\frac{[\nu^*+1]}{(1-\Delta)}+\frac{\mu^*}{(1-\Delta)}\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right)$, then, the Ulam-Hyers stability condition is satisfied. More generally, for $\Phi_{\mathbb{F}}(\varepsilon)=\left(\frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\frac{[\nu^*+1]}{(1-\Delta)}+\frac{\mu^*}{(1-\Delta)}\frac{(\psi(\xi)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\right)\varepsilon$; $\Phi(0)=0$ the generalized Ulam-Hyers stability condition is also satisfied. This completes the proof. □

5 Example

Example 5.1. Let $E=l^1=\{\omega=(\omega_1,\omega_2,...,\omega_n,...):\sum_{n=1}^{\infty}|\omega_n|<\infty\}$ with the norm $\|\omega\|_E=\sum_{n=1}^{\infty}|\omega_n|$. Consider the following BVP of a LFDE:

$$
\begin{cases}
\mathbb{D}_{a^+}^{\frac{3}{2};\varsigma}(\mathbb{D}_{a^+}^{\frac{1}{2};\varsigma}+\frac{1}{4})\omega(\varsigma)=\mathbb{F}(\varsigma,\omega(\varsigma)), \quad \varsigma\in \mathrm{J}=[0,1],\\
\omega(0)=0,\\
\omega(1)+\frac{1}{4}\mathbb{I}_{a^+}^{\frac{1}{2};\varsigma}\omega(1)=0,\\
\mathbb{D}_{a^+}^{\frac{1}{2};\varsigma}\omega(\frac{1}{3})+\frac{1}{4}\omega(\frac{1}{3})=0, \quad \xi\in]0,1].
\end{cases} \tag{5.1}
$$

where

$$
\beta=\frac{3}{2},\alpha=\frac{1}{2},b=1,a=0,\ \psi(\varsigma)=\varsigma,,\lambda=\frac{1}{4};\xi=\frac{1}{3}. \tag{5.2}
$$

Using the given values of the parameters in (3.6), by the Matlab program, we find that $\Theta=\omega_2-\omega_1\omega_3\simeq 0.3462\neq 0$. In order to illustrate Theorem 3.5 and Theorem 3.6, we take

$$
f(\varsigma,\omega(\varsigma))=\frac{\varsigma^2-1}{(4-\sin^2\pi\varsigma)^2}\left(\frac{|\omega(\varsigma)|}{1+|\omega(\varsigma)|}\right), \tag{5.3}
$$

and note that $\|f(\varsigma,\omega)-f(\varsigma,v)\|\leq\frac{1}{8}\|\omega-v\|$. Hence the conditions (H1)-(H2) and (A1)-(A3) holds with $\mathcal{L}=\frac{1}{8}$. Further from the above given data it is easy to calculate $\Lambda\simeq 0.1325<1$, by Theorem 3.6, the BVP (5.1) with the data (5.2) and (5.3) has at least a solution ω. Furthermore $\Delta\simeq 0.2265<1$. Hence by Theorem 3.5 the BVP (5.1) with the data (5.2) and (5.3) has a unique solution. Moreover, Theorem 4.4 ensures that the BVP (5.1) is HU stable and generalized HU stable.

6 Conclusion

In this research work, we have investigated the sufficient conditions for the existence and uniqueness of solutions for Langevin equations using the ψ-Caputo fractional derivative operator according to boundary conditions. Two corresponding theoretical findings have been investigated by using Mönch and Banach fixed point theorem. We expand the results studied in the earlier research by [19, 20] using a new fractional operator. To our knowledge, the proposed technique has not hitherto been used to address problems of this nature. We believe that our work will make a contribution to the literature by giving an adequate description of a number of dynamical processes and applications outlined in some fractal mediums. In future work, we will study the stability results in the Ulam sense of the Langevin equations and extend the results by using new general operators so called Piecewise differential and integral operators.

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