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A new extension of the Darbo theorem for the Schauder type selections with an application

Ehsan Lotfali Ghasab, Hamid Majani*

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran

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Abstract

In the present article, we provide a new nonlinear contraction for the Schauder type selections of multi-valued mappings in metric spaces which is a new spread of the Darbo theorem. Meanwhile, we apply the main results in coupled fixed-point theory and functional integral equation.

Keywords: Measure of noncompactness, Schauder type selections, Darbo theorem, multi-valued mapping, complete metric spaces
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1 Introduction and preliminaries

In 1930, the study of the measure of noncompactness (briefly, MoNC) was started by several researchers (see [1, 2, 3, 4, 6, 15] and references therein). At the same time, Schauder [16] recommended his fixed-point principle. In 1955, Darbo [9] applied the concept of MoNC to prove the existence of fixed-points of the condensing mappings. Note that him result generalized both the classical Banach principle and the Schauder fixed-point theorem. Also, his theorem has many applications to prove the existence of solutions for a big category of differential and integral equations (see [5, 10]). On the other hand, Nadler [14] expressed the contraction principle for multi-valued mappings. These mappings and related selection theorems are useful tools in many sections of applied sciences.

In the present article, we establish the existence selections for generalized multi-valued and single-valued mappings on complete metric spaces using some new generalizations of Darbo theorem. Meanwhile, we obtain a relationship between coupled fixed-point and fixed-point. Finally, we apply our main theorem in a functional integral equation. For these, we need some notations and definitions which are expressed below.

Notation.

- $\mathcal{F}$ is a Banach space with the norm $\|.\|$;
- $B(a, r)$ is the closed ball in $\mathcal{F}$ with center a and radius r;
- for $A \subset \mathcal{F}$, $\bar{A}$ and $Conv\, A$ are the closure and the closed convex hull of A;

*Corresponding author
Email addresses: e-lotfali@stu.scu.ac.ir; e.l.ghasab@gmail.com (Ehsan Lotfali Ghasab), h.majani@scu.ac.ir; majani.hamid@gmail.com (Hamid Majani)

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- $A+B$ and λA with $\lambda \in \mathbb{R}$ are algebraic operations on the sets A and B;
- $N(A)$ is the collection of all nonempty subsets of A;
- $M_{\mathcal{F}}$ is the collection of all nonempty and bounded subsets of $\mathcal{F}$ and $N_{\mathcal{F}}$ is its sub-collection including all relatively compact set.

Definition 1.1. [6] Consider a mapping $\nu : M_{\mathcal{F}} \to \mathbb{R}_+ = [0,\infty)$ provided that the following cases are held:

i) The family $ker\nu = \{A \in M_{\mathcal{F}} : \nu(A) = 0\}$ is nonempty and $ker\nu \subset N_{\mathcal{F}}$, where $\ker\nu$ is the kernel of the MoNC ν;

ii) $A \subset B \Rightarrow \nu(A) \leq \nu(B)$;

iii) $\nu(\bar{A}) = \nu(A)$;

iv) $\nu(Conv\, A) = \nu(A)$;

v) $\nu(\lambda A + (1-\lambda)B) \leq \lambda\nu(A) + (1-\lambda)\nu(B)$ for $\lambda \in [0,1]$;

vi) If (A_n) is a nested sequence of closed sets from $M_{\mathcal{F}}$ so that $\lim_{n\to\infty} \nu(A_n) = 0$, then $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty.

Then ν is called a MoNC in $\mathcal{F}$.

Note that A_∞ in axiom (vi) is a member of the $\ker\nu$.

Definition 1.2. [8] Consider a multi-valued mapping G from $\mathcal{F}$ to $N(\mathcal{F})$.

- A selection from G is a function $f : \mathcal{F} \to \mathcal{F}$ with $f(a) \in G(a)$ for any $a \in \mathcal{F}$.
- $G^{-1}(b)$ is the set of all a belonging to $\mathcal{F}$ such that b is belongs to $G(a)$ for each $b \in \mathcal{F}$.

Theorem 1.3. (Browder-Ky Fan Theorem)[8] Assume that $G : \mathcal{F} \to BC(\mathcal{F})$ is a multi-valued mapping having convex values and $G^{-1}(b)$ is open for all b. Then there exists a continuous function $f : \mathcal{F} \to \mathcal{F}$ such that $f(a) \in G(a)$ for all a.

2 Results

In this section, $A \neq \emptyset$ is a bounded, closed and convex subset of $\mathcal{F}$. Moreover, suppose that Φ is the class of all nondecreasing, subadditive, bounded from below and upper semi-continuous functions $\phi : [0,+\infty) \to [0,+\infty)$ such that $\lim_{n\to\infty} \phi^n(i) = 0$ for every $i \geq 0$. Also, we consider $\beta : [0,+\infty) \to [0,+\infty)$ is a subadditive, continuous and nondecreasing function with $\beta^{-1}(0) = (0)$.

Theorem 2.1. Suppose that $W : A \to N(A)$ is a multi-valued mapping having convex values so that $W^{-1}(b)$ is open for all b, $\phi \in \Phi$ and

$$\beta(\nu(WA)) \leq \phi(\beta(\nu(A))) - \phi(\beta((\nu(WA)))). \tag{2.1}$$

Then W has a fixed-point.

Proof . Using Lemma 1.3, there exists selection of $f : A \to A$ such that $fa \in Wa$ for all $a \in A$. Suppose $E_n = Conv f E_{n-1}$ for $n = 1, 2, ...$, where $E_0 = A$. Then, we have $E_n = Conv f(E_{n-1}) \subset W(E_{n-1})$. Now, from (2.1), we get

$$\beta(\nu(E_1)) \leq \phi(\beta(\nu(E_0))) - \phi(\beta((\nu(E_1)))).$$

Also, for $E_1 \subset A$, there exists $E_2 \subset W E_1$ with $E_1 \neq E_2$ and

$$\beta(\nu(E_2)) \leq \phi(\beta(\nu(E_1))) - \phi(\beta((\nu(E_2)))).$$

Continue this process, we obtain a sequence $\{E_n\}$, where $E_n \subset WE_{n-1}$ and

$$\beta(\nu(E_n)) \leq \phi(\beta(\nu(E_{n-1}))) - \phi(\beta((\nu(E_n)))). \tag{2.2}$$

If there exists $n_0 \in \mathbb{N}$ provided that $\nu(E_{n_0}) = 0$, then E_{n_0} will be compact. In this manner, Schauder theorem induces that f has a fixed-point. Now, from (2.2), we have $\phi(\beta(\nu(E_{n-1}))) \geq \phi(\beta(\nu(E_n)))$ for all n. Hence, $\{\phi(\beta(\nu(E_n)))\}$ is a decreasing sequence. Since ϕ is bounded from below, this sequence is convergence. On the other, from Remark 3 of [12] and Remark 2 of [13], we get

$$\lim_{i\to 0^+} \frac{\beta(i)}{i} = sup\{\frac{\beta(i)}{i} : i > 0\},$$

so

$$\lim \inf_{i\to 0^+} \frac{\beta(i)}{i} > 0. \tag{2.3}$$

By (2.3), there exists $\delta > 0$ and $c > 0$ such that

$$\beta(i) \geq ci, \tag{2.4}$$

for all $i \in [0, \delta]$. Since β is nondecreasing, then $\beta(i) \geq \beta(\delta)$ for all $i \in [\delta, +\infty)$. Let $0 < \epsilon < \beta(\delta)$. Then $\beta(i) > \epsilon$ for any $i \in [\delta, +\infty)$, i.e. if $\beta(i) \leq \epsilon$, then $i \in [0, \delta]$. Therefore, we have

$$\{i \geq 0 : \beta(i) \leq \epsilon\} \subset [0, \delta],$$

which together with (2.4) implies that

$$\beta(i) \geq ci \tag{2.5}$$

for all $i \in \{i \geq 0 : \beta(i) \leq \epsilon\}$. Now, notice that $\{\phi(\beta(\nu(E_n)))\}$ is convergent. Thus, there exists some $N \in \mathbb{N}$ so that

$$\beta(\nu(E_n)) \leq \phi(\beta(\nu(E_{n-1}))) - \phi(\beta(\nu(E_n))) < \epsilon$$

for each $n \geq N$, which induces that $\beta(\nu(E_n)) \to 0$ as $n \to \infty$. Moreover, by (2.5), we get

$$c\nu(E_n) \leq \beta(\nu(E_n)) \leq \phi(\beta(\nu(E_{n-1}))) - \phi(\beta(\nu(E_n)))$$

for every $n \geq N$, which induces that $\nu(E_n) \to 0$. Now, by axiom (vi) of Definition 1.1, we conclude that $E_\infty \subset A$ is a nonempty, closed, convex set, where $E_\infty = \bigcap_{n=1}^{\infty} E_n$. Furthermore, E_∞ is invariant under function f and $E_\infty \in \ker \nu$. Now, by applying the Schauder theorem, the proof ends (because f has a fixed-point and since $fa \in Wa$, W has a fixed-point). □

Theorem 2.2. Suppose $W : A \to A$ is a mapping provided that

$$\beta(\nu(WA)) \leq \phi(\beta(\nu(A))) - \phi(\beta(\nu(WA))),$$

where $\phi \in \Phi$. Then W has a fixed-point.

Proof . The proof is analogous on the argument of Theorem 2.1 and left to the reader. □

Corollary 2.3. Assume that $W : A \to A$ is a mapping so that

$$\beta(||Wa - Wb||) \leq \phi(\beta(||a - b||)) - \phi(\beta(||Wa - Wb||)),$$

where $||\cdot||$ is the same usual norm and $\phi \in \Phi$. Then W has a fixed-point.

Proof . Let $\nu : M_{\mathcal{F}} \to \mathbb{R}_+$ defined by $\nu(A) = diamA$, where $diamA = \sup\{||a-b|| : a, b \in A\}$ stands for the diameter of A. Note that ν is a MoNC in $\mathcal{F}$. So, we have

$$\sup_{a,b\in A} \beta(||Wa - Wb||) \leq \sup_{a,b\in A} \phi(\beta(||a - b||)) - \sup_{a,b\in A} \phi(\beta(||Wa - Wb||)).$$

By the continuity of the function β, we derive that

$$\beta(\sup_{a,b\in A}||Wa-Wb||)\leq\phi(\beta(\sup_{a,b\in A}||a-b||))-\phi(\beta(\sup_{a,b\in A}||Wa-Wb||)).$$

This yields that $\beta(\nu(WA))\leq\phi(\beta(\nu(A)))-\phi(\beta(\nu(WA)))$. Now, using Theorem 2.2, W has a fixed-point. □

As you know, the theory of coupled fixed-points was started by Bhaskar and Lakshmikantham's article [7]. After that, many researchers generalized this concept. For more details on n-tuple fixed-point theorems, we refer to [11, 17] and significantly some references therein.

Theorem 2.4. [4] Let $\nu_1,\nu_2,\ldots,\nu_n$ be M(s)oNC in Banach spaces $\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n$, respectively. Also, suppose that $\mathcal{W}:[0,\infty)^n\to[0,\infty)$ is a convex function so that $\mathcal{W}(l_1,\ldots,l_n)=0$ iff $l_i=0$ for $i=1,2,\ldots,n$. Then $\tilde{\nu}(A)=\mathcal{W}(\nu_1(A_1),\nu_2(A_2),\ldots,\nu_n(A_n))$ defines a MoNC in $\mathcal{F}_1\times\mathcal{F}_2\times\cdots\times\mathcal{F}_n$, where A_i are the natural projection of A into $\mathcal{F}_i$ for $i=1,2,\ldots,n$.

Notice that $\tilde{\nu}(A)=\nu(A_1)+\nu(A_2)$ is a MoNC, where A_1 and A_2 denote the natural projections of A into $\mathcal{F}$ (see [6]).

Theorem 2.5. Assume that $W:A\times A\to A$ is a mapping so that for any subset A_1,A_2 of A, we have

$$\beta(\nu(W(A_1\times A_2)))\leq\frac{1}{2}[\phi(\beta(\nu(A_1)+\nu(A_2)))]-\phi(\beta(\nu(W(A_1\times A_2)))),$$

where $\phi\in\Phi$. Then W has a coupled fixed-point.

Proof . Define the mapping $\mathcal{W}:A^2\to A^2$ by $\mathcal{W}(a,b)=(W(a,b),W(b,a))$. Now, we have

$$\begin{aligned}
\beta(\tilde{\nu}(\mathcal{W}(A)))&=\beta(\tilde{\nu}((W(A_1\times A_2),W(A_2\times A_1))))\\
&\leq\beta(\nu(W(A_1\times A_2)))+\beta(\nu(W(A_2\times A_1)))\\
&\leq\frac{1}{2}[\phi(\beta(\nu(A_1)+\nu(A_2)))]-\phi(\beta(\nu(W(A_1\times A_2))))+\frac{1}{2}[\phi(\beta(\nu(A_2)+\nu(A_1)))]-\phi(\beta(\nu(W(A_2\times A_1))))\\
&=\frac{1}{2}[\phi(\beta(\tilde{\nu}(A)))]-\phi(\beta(\nu(W(A_1\times A_2))))+\frac{1}{2}[\phi(\beta(\tilde{\nu}(A)))]-\phi(\beta(\nu(W(A_2\times A_1))))\\
&=\phi(\beta(\tilde{\nu}(A)))-[\phi(\beta(\nu(W(A_1\times A_2))))+\phi(\beta(\nu(W(A_2\times A_1))))]\\
&\leq\phi(\beta(\tilde{\nu}(A)))-[\phi(\beta(\nu(W(A_1\times A_2))+\nu(W(A_2\times A_1))))]\\
&=\phi(\beta(\tilde{\nu}(A)))-\phi(\beta(\tilde{\nu}(W(A_1\times A_2),W(A_2\times A_1))))\\
&=\phi(\beta(\tilde{\nu}(A)))-\phi(\beta(\tilde{\nu}(\mathcal{W}(A)))).
\end{aligned}$$

Continue the same argument as in the proof of Theorem 2.2. Thus, $\mathcal{W}$ has a fixed-point, which induces that W has a coupled fixed-point. □

3 Application

In this section we provide applications of the generalization of Darbo fixed-point theorem contained in Theorem 2.1 to prove the existence of solutions of a functional integral equation. For this, assume that $BC(\mathbb{R}_+)$ is the Banach space of all real, continuous and bounded functions on the positive real number with $\|y\|=sup\{|y(i)|:i\geq0\}$. Now, let A be a nonempty and bounded subset of $BC(\mathbb{R}_+)$ and $L>0$. For $y\in A$ and $\varrho>0$, we consider the following notations:

$$\begin{aligned}
\mathcal{M}^L(y,\varrho)&=\sup\{|y(i)-y(j)|:i,j\in[0,L],|i-j|\leq\varrho\},\\
\mathcal{M}^L(A,\varrho)&=\sup\{\mathcal{M}^L(y,\varrho):y\in A\},\\
\mathcal{M}_0^L(A)&=\lim_{\varrho\to0}\mathcal{M}^L(A,\varrho),\\
\mathcal{M}_0(A)&=\lim_{L\to\infty}\mathcal{M}_0^L(A).
\end{aligned}$$

Further, for $i \in \mathbb{R}_+$, put $A(i) = \{y(i) : y \in A\}$. Finally, define the mapping ν on the family $M_{BC(\mathbb{R}_+)}$ by

$$\nu(A) = \mathcal{M}_0(A) + \limsup_{i\to\infty} diamA(i),$$

where $diamA(i)$ is understood as

$$diamA(i) = \sup\{|y(i) - z(i)| : y, z \in A\}.$$

The mapping ν is a MoNC in $BC(\mathbb{R}_+)$ (see [6]). Also, $ker\nu$ includes nonempty and bounded sets A so that functions in A are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle organized by the graphs of functions in A arrives to 0 at infinity.

Theorem 3.1. Consider the following conditions:

(i) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping and the mapping $i \to f(i, 0)$ located in $BC(\mathbb{R}_+)$;

(ii) There is $\phi \in \Phi$ provided that for every $i \in \mathbb{R}_+$ and any $a, b \in \mathbb{R}$, we have

$$|f(i, a) - f(i, b)| \le \phi(|a - b|) - \phi(|f(i, a) + \int_0^i g(i, j, a)ds - f(i, b) - \int_0^i g(i, j, b)dj|).$$

Further, suppose that ϕ is superadditive;

(iii) There are continuous mappings $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $o, h : \mathbb{R}_+ \to \mathbb{R}_+$ provided that $\lim_{i\to\infty} o(i) \int_0^i h(j)dj = 0$ and $|g(i, j, a)| \le o(i)h(j)$ for $i, j \in [0, \infty)$ with $j \le i$ and for any $a \in \mathbb{R}$;

(iv) There is a positive solution r_0 of the relation $\phi(r) + q \le r$, with $q = \sup\{|f(i, 0)| + o(i) \int_0^i h(j)dj : i \ge 0\}$.

Then the functional integral equation

$$y(i) = f(i, y(i)) + \int_0^i g(i, j, y(j))dj \tag{3.1}$$

has a solution in $BC(\mathbb{R}_+)$.

Proof . Consider $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ by

$$(Ty)(i) = f(i, y(i)) + \int_0^i g(i, j, y(j))dj$$

for $i \in \mathbb{R}_+$ and $W : BC(\mathbb{R}_+) \to N(BC(\mathbb{R}_+))$ by $W(y) = \{(Ty)(i)\}$. By assumptions, the function Ty is continuous on $\mathbb{R}_+$. Moreover, for an optional $y \in BC(\mathbb{R}_+)$, we get

$$\begin{aligned}|(Ty)(i)| &\le |f(i, y(i)) - f(i, 0)| + |f(i, 0)| + \int_0^i |g(i, j, y(j))|dj \\ &\le \phi(|y(i)|) - \phi(|f(i, y(i)) + \int_0^i g(i, j, y(j))dj - f(i, 0) - \int_0^i g(i, j, 0)dj|) + |f(i, 0)| + c(i) \\ &\le \phi(|y(i)|) + |f(i, 0)| + c(i),\end{aligned}$$

which $c(i) = o(i) \int_0^i h(j)dj$. Since the function ϕ is nondecreasing, $\|Ty\| \le \phi(\|y\|) + q$, where q is defined in (iv). Further, we deduce that T is a self-mapping on B_{r_0}, where r_0 is a constant extant in (iv). Here, we present T is continuous on B_{r_0}. For this, select an optional number $\varrho > 0$. Then, by a normal calculation, we gain

$$|(Ty)(i) - (Tz)(i)| \le \phi(\varrho) - \phi(|f(i, y(i)) + \int_0^i g(i, j, y(j))dj - f(i, z(i)) - \int_0^i g(i, j, z(j))dj|) + 2c(i) \tag{3.2}$$

for $y, z \in B_{r_0}$ so that $\|y - z\| \le \varrho$ and for any $i \in \mathbb{R}_+$. Moreover, by hypothesis (iii), there exists a number $L > 0$ so that

$$2o(i) \int_0^i h(j)dj \le \varrho \tag{3.3}$$

for each $i \geq L$. Thus, by (3.2) and (3.3), we obtain

$$|(Ty)(i) - (Tz)(i)| \leq 2\varrho - \phi(|f(i,y(i)) + \int_0^i g(i,j,y(j))dj - f(i,z(i)) - \int_0^i g(i,j,z(j))dj|) < 2\varrho \tag{3.4}$$

for an arbitrary $i \geq L$. Now, let us define the quantity $\mathcal{M}^L(g,\varrho)$ and $\mathcal{M}^L(f,\varrho)$ by putting

$$\mathcal{M}^L(g,\varrho) = \sup\{|g(i,j,a) - g(i,j,b)| : i,j \in [0,L], a,b \in [-r_0,r_0], |a-b| \leq \varrho\},$$

$$\mathcal{M}^L(f,\varrho) = \sup\{|f(i,y(i)) + \int_0^i g(i,j,y(j))dj - f(i,z(i)) - \int_0^i g(i,j,z(j))dj| : i,j \in [0,L], y,z \in B_{r_0}, ||y-z|| \leq \varrho\}.$$

Because of the uniformly continuity of $g(i,j,a)$ on $[0,L] \times [0,L] \times [-r_0,r_0]$, $\mathcal{M}^L(g,\varrho) \to 0$ as $\varrho \to 0$. Now, using (3.2), we obtain

$$|(Ty)(i) - (Tz)(i)| \leq \phi(\varrho) - \phi(\mathcal{M}^L(f,\varrho)) + \int_0^L \mathcal{M}^L(g,\varrho)dj < \phi(\varrho) + L\mathcal{M}^L(g,\varrho) \tag{3.5}$$

for an optional fixed $i \in [0,L]$. Finally, combining (3.4) and (3.5), the operator T will be continuous on the ball B_{r_0}. Now, select an arbitrary nonempty subset A of B_{r_0} also, choose arbitrarily $i,j \in [0,L]$ with $j < i$ so that $|i-j| \leq \varrho$. Then, for $y \in A$, we get

$$\begin{aligned}
|(Ty)(i) - (Ty)(j)| &= |f(i,y(i)) + \int_0^i g(i,\tau,y(\tau))d\tau - f(j,y(j)) - \int_0^j g(j,\tau,y(\tau))d\tau| \\
&\leq |f(i,y(i)) - f(j,y(i))| + |f(j,y(i)) - f(j,y(j))| \\
&\quad + |\int_0^i g(i,\tau,y(\tau))d\tau - \int_0^i g(j,\tau,y(\tau))d\tau| + |\int_0^i g(j,\tau,y(\tau))d\tau - \int_0^j g(j,\tau,y(\tau))d\tau| \\
&\leq \mathcal{M}_1^L(f,\varrho) + \phi(|y(i) - y(j)|) - \phi(|f(i,y(i)) + \int_0^i g(i,j,y(j))dj - f(i,y(j)) \\
&\quad - \int_0^i g(i,j,y(j))dj|) + \int_0^i |g(i,\tau,y(\tau)) - g(j,\tau,y(\tau))|d\tau + \int_j^i |g(j,\tau,y(\tau))|d\tau \\
&\leq \mathcal{M}_1^L(f,\varrho) + \phi(\mathcal{M}^L(y,\varrho)) - \phi(\mathcal{M}^L(f,\varrho)) + \int_0^i \mathcal{M}_1^L(g,\varrho)d\tau + o(j)\int_j^i h(\tau)d\tau \\
&\leq \mathcal{M}_1^L(f,\varrho) + \phi(\mathcal{M}^L(y,\varrho)) - \phi(\mathcal{M}^L(f,\varrho)) + L\mathcal{M}_1^L(g,\varrho) + \varrho\sup\{o(j)h(i) : i,j \in [0,L]\},
\end{aligned} \tag{3.6}$$

in which

$$\mathcal{M}_1^L(f,\varrho) = \sup\{|f(i,y) - f(j,y)| : i,j \in [0,L], y \in [-r_0,r_0], |i-j| \leq \varrho\},$$
$$\mathcal{M}_1^L(g,\varrho) = \sup\{|g(i,\tau,y) - g(j,\tau,y)| : i,j,\tau \in [0,L], y \in [-r_0,r_0], |i-j| \leq \varrho\}.$$

Note that f and g are uniform continuous on $[0,L] \times [-r_0,r_0]$ and $[0,L] \times [0,L] \times [-r_0,r_0]$, respectively. Thus, $\mathcal{M}_1^L(f,\varrho), \mathcal{M}_1^L(g,\varrho) \to 0$ as $\varrho \to 0$. Further, by the continuity of the mappings $o = o(i)$ and $h = h(i)$ on $\mathbb{R}_+$, we find that $\sup\{o(j)h(i) : i,j \in [0,L]\}$ is a finite value. Hence, by (3.6), we arrive

$$\mathcal{M}_0^L(TA) \leq \lim_{\varrho\to 0} \phi(\mathcal{M}^L(A,\varrho)) - \lim_{\varrho\to 0} \phi(\mathcal{M}^L(TA,\varrho)).$$

Now, since ϕ is upper semicontinuous, we get

$$\mathcal{M}_0^L(TA) \leq \phi(\mathcal{M}_0^L(A)) - \phi(\mathcal{M}_0^L(TA))$$

and consequently,

$$\mathcal{M}_0(TA) \leq \phi(\mathcal{M}_0(A)) - \phi(\mathcal{M}_0(TA)). \tag{3.7}$$

Now, select two optional functions $y, z \in A$. By simple calculation, we gain

$$|(Ty)(i) - (Tz)(i)| \leq \phi(|y(i) - z(i)|) - \phi(|f(i,y(i)) + \int_0^i g(i,j,y(j))dj - f(i,z(i)) - \int_0^i g(i,j,z(j))dj|) + 2c(i)$$

for $i \in \mathbb{R}$. It follows for this estimate that

$$diam(TA)(i) \leq \phi(diamA(i)) - \phi(diamTA(i)) + 2c(i).$$

Now, because of the upper semicontinuity of ϕ we obtain

$$\limsup_{i\to\infty} diam(TA)(i) \leq \phi(\limsup_{i\to\infty} diamA(i)) - \phi(\limsup_{i\to\infty} diamTA(i)). \tag{3.8}$$

Now, combining (3.7) and (3.8), applying the superadditivity of ϕ and using (iii), we gain

$$\mathcal{M}_0(TA) + \limsup_{i\to\infty} diam(TA)(i) \leq \phi(\mathcal{M}_0(A) + \limsup_{i\to\infty} diamA(i)) - \phi(\mathcal{M}_0(TA) - \limsup_{i\to\infty} diamTA(i)),$$

that results

$$\nu(TA) \leq \phi(\nu(A)) - \phi(\nu(TA)), \tag{3.9}$$

in which ν is the MoNC introduced in $BC(\mathbb{R}_+)$. Finally, applying (3.9) and Theorem 2.1, and putting $\beta(i) = i$, the proof ends. □

4 Conclusions

In this paper, established the existence selections for generalized multi-valued and single-valued mappings on complete metric spaces using some new generalizations of Darbo theorem. Also, obtained a relationship between coupled fixed-point and fixed-point. Finally, the main theorem was applied to a functional integral equation.

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