Int. J. Nonlinear Anal. Appl. 15 (2024) 9, 263–270
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.31491.4644

Convolution properties for some subclasses of meromorphic p-valent functions of complex order associated with q-derivative

Mohammad Hassan Golmohammadi, Aboalghasem Alishahi*

Department of Mathematics, Payame Noor University, P.O. Bax 19395–4697, Tehran, Iran

(Communicated by Ali Jabbari)

Abstract

In this present investigation, for functions of the form $f(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}a_kz^k$, which are analytic in the punctured unit disk $\mathbb{U}^*=\{z\in\mathbb{C}:\ 0<|z|<1\}$, we introduce a new subclass of meromorphically p-valent functions and investigate convolution properties, Coefficient estimates and contianment for this subclass.

Keywords: q-derivative, meromorphic function, coefficient bound, extreme point, convex set, partial sum
2020 MSC: 30D30

1 Introduction

Let Σ_p denote the class of mermorphic p-valent functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^*=\{z\in\mathbb{C}:\ 0<|z|<1\}.$$

If $f\in\Sigma_p$ is given by(1.1) and $g\in\Sigma_p$ given by

$$g(z)=\frac{1}{z^p}+\sum_{k=1-p}^{+\infty}b_kz^k,$$

then the Hadamard product (or convolution) $f*g$ of f and g is defined by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{k=1-p}^{+\infty}a_kb_kz^k. \tag{1.2}$$

*Corresponding author
Email addresses: golmohamadi@pnu.ac.ir (Mohammad Hassan Golmohammadi), a_alishahy@pnu.ac.ir (Aboalghasem Alishahi)

Received: March 2023 *Accepted:* August 2023

For two function f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write $f \prec g$ or $f(z) \prec g(z) \quad (z \in \mathbb{U})$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, shuch that $f(z) = \omega(g(z))$, (see [9, 11]). Gasper and Rahman in [3] defined the q-derivative of a function $f(z)$ of the from (1.1) by

$$D_q f(z) := \frac{f(qz) - f(q)}{(q-1)z}, \tag{1.3}$$

where $z \in \mathbb{U}^*$ and $0 < q < 1$. From (1.3) for a function $f(z)$ given by (1.1) we get

$$D_q f(z) = \frac{q^{-p} - 1}{(q-1)z^{p+1}} + \sum_{k=1}^{+\infty} [k]_q a_k z^{k-1}, \qquad z \in \mathbb{U}^*, \tag{1.4}$$

where

$$[k]_q := \frac{q^k - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{k-1}. \tag{1.5}$$

also $[k]_q \to 1$ as $q \to \overline{1}$.So we conclude $\lim_{q \to \overline{1}} D_q f(z) = f'(z), z \in \mathbb{U}^*$. Many important properties of certain subclasses of meromorphic p-valent functions were studied by several authors including Aouf and Srivastava [2], Joshi and Srivastava [4], Liu and Srivastava[7], Liu and Owa [6], Liu and Srivastava [8], Ravichandran, Sivaprasadkumar and Subramanian [12].

2 Preliminaries

Using the subclasses defined by Mostafa, Aouf, Zayed and Bulboaca in[10], Now we introduce new subclasess of mermorphic p-valent functions a and investigate convolution properties and cofficien estimates for these subclasses as follows:

Definition 2.1. For $0 \leq \lambda < 1$, $-1 \leq B < A \leq 1$, and $b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$, let $\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$ be the subclass of Σ_p consisting of function $f(z)$ of the form(1.1) and satisfying the analytic criterion

$$q\frac{1-q^{-p}}{q-1} - \frac{1}{b}\left[\frac{zD_q f(z)}{(1 - \lambda\frac{1-q^{-p}}{q-1})f(z) - \lambda z D_q f(z)} + \frac{1-q^{-p}}{q-1}\right] \prec \frac{1+Az}{1+Bz} \tag{2.1}$$

Also, let $\Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$ be the subclass of Σ_p consisting of function $f(z)$ of the form(1.1) and satisfying

$$q\frac{1-q^{-p}}{q-1} - \frac{1}{b}\left[\frac{zD_q(\frac{1-q^{-p}}{q-1} zD_q f(z))}{(1 - \lambda\frac{1-q^{-p}}{q-1})(\frac{1-q^{-p}}{q-1} zD_q f(z)) - \lambda z D_q(\frac{1-q^{-p}}{q-1} zD_q f(z))} + \frac{1-q^{-p}}{q-1}\right] \prec \frac{1+Az}{1+Bz}. \tag{2.2}$$

It is easy to verify from (2.1) and (2.2) that

$$f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B] \Longleftrightarrow \frac{1-q^{-p}}{q-1} zD_q f(z) \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]. \tag{2.3}$$

we note that

1. For $p = 1$ we get $\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B] = \Sigma_1 \mathcal{S}^*_{q,\lambda}[b; A, B] = \Sigma \mathcal{S}^*_{q,\lambda}[b; A, B]$, (see [1]).
2. For $p = 1$ we get $\Sigma_p \mathcal{K}_{q,\lambda}[b; A, B] = \Sigma_1 \mathcal{K}_{q,\lambda}[b; A, B] = \Sigma \mathcal{K}_{q,\lambda}[b; A, B]$, (see [1]).

3 Main Result

In this section we give some new subclasses of mermorphic p-valent functions.

3.1 subclasses $\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ and $\Sigma_p\mathcal{K}_{q,\lambda}[b;A,B]$

In the first theorem we give some necessary and sufficieint conditions for member of subclass $\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$.

Theorem 3.1. If $f\in\Sigma_p$, then $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z^p\left[f(z)*\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)M(\theta,p)-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)}\right]\neq 0 \tag{3.1}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$. Where

$$M(\theta,p)=\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]} \tag{3.2}$$

Proof . It is easy to verify that for any $f\in\Sigma_p$ the next relations hold:

$$f(z)*\frac{1}{z^p(1-z)}=f(z), \tag{3.3}$$

and

$$f(z)*\left[\frac{1}{z^p(1-z)(1-qz)}-\frac{1+\frac{1-q}{q^{-p}-1}}{z^{p-1}(1-z)(1-qz)}\right]=\frac{1-q}{q^{-p}-1}zD_qf(z). \tag{3.4}$$

First, if $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$, in order to prove that (3.1) holds we will write (2.1) by using the definition of the subordination, that is

$$\frac{zD_qf(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)}=\frac{\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]}{1+B\omega(z)}+\frac{\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]\omega(z)}{1+B\omega(z)}, \tag{3.5}$$

where ω is a Schwarz function, hence

$$\begin{aligned}&z\Big\{[1+Be^{i\theta}]zD_qf(z)-[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}]\\&-[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}]e^{i\theta}[(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)]\Big\}\neq 0\end{aligned} \tag{3.6}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,\pi)$. Using (3.3) and (3.4), the relation (3.6) may be written as

$$\begin{aligned}&z\Bigg\{\left(-\frac{1-q^{-p}}{q-1}-B\frac{1-q^{-p}}{q-1}z-\lambda\frac{1-q^{-p}}{q-1}\times\left(\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]+\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}\right)\right)\\&f(z)*\left[\frac{1}{z^p(1-z)(1-qz)}-\frac{1+\frac{1-q}{q^{-p}-1}}{z^{p-1}(1-z)(1-qz)}\right]+\left\{\left(\lambda\frac{1-q^{-p}}{q-1}-1\right)\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]\right.\\&\left.+\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}\right\}\left[f(z)*\frac{1}{z^p(1-z)}\right]\Bigg\}\neq 0,\end{aligned} \tag{3.7}$$

which is equivalent to

$$z\left[f(z)*\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)}\right]\neq 0, \tag{3.8}$$

where $z\in\mathbb{U}$, $\theta\in[0,2\pi)$ and thus the first part of Theorem (3.1) was proved. Reversely, suppose that $f\in\Sigma_p$ satisfy the condition (3.1). Like it was previously shown, the assumption (3.1) is equivalent to (3.6), hence

$$\frac{zD_qf(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)}\neq\frac{\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]}{1+Be^{i\theta}}+\frac{\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}}{1+Be^{i\theta}}, \tag{3.9}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$. Denoting

$$\varphi(z) = \frac{zD_qf(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z) - \lambda zD_qf(z)}$$

and

$$\psi(z) = \frac{\left[b(q\frac{1-q^{-p}}{q-1} - 1) - \frac{1-q^{-p}}{q-1}\right] + \left[b(qB\frac{1-q^{-p}}{q-1} - A) - B\frac{1-q^{-p}}{q-1}\right]z}{1+Bz},$$

The relation (3.9) means that

$$\varphi(\mathbb{U}) \cap \psi(L(\mathbb{U})) = \emptyset$$

and

$$(L(z) = \Psi(z) - \left[b(q\frac{1-q^{-p}}{q-1} - 1) - \frac{1-q^{-p}}{q-1}\right]).$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(L(\mathbb{U}))$. Therefore, using the fact that $\varphi(0) = \psi(L(0))$ and the p-valent function ψ, it follows that $\varphi(z) \prec \psi(z)$, which implies that $f \in \Sigma_p\mathcal{S}^*_{q,\lambda}[b; A, B]$. Thus, the proof of Theorem (3.1) is completed. □

Theorem 3.2. If $f \in \Sigma_p$, then $f \in \Sigma_p\mathcal{K}_{q,\lambda}[b; A, B]$ if and only if

$$z^p\left[f(z) * \frac{1 - \frac{1-q^{p+2}}{1-q^p}z + \left[\frac{q-q^p}{1-q^p}z + qz^2\frac{q^{p+1}-1}{1-q^p}\right]\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)M(\theta, p) - (q + \frac{\lambda}{q^p})\right\}}{z^p(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \tag{3.10}$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$, where $M(\theta, p)$ is given by (3.2).

Proof . From (2.3) it follows that $f \in \Sigma_p\mathcal{K}_{q,\lambda}[b; A, B]$ if and only if $\Phi_q(z) := \frac{q-1}{1-q^{-p}}zD_qf(z) \in \Sigma_p\mathcal{S}^*_{q,\lambda}[b; A, B]$. Then, accoding to Theorem (3.1), the function Φ_q belongs to $\Sigma_p\mathcal{S}^*_{q,\lambda}[b; A, B]$ if and only if

$$z[\Phi_q(z) * g(z)] \neq 0, \tag{3.11}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$, where

$$g(z) = \frac{1 + \left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1} + (A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]} - (q + \frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)}. \tag{3.12}$$

A simple computation shows that

$$\begin{aligned} D_qg(z) = \frac{g(qz) - g(z)}{(q-1)z} &= \frac{(1-q^p) - (1-q^{p+2})z}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2z)} \\ &+ \frac{[(q-q^p)z + qz^2(q^{p+1}-1)]\left(\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1} + (A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}\right.}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2z)} \\ &- \frac{\left.(q + \frac{\lambda}{q^p})\right)}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2z)} \end{aligned} \tag{3.13}$$

and therefore

$$\begin{aligned} \frac{1-q}{1-q^{-p}}zD_qg(z) &= \frac{1 - \frac{1-q^{p+2}}{1-q^p}z + [\frac{q-q^p}{1-q^p}z + \frac{q^{p+1}-1}{1-q^p}qz^2][\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1} + (A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}}{(1-z)(1-qz)(1-q^2z)} \\ &- \frac{(q + \frac{\lambda}{q^p}))]}{(1-z)(1-qz)(1-q^2z)} \end{aligned}$$

Using the above relation and the identity

$$[\frac{q-1}{1-q^{-p}}zD_qf(z)]*g(z)=f(z)*[\frac{q-1}{1-q^{-p}}zD_qg(z)] \tag{3.14}$$

it is easy to check that (3.11) is equivalent to (3.10). □

Theorem 3.3. If $f\in\Sigma_p$, then $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$1+\sum_{k=1}^{\infty}\Big[\frac{(1-\frac{\lambda}{q^p}[k]_q)\left[(1-q\frac{1-q^{-p}}{q-1})+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]bq^p}{q^pb\left[\left(1-q\frac{1-q^{-p}}{q-1}\right)+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]}+\frac{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)(1+Be^{i\theta})[k]_q}{q^pb\left[\left(1-q\frac{1-q^{-p}}{q-1}\right)+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]}\Big]a_kz^{k+p}\neq 0$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$.

Proof . If $f\in\Sigma_p$, then from Theorem (3.1) we have

$$z^p[f(z)*\frac{1+(1-\lambda\frac{1-q^{-p}}{q-1})\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})z}{z^p(1-z)(1-qz)}]\neq 0 \tag{3.15}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$, since

$$\frac{1}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}[k+p]_qz^k \tag{3.16}$$

it follows that

$$\frac{1+(1-\lambda\frac{1-q^{-p}}{q-1})\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})z}{z^p(1-z)(1-qz)}$$
$$=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}\Big[1+[\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-\frac{\lambda}{q^p}][k+p]_q\Big]z^k \tag{3.17}$$

and we may that easily check that (**??**) is equvalent to (3.15). □

3.2 Duality

In this section, we by using the definitions of the duality in[5], for a set $V\subset\mathbb{A}$,The dual set V, by $V^*$is defined as

$$V^*=\Big\{g\in\mathbb{A};\ \frac{1}{z}(f*g)(z)\neq 0\ for\ all f\in V\ and\ z\in\mathbb{U}\Big\}.$$

Now, for a set $W\subset\Sigma_p$, the dual W, denoted by W^*, is defined as

$$W^*=\Big\{g\in\Sigma_p;\ z^p(f*g)(z)\neq 0\ for\ all f\in W\ and\ z\in\mathbb{U}\Big\}.$$

The standard reference to duality for convolutions is the morograph by Rucheweyh [14], and his paper [13]. Assume that $f\in\Sigma_p$. By Theorem(3.1), $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z^2(f(z)*h_\theta(z))\neq 0,\qquad z\in\mathbb{U}^*, \tag{3.18}$$

where

$$h_\theta(z)=\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)} \tag{3.19}$$

and

$$M(\theta,p)=\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}. \tag{3.20}$$

Moreover, for $f \in \Sigma_p$. By Theorem(3.1), $f \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$ if and only if

$$z^2(f(z) * L_\sigma(z)) \neq 0, \qquad z \in \mathbb{U}^*,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$., where $M(\theta, p)$ is given by 1, where

$$L_\sigma(z) = \frac{1 - \frac{1-q^{p+2}}{1-q^p} z + \left[\frac{q-q^p}{1-q^p} z + qz^2 \frac{q^{p+1}-1}{1-q^p}\right] \left\{\left(1 - \lambda \frac{1-q^{-p}}{q-1}\right) M(\theta, p) - (q + \frac{\lambda}{q^p})\right\}}{z^p(1-z)(1-qz)(1-q^2z)}.$$

Definition 3.4. We define W^* as follows:

$$\begin{aligned} W^*_\theta &= (\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B])^* \\ &= \Big\{h_\theta(z) \in \Sigma_p;\ z^p(f(z) * h_\theta(z))(z) \neq 0\ ,\ f \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]\ ,\ \theta \in [0, 2\pi)\Big\}. \end{aligned}$$

and

$$\begin{aligned} W^*_\zeta &= (\Sigma_p \mathcal{K}_{q,\lambda}[b; A, B])^* \\ &= \Big\{l_\zeta(z) \in \Sigma_p;\ z^p(f(z) * l_\zeta(z))(z) \neq 0\ ,\ f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]\Big\}. \end{aligned}$$

Theorem 3.5. Let function $h_\theta(z) = \dfrac{1}{z^p} + \sum_{k=1-p}^{\infty} c_k z^k \in W^*_\theta$. The

$$\begin{aligned} |c_k| &\leq (1 + q + q^2 + \cdots + q^{k+p-1})\Big\{(1 - \lambda \frac{1-q^{-p}}{q-1}) \frac{1+B}{q^p b[1 - q\frac{1-q^{-p}}{q-1} + (A - Bq\frac{1-q^{-p}}{q-1})]} \\ &\quad -(q + \frac{\lambda}{q^p})\Big\} - (1 + q + q^2 + \cdots + q^{k+p-1}) \end{aligned}$$

Proof . Let $h_\theta \in W^*$. Then we have

$$\begin{aligned} h_\theta(z) &= \frac{1}{z^p(1-z)(1-qz)} + \frac{\Big\{(1 - \lambda\frac{1-q^{-p}}{q-1})M(\theta, p) - (q + \frac{\lambda}{q^p})\Big\} z}{z^p(1-z)(1-qz)} \\ &= \frac{1}{z^p}(1 + (1+q)z + (1 + q + q^2)z^2 + \cdots + q^{k+p-1}) \\ &\quad + \frac{\Big\{(1 - \lambda\frac{1-q^{-p}}{q-1})M(\theta, p) - (q + \frac{\lambda}{q^p})\Big\}}{z^{p-1}}(1 + q + q^2)z^2 + \cdots + q^{k+p-1}) \\ &= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} c_k z^k \end{aligned}$$

where

$$c_k = (1 + q + q^2 + \cdots + q^{k+p-1})\Big\{(1 - \lambda\frac{1-q^{-p}}{q-1})M(\theta, p) - (q + \frac{\lambda}{q^p})\Big\} - (1 + q + q^2 + \cdots + q^{k+p-1})$$

and so

$$|c_k| \leq (1 + q + q^2 + \cdots + q^{k+p-1})\Big\{(1 - \lambda\frac{1-q^{-p}}{q-1})M(\theta, p) - (q + \frac{\lambda}{q^p})\Big\} - (1 + q + q^2 + \cdots + q^{k+p-1})$$

where

$$M(\theta, p) = \frac{1 + Be^{i\theta}}{q^p b[1 - q\frac{1-q^{-p}}{q-1} + (A - Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}.$$

□

Corollary 3.6. Let $f(z) = \dfrac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \in \Sigma_p$. if

$$\sum_{k=1-p}^{\infty} \Big[(1+q+q^2+\cdots+q^{k+p-1})\Big((1-\lambda\frac{1-q^{-p}}{q-1})|M(\theta,p)|-(q+\frac{\lambda}{q^p})\Big)+(q+\frac{\lambda}{q^p})\Big]$$
$$+(1+q+q^2+\cdots+q^{k+p})|c_k| \leq 1,$$

Then $f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$.

Proof . Let $h_\theta(z) = \dfrac{1}{z^p} + \sum_{k=1-p}^{\infty} c_k z^k \in W_\theta^*$. The we have

$$\begin{aligned} z^p|(f(z)*h_\theta(z))| &= |1+\sum_{k=1-p}^{\infty} a_k c_k z^k| \\ &\geq 1-\sum_{k=1-p}^{\infty} |a_k||c_k||z| \\ &> 1-\sum_{k=1-p}^{\infty} |a_k||c_k \\ &> 0. \end{aligned}$$

Thus $z^p(f(z)*h_\theta(z)) \neq 0$ and now form Theorem 2.1 we have $f \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$. $\square$

References

[1] M.K. Aouf, A.O. Mostafa, and H.M. Zayed, *Convolution properties for some subclasses of meromorphic functions of complex order*, Abstr. Appl. Anal. **2015** (2015), Article ID 973613, 1–6.

[2] M.K. Aouf and H.M. Srivastava, *A new criterion for meromorphivally p-valent convex functions of order α*, Math. Sci. Res. Hot. Line **8** (1997), no. 1, 7–12.

[3] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[4] S.B. Joshi and H.M. Srivastava, *A certain family of meromorphivally multivalent functions*, Comput. Math. Appl. **38** (1999), no. 3/4, 201–211.

[5] R. Kargar, A. Ebadian, and J. Sokol, *Some properties of analytic tunctions related with bounded positive real part*, Int. J. Nonlinear Anal. Appl. **8** (2017), 235–344.

[6] J.L. Liu and S. Owa, *On a class of meromorphivally p-valent functions involving certain linear operators*, Int. J. Math. Math. Sci. **32** (2002), 271–180.

[7] J.L. Liu and H.M. Srivastava, *A linear operator and associated families of meromorphivally multivalent functions*, J. Math. Anal. Appl. **259** (2001), 566–581.

[8] J.L. Liu and H.M. Srivastava, *Some convolution conditions for starlikeness and convexity of meromorphivally multivalent functions*, Appl. Math. Lett. **16** (2003), 13–16.

[9] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.

[10] A.O. Mostafa, M.k. Aouf, H.M. Zayed and T. Bulboaca, *Convolution conditions for subelasse of mermorphic functions of complex order associated with basic Bessel functions*, J. Egypt. Math. Soc. **25** (2017), 286–290.

[11] S.S. Miller and P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Appl. Math. No.255 Marcel Dekker, Inc., New York, 2000.

[12] V. Ravichandran, S. Sivaprasadkumar, and K.G. Subramanian, *Convolution conditions for spirallikeness and convex spirallikeness of certain mermorphic p-valent functions*, J. Pure. Appl. Math. **11** (2004), 1–7.

[13] S. Ruscheweyh, *Duality for Hadamaed products applications to extremal proplems for functions regular in the unit disc*, Trans. Amer. Math. Soc. **210** (1975), 63–74.

[14] S. Ruscheweyh, *Convolutions in Geometric Functions Theory*, Les Presses De Montreal, Montreal, 1982.

[15] Z.-G. Wang and M.-L. Li, *Some properties of certain family of multiplier transforms*, Filomat **31** (2017), no. 1, 159–173.