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Inequalities of Simpson-type for twice-differentiable convex functions via conformable fractional integrals

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Abstract

This paper proves an equality for the case of twice-differentiable convex functions involving conformable fractional integrals. Using the established equality, we give new Simpson-type inequalities for the case of twice-differentiable convex functions via conformable fractional integrals. We also consider some special cases which can be deduced from the main results.

Keywords: Simpson-type inequality, fractional conformable integrals, fractional calculus, convex function
2020 MSC: 26D10, 26D15, 26A51

1 Introduction and Preliminaries

The convexity of functions is a very important and fundamental concept in both areas of pure and applied mathematics. This function has attracted considerable attention and has been applied to various inequalities by many researchers. The most famous inequality which has been used with convex functions is Simpson type inequality. This inequality, a well-known technique of numerical integration and approximations for definite integrals, was discovered by Thomas Simpson.

Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable function on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Sarikaya et al. [19] introduced Simpson-type inequality for the case of twice-differentiable convex function, and they used the following lemma to prove the main inequalities.

Lemma 1.2. [19] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice-differentiable function on I^o such that $f'' \in L_1[a, b]$, where a,b $\in I$ with $a < b$. Then, we have the following equality

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx$$

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$$= (b-a)^2 \left[\int_0^{\frac{1}{2}} \frac{t}{2} \left(\frac{1}{3} - t \right) f''(tb + (1-t)a)\, dt + \int_{\frac{1}{2}}^{1} (1-t) \left(\frac{t}{2} - \frac{1}{3} \right) f''(tb + (1-t)a)\, dt \right].$$

Using Lemma 1.2, Sarikaya et al. [19] established the inequalities as follows

Theorem 1.3. [19] Assume that the assumptions of Lemma 1.2 hold. If $|f''|$ is convex on $[a,b]$, then the following inequality

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{162} \left[|f''(a)| + |f''(b)| \right].$$

is valid.

Theorem 1.4. [6] Let us consider that the conditions of Lemma 1.2 hold. If $|f''|^q$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &\leq \frac{(b-a)^2}{6} \left(\int_0^{\frac{1}{2}} t^p |1-3t|^p\, dt \right)^{\frac{1}{p}} \left[\left(\frac{|f''(a)|^q + 3|f''(b)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3|f''(a)|^q + |f''(b)|^q}{8} \right)^{\frac{1}{q}} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.5. Suppose that the assumptions of Lemma 1.2 hold. If $|f''|^q$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{(b-a)^2}{162} \left[\left(\frac{59|f''(a)|^q + 133|f''(b)|^q}{3 \times 2^6} \right)^{\frac{1}{q}} + \left(\frac{133|f''(a)|^q + 59|f''(b)|^q}{3 \times 2^6} \right)^{\frac{1}{q}} \right], \end{aligned}$$

which is given by [18, Proposition 7].

Simpson-type inequality via fractional calculus have been considered widely by many interested researchers. Fractional calculus is a field of mathematics that expands the traditional derivative and integral ideas to non-integer orders. The popularity of this topic among mathematicians continues to increase very strongly in resent years (see [5, 3]). Fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes, physics and engineering problems [9, 4, 8].

New studies have been discussed on developing a class of fractional integral operators and their applicability in a variety of scientific disciplines. Using only the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative was developed in paper [16]. Moreover, some important requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. By the way, the author [1] proved that the conformable approach in [16] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [20, 12].

Kilbas et al. [17] defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 1.6. [17] The Riemann-Liouville integrals $J_{a+}^{\beta} f(x)$ and $J_{b-}^{\beta} f(x)$ of order $\beta > 0$ are respectively given by

$$J_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)dt, \quad x < b, \tag{1.2}$$

where $f \in L_1[a,b]$ and Γ denotes the Gamma function. The Riemann-Liouville integrals reduces to the classical integrals in the case $\beta = 1$.

Budak et al. [6] obtained an equality for twice differentiable functions via Riemann-Liouville fractional integrals. Then, the authors established several fractional Simpson type inequalities for the case of functions whose second derivatives in absolute value are convex.

Lemma 1.7. [6] Let $f:[a,b]\to R$ be twice-differentiable function such that $f''\in L_1[a,b]$. Then, the following equality holds:

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}-}f(a)+J^{\beta}_{\frac{a+b}{2}+}f(b)\right]$$
$$=\frac{(b-a)^2}{6}\left[\int_0^{\frac{1}{2}}t\left(1-3\frac{2^{\beta}}{\beta+1}t^{\beta}\right)f''(tb+(1-t)a)\,dt+\int_{\frac{1}{2}}^{1}(1-t)\left(1-3\frac{2^{\beta}}{\beta+1}(1-t)^{\beta}\right)f''(tb+(1-t)a)\,dt\right].$$

Budak et al. [6] presented Simpson type inequality for the case of twice-differentiable convex mappings via fractional integrals using Lemma 1.7 to establish their main equalities. The authors [6] obtained the following Simpson type inequalities for convex functions.

Theorem 1.8. [6] Assume that the assumptions of Lemma 1.7 hold. Assume also that the mapping $|f''|$ is convex on $[a,b]$. Then, we have the following inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}+}f(b)+J^{\beta}_{\frac{a+b}{2}-}f(a)\right]\right|\le\frac{(b-a)^2}{6}\phi_1(\beta)\left[|f''(a)|+|f''(b)|\right],$$

where

$$\phi_1(\beta)=\frac{1}{4(\beta+2)}\left(\beta\left(\frac{\beta+1}{3}\right)^{\frac{2}{\beta}}+\frac{3}{\beta+1}\right)-\frac{1}{8}. \tag{1.3}$$

Theorem 1.9. [6] Suppose that the assumptions of Lemma 1.7 hold. Suppose also that the mapping $|f''|^q$, $q>1$ is convex on $[a,b]$. Then, the following inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}+}f(b)+J^{\beta}_{\frac{a+b}{2}-}f(a)\right]\right|$$
$$\le\frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}t^p\left|1-\frac{3\cdot2^{\beta}}{\beta+1}t^{\beta}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{8}\right)^{\frac{1}{q}}\right]$$

is valid. Here, $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.10. [6] Assume that the assumptions of Lemma 1.7 hold. If the mapping $|f''|^q$, $q\ge1$ is convex on $[a,b]$, then we have the following inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}+}f(b)+J^{\beta}_{\frac{a+b}{2}-}f(a)\right]\right|$$
$$\le\frac{(b-a)^2}{6}(\phi_1(\beta))^{1-\frac{1}{q}}\left[\left(\phi_2(\beta)|f''(b)|^q+(\phi_1(\beta)-\phi_2(\beta))|f''(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((\phi_1(\beta)-\phi_2(\beta))|f''(b)|^q+\phi_2(\beta)|f''(a)|^q\right)^{\frac{1}{q}}\right],$$

where $\phi_1(\beta)$ is defined as in (1.3) and

$$\phi_2(\beta)=\frac{1}{4(\beta+3)}\left[\frac{\beta}{3}\left(\frac{\beta+1}{3}\right)^{\frac{3}{\alpha}}+\frac{3}{2(\beta+1)}\right]-\frac{1}{24}.$$

Remark 1.11. For classical integrals,

(i) if we put $\beta = 1$, then Lemma 1.7 leads to Lemma 1.2.
(ii) By setting $\beta = 1$, then Theorem 1.8 reduces to Theorem 1.3.
(iii) Let us consider $\beta = 1$. Then, Theorem 1.9 becomes to Theorem 1.4.
(iv) Considering $\beta = 1$, then Theorem 1.10 equals to Theorem 1.5.

Jarad et al. [13] introduced the following fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined by as follows.

Definition 1.12. [13] For $f \in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}_{a}J^{\alpha} f(x)$ and ${}^{\beta}J^{\alpha}_{b} f(x)$ of order $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$ and $\alpha \in (0,1]$ are respectively given by

$$ {}^{\beta}J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \qquad t > a, \tag{1.4} $$

and

$$ {}^{\beta}J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt, \qquad t < b. \tag{1.5} $$

Note that, the fractional integral in (1.4) coincides with the Riemann-Liouville fractional integral in (1.1) when $a = 0$ and $\alpha = 1$. Moreover, the fractional integral in (1.5) coincides with the Riemann-Liouville fractional integral in (1.2) when $b = 0$ and $\alpha = 1$. Some recent results connected with fractional integral inequalities, see [10, 11, 7, 15, 14, 2] and the references cited therein.

With the help of the continuing studies and mentioned papers above, we will prove some new Simpson type inequalities associated with twice-differentiable convex function via conformable fractional integrals. The entire form of the study takes the form of three sections including introduction and Preliminaries. In section 2, we will prove an equality for the case twice-differentiable functions involving the conformable fractional integrals. Moreover, we will also show that the newly established equalities are the generalization of the existing Simpson type inequalities. Finally, summary and concluding remarks are given in Section 3.

2 Main Results

Lemma 2.1. If $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on (a,b) so that $f'' \in L_1([a,b])$ with $a < b$, then the following equality holds:

$$ \begin{aligned} & \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}} \left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-} f(a) + {}^{\beta}J^{\alpha}_{\frac{a+b}{2}+} f(b) \right] - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ & = \frac{(b-a)^2 \alpha^{\beta}}{2} \left\{ \int_0^{\frac{1}{2}} \left(\int_0^t \left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{3\alpha^{\beta}} \right] ds \right) f''(tb+(1-t)a)\, dt \right. \\ & \left. + \int_0^{\frac{1}{2}} \left(\int_0^t \left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{3\alpha^{\beta}} \right] ds \right) f''(ta+(1-t)b)\, dt \right\}. \end{aligned} \tag{2.1} $$

Proof . Let us consider

$$ \begin{aligned} & \frac{(b-a)^2 \alpha^{\beta}}{2} \left\{ \int_0^{\frac{1}{2}} \left(\int_0^t \left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{3\alpha^{\beta}} \right] ds \right) f''(tb+(1-t)a)\, dt \right. \\ & \left. + \int_0^{\frac{1}{2}} \left(\int_0^t \left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{3\alpha^{\beta}} \right] ds \right) f''(ta+(1-t)b)\, dt \right\} = \frac{(b-a)^2 \alpha^{\beta}}{2} \{A_1 + A_2\}. \end{aligned} \tag{2.2} $$

If we use the facts of the fundamental rules of integration by parts, then we obtain

$$
\begin{aligned}
A_1 &= \int_0^{\frac{1}{2}} \left(\int_0^t \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f''(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a} \left(\int_0^t \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'(tb+(1-t)a) \Big|_0^{\frac{1}{2}} \\
&\quad - \frac{1}{b-a} \int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2t)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] f'(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'\left(\frac{a+b}{2}\right) \\
&\quad - \frac{1}{b-a} \left\{ \frac{1}{b-a} \left[\left(\frac{1-(1-2t)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] f(tb+(1-t)a) \Big|_0^{\frac{1}{2}} \right. \\
&\quad \left. - \frac{2\beta}{b-a} \int_0^{\frac{1}{2}} \left(\frac{1-(1-2t)^\alpha}{\alpha} \right)^{\beta-1} (1-2t)^{\alpha-1} f(tb+(1-t)a)\,dt \right\} \\
&= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^2 3\alpha^\beta} \left(2f\left(\frac{a+b}{2}\right) + f(a) \right) \\
&\quad + \frac{2\beta}{(b-a)^3} \int_a^{\frac{a+b}{2}} \left(\frac{1-\left(\frac{2}{b-a}\right)^\alpha \left(\frac{a+b}{2}-x\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{2}{b-a} \right)^{\alpha-1} \frac{f(x)}{\left(\frac{a+b}{2}-x\right)^{1-\alpha}} dx \\
&= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^2 3\alpha^\beta} \left(2f\left(\frac{a+b}{2}\right) + f(a) \right) \\
&\quad + \left(\frac{2}{b-a} \right)^{\alpha\beta} \frac{\Gamma(\beta+1)}{(b-a)^2} \frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(x)}{\left(\frac{a+b}{2}-x\right)^{1-\alpha}} dx.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
A_1 &= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'\left(\frac{a+b}{2}\right) \\
&\quad - \frac{1}{(b-a)^2 3\alpha^\beta} \left(2f\left(\frac{a+b}{2}\right) + f(a) \right) + \left(\frac{2}{b-a} \right)^{\alpha\beta} \frac{\Gamma(\beta+1)}{(b-a)^2} \, {}^\beta J^\alpha_{\frac{a+b}{2}-} f(a).
\end{aligned} \tag{2.3}
$$

Then, similar to foregoing process, we have

$$
\begin{aligned}
A_2 &= -\frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \left[\left(\frac{1-(1-2s)^\alpha}{\alpha} \right)^\beta - \frac{1}{3\alpha^\beta} \right] ds \right) f'\left(\frac{a+b}{2}\right) \\
&\quad - \frac{1}{(b-a)^2 3\alpha^\beta} \left(2f\left(\frac{a+b}{2}\right) + f(b) \right) + \left(\frac{2}{b-a} \right)^{\alpha\beta} \frac{\Gamma(\beta+1)}{(b-a)^2} \, {}^\beta J^\alpha_{\frac{a+b}{2}+} f(b).
\end{aligned} \tag{2.4}
$$

If we substitute equalities (2.3) and (2.4) in the equality (2.2), then we can easily have

$$\frac{(b-a)^2\alpha^\beta}{2}\{A_1+A_2\}=\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\frac{a+b}{2}-}f(a)+{}^\beta J^\alpha_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

This completes the proof of Lemma 2.1. □

Remark 2.2. In Lemma 2.1, we have the equalities as follows:

(i) if we select $\alpha=1$ in (2.1), then Lemma 2.1 is equal to Lemma 1.7.
(ii) Let us consider $\alpha=1$ and $\beta=1$ in (2.1). Then, Lemma 2.1 reduces to Lemma 1.2.

Theorem 2.3. Let us consider that $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) such that $f''\in L_1([a,b])$. If $|f''|$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\frac{a+b}{2}-}f(a)+{}^\beta J^\alpha_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^\beta}{2}\varphi_1(\alpha,\beta)\left[|f''(a)|+|f''(b)|\right],\end{aligned}\tag{2.5}$$

where

$$\varphi_1(\alpha,\beta)=\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\right]ds\right|dt.$$

Proof . Let us take the absolute value of both sides of (2.1). Then, we get

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\frac{a+b}{2}-}f(a)+{}^\beta J^\alpha_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\right]ds\right||f''(tb+(1-t)a)|\,dt\right.\\&\left.+\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\right]ds\right||f''(ta+(1-t)b)|\,dt\right\}.\end{aligned}\tag{2.6}$$

Since $|f''|$ is convex on $[a,b]$, we get

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\frac{a+b}{2}-}f(a)+{}^\beta J^\alpha_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\right]ds\right|\left[t|f''(b)|+(1-t)|f''(a)|\right]dt\right.\\&\left.+\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^\alpha}{\alpha}\right)^\beta-\frac{1}{3\alpha^\beta}\right]ds\right|\left[t|f''(a)|+(1-t)|f''(b)|\right]dt\right\}.\end{aligned}$$

Thus, the proof of Theorem 2.3 is finished. □

Remark 2.4. In Theorem 2.3, we have the inequalities as follows:

(i) if it is chosen $\alpha=1$ in (2.5), then Theorem 2.3 becomes to Theorem 1.8.

(ii) For $\alpha=1$ and $\beta=1$ in (2.5), then Theorem 2.3 reduces to Theorem 1.3.

Theorem 2.5. Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) so that $f''\in L_1([a,b])$. If $|f''|^q$ is convex on $[a,b]$ with $q>1$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^{\beta}}{2}\psi^{\beta}_{\alpha}(p)\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f''(a)|^q+3|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.7}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi^{\beta}_{\alpha}(p)=\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|^p dt\right)^{\frac{1}{p}}.$$

Proof . If we apply Hölder inequality in (2.6), then we have

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|^p dt\right)^{\frac{1}{p}}\\&\quad\times\left[\left(\int_0^{\frac{1}{2}}|f''(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

From the fact that $|f''|^q$ is convex on $[a,b]$, it yields

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|^p dt\right)^{\frac{1}{p}}\\&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left(t|f''(b)|^q+(1-t)|f''(a)|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left(t|f''(a)|^q+(1-t)|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\right]\\&=\frac{(b-a)^2\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|^p dt\right)^{\frac{1}{p}}\\&\quad\times\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f''(a)|^q+3|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right].\end{aligned}$$

This ends the proof of Theorem 2.5. □

Remark 2.6. In Theorem 2.5, we obtain the inequalities as follows:

(i) if we assign $\alpha=1$ in (2.7), then Theorem 2.5 equals to Theorem 1.9.

(ii) Let us note $\alpha=1$ and $\beta=1$ in (2.7). Then, Theorem 2.5 reduces to Theorem 1.4.

Theorem 2.7. Assume $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) so that $f''\in L_1([a,b])$. If $|f''|^q$ is convex on $[a,b]$ with $q\geq 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq\frac{(b-a)^2\alpha^{\beta}}{2}(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}}\left[\left(\varphi_2(\alpha,\beta)|f''(b)|^q+(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))|f''(a)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\varphi_2(\alpha,\beta)|f''(a)|^q+(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))|f''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.8}$$

Here,

$$\varphi_2(\alpha,\beta)=\int_0^{\frac{1}{2}}t\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|dt.$$

Proof . Applying power-mean inequality in (2.6), we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right||f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right||f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is known that $|f''|^q$ is convex on $[a,b]$. Then, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\frac{a+b}{2}-}f(a)+{}^{\beta}J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|\left(t|f''(b)|^q+(1-t)|f''(a)|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|\left(t|f''(a)|^q+(1-t)|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\right]\\
&=\frac{(b-a)^2\alpha^{\beta}}{2}(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}}\left[\left(\varphi_2(\alpha,\beta)|f''(b)|^q+(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))|f''(a)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\varphi_2(\alpha,\beta)|f''(a)|^q+(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))|f''(b)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which completes the proof of Theorem 2.7. □

Remark 2.8. In Theorem 2.7, we get the inequalities as follows:

(i) Consider $\alpha = 1$ in (2.8). Then, Theorem 2.7 becomes to Theorem 1.10.
(ii) if we take $\alpha = 1$ and $\beta = 1$ in (2.8), then Theorem 2.7 reduces to Theorem 1.5.

3 Summary and concluding remarks

In this work, we established new estimates of Simpson type inequalities via conformable fractional integrals for the case of twice-differentiable convex functions. Our main results were proven to be generalizations of the Riemann-Liouville fractional integrals related to Simpson type inequalities. In future works, researchers can obtain similar inequalities of Simpson-type inequalities via conformable fractional integrals for convex functions by using quantum calculus. Furthermore, the ideas and strategies for our results concerning Simpson type inequalities via conformable fractional integrals may open new avenues for further research in this area.

References

[1] A.A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fract. Calc. Appl. Anal. **22** (2019), 242–254.

[2] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[3] G.A. Anastassiou, *Generalized Fractional Calculus: New Advancements and Applications*, Springer, Switzerland, 2021.

[4] N. Attia, A. Akgul, D. Seba, and A. Nour, *An efficient numerical technique for a biological population model of fractional order*, Chaos Solitons Fractals **141** (2020), 110349.

[5] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2016.

[6] H. Budak, H. Kara, and F. Hezenci, *Fractional Simpson-type inequalities for twice differentiable functions*, Sahand Commun. Math. Anal. **20** (2023), no. 3, 97–108.

[7] H. Desalegn, J.B. Mijena, E.R. Nwaeze, and T. Abdi, *Simpson's type inequalities for s-convex functions via generalized proportional fractional integral*, Foundations **2** (2022), no. 3, 607–616.

[8] A. Gabr, A.H. Abdel Kader, and M.S. Abdel Latif, *The effect of the parameters of the generalized fractional derivatives on the behavior of linear electrical circuits*, Int. J. Appl. Comput. Math. **7** (2021), 247.

[9] N. Iqbal, A. Akgul, R. Shah, A. Bariq, M.M. Al-Sawalha, and A. Ali, *On solutions of fractional-order gas dynamics equation by effective techniques*, J. Funct. Spaces **2022** (2022), 3341754.

[10] F. Hezenci, H. Budak, and H. Kara, *New version of fractional Simpson type inequalities for twice differentiable functions*, Adv. Difference Equ. **2021** (2021), no. 1, 1–10.

[11] H. Budak, F. Hezenci, and H. Kara, *On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integrals*, Math. Methods Appl. Sci. **44** (2021), no. 17, 12522–12536.

[12] A. Hyder and A.H. Soliman, *A new generalized θ-conformable calculus and its applications in mathematical physics*, Physica Scripta **96** (2020), 015208.

[13] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Adv. Difference Equ. **2017** (2017), 247.

[14] U.N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014), 1–15.

[15] S. Kermausuor, *Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions*, Kragujev. J. Math. **45** (2021), 709–720.

[16] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[17] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[18] M.Z. Sarikaya and N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Modell. **54** (2011), no. 9-10, 2175–2182.

[19] M.Z. Sarikaya, E. Set, and M.E. Ozdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, J. Appl. Math. Stat. Inf. **9** (2013), no. 1, 37–45.

[20] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo **54** (2017), 903–917.