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Fekete-Szegö problem for two new subclasses of bi-univalent functions defined by Bernoulli polynomial

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Abstract

This investigation deals with two new subclasses of analytic and bi-univalent functions defined by Bernoulli polynomial. In this paper, coefficient estimation and Fekete-Szegö problems are solved for these newly defined function subclasses. In addition, certain remarks are indicated for the subclasses of bi-starlike and bi-convex functions.

Keywords: Bi-univalent function, coefficient estimates, Fekete-Szegö functional, Bernoulli polynomials
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1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, which are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \Delta). \tag{1.1}$$

It is clear that the function f of the form (1.1) satisfy normalization conditions $f(0) = 0$ and $f'(0) = 1$. By $\mathcal{S}$ we show the subclass of $\mathcal{A}$ consisting of all functions, which are univalent in Δ. It is well-known that the familiar Koebe-$\frac{1}{4}$ theorem [9] makes sure that the image of Δ under every function $f \in \mathcal{S}$ contains a disk with radius $\frac{1}{4}$. Thus, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

It is emphasize here that every inverse function f^{-1} need not to be univalent in Δ. If f and f^{-1} are univalent in Δ, then $f \in \mathcal{S}$ is said to be bi-univalent in Δ, and the class of all analytic and bi-univalent functions defined in the unit disk Δ is donated by Σ. By using series expression of the function f of the form (1.1) one can see that inverse function f^{-1} may be expressed as below:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots := g(w). \tag{1.2}$$

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It is known that the functions

$$l_1(z) = \frac{z}{1-z} \text{ and } l_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

are in the function class $\mathcal{S}$. Moreover, the inverses of these functions are, respectively,

$$l_1^{-1}(w) = \frac{w}{1+w} \text{ and } l_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}.$$

Also, the functions $l_1^{-1}(w)$ and $l_2^{-1}(w)$ are in the class $\mathcal{S}$. So, these functions are in the function class Σ and the function class Σ is a non-empty set. In addition, the familiar Koebe function $k(z) = \frac{z}{(1-z)^2} \notin \Sigma$, since the third coefficient of the function $k^{-1}(z)$ is -5 and it does not satisfy Bieberbach conjecture.

There are a wide literature on some properties of analytic and bi-univalent functions. In the recent years, numerious papers have been published on this topic. For the recent developments in this field the interested readers can refer to the papers [1, 8, 10, 12, 13, 17, 18, 20, 21, 22, 23, 24, 25, 26] and references therein.

If the functions f and $F \in \mathcal{A}$, then f is said to be subordinate to F if there exists a Schwarz function w such that

$$w(0) = 0, |w(z)| < 1 \text{ and } f(z) = F(w(z)) \qquad (z \in \Delta).$$

This subordination is shown by

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \Delta).$$

If F is univalent function in Δ, then this subordination is equivalent to

$$f(0) = F(0), \quad f(\Delta) \subset F(\Delta).$$

There are comprehensive information about the subordination notion in Monographs written by Miller and Mocanu (see [15]).

In univalent function theory, one of the most attractive problems is known as the Fekete-Szegö problem [11, 14]. This problem is related to coefficients of the functions in the class $\mathcal{S}$ and it is expressed below:

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), \text{ for } 0 \leq \mu < 1.$$

The fundemental inequality $\left|a_3 - \mu a_2^2\right| \leq 1$ is obtained as $\mu \to 1$. The coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the open unit disk Δ has a significant impact on univalent function theory. The Fekete-Szegö problem is known as the maximization problem for functional $|F_\mu(f)|$.

Orthogonal polynomials such as Hermite, Laguerre, Jacobi and Bernoulli polynomials are of great importance in applied sciences. In recent years, mathematicians have built a bridge between geometric function theory and orthogonal polynomials. In [2, 3, 4, 6, 7, 19] the authors defined some new subclasses of analytic and univalent functions by using some orthogonal polynomials and they investigated coefficient estimation and Fekete-Szegö problems for the functions belonging to these function classes. In this paper, we define two new subclasses of analytic and bi-univalent functions by using Bernoulli polynomial and investigate initial coefficient estimation and Fekete-Szegö problems for the functions belonging to new classes.

Bernoulli polynomials are defined the following generating functions (see [16]):

$$F(x,z) = \frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k, \quad |z| < 2\pi, \tag{1.3}$$

where $B_k(x)$ is the k-th Bernoulli polynomial in variable x.

2 The Class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda,\delta)$

In this section we introduce a new function class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda,\delta)$ and investigate initial coefficient bounds estimations and the Fekete-Szegö inequality for this class.

Definition 2.1. Let $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. If the function $f(z) \in \Sigma$ of the form (1.1) satisfies the following conditions, then it is called in the class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda,\delta)$:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) \prec \frac{ze^{xz}}{e^z-1} = F(x,z), \tag{2.1}$$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) \prec \frac{we^{xw}}{e^w-1} = F(x,w), \tag{2.2}$$

where $\xi = \frac{2\lambda+\mu}{2\lambda+1}$ and the function g is of the form (1.2).

Remark 2.2. Taking $\lambda = 1$ and $\delta = \mu = 0$ in Definition 2.1 we obtain the class $\mathcal{Y}_{\Sigma}^{0}(1,0)$ of bi-starlike functions and it satisfies the following subordinations:

$$\frac{zf'(z)}{f(z)} \prec \frac{ze^{xz}}{e^z-1} = F(x,z), \tag{2.3}$$

$$\frac{wg'(w)}{g(w)} \prec \frac{we^{xw}}{e^w-1} = F(x,w). \tag{2.4}$$

Theorem 2.3. Suppose that $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. If $f \in \mathcal{Y}_{\Sigma}^{\mu}(\lambda,\delta)$, then

$$|a_2| \leq \frac{|B_1(x)|\sqrt{2|B_1(x)|}}{\sqrt{\left|B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\xi\delta)^2\right|}}, \tag{2.5}$$

$$|a_3| \leq \frac{B_1^2(x)}{(\lambda+\mu+2\xi\delta)^2} + \frac{|B_1(x)|}{\left|(\mu+2\lambda)(1+\frac{6\delta}{12\lambda+1})\right|} \tag{2.6}$$

and for $\eta \in \mathbb{R}$

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{|B_1(x)|}{(\mu+2\lambda)(1+\frac{6\delta}{12\lambda+1})}, & |1-\eta| \leq T(x,\lambda,\mu,\delta) \\ \frac{2|B_1(x)|^3|1-\eta|}{\left|B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\xi\delta)^2\right|}, & |1-\eta| \geq T(x,\lambda,\mu,\delta) \end{cases}, \tag{2.7}$$

where $T(x,\lambda,\mu,\delta) = \frac{\left|B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\xi\delta)^2\right|}{2B_1^2(x)(\mu+2\lambda)(1+\frac{6\delta}{12\lambda+1})}$.

Proof . Let $f(z) \in \mathcal{Y}_{\Sigma}^{\mu}(\lambda,\delta)$, $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. By Definition 2.1, there are two Schwarz functions $p, r : \Delta \to \Delta$,

$$p(z) = p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{2.8}$$

$$r(w) = r_1 w + r_2 w^2 + r_3 w^3 + \cdots \tag{2.9}$$

such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) = F(x,p(z)) \tag{2.10}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) = F(x,r(w)), \tag{2.11}$$

where $z, w \in \Delta$. It is well-known the definition of Schwarz function that $|p_i| \leq 1$ and $|r_i| \leq 1$ for $\forall i \in \mathbb{N}$. A basic calculation yields that right hand sides of the equations (2.10) and (2.11) are, respectively,

$$F(x,p(z)) = B_0(x) + [B_1(x)p_1]z + [B_1(x)p_2 + \frac{B_2(x)}{2!}p_1^2]z^2 + \left[B_1(x)p_3 + B_2(x)p_1p_2 + \frac{B_3(x)}{3!}p_1^3\right]z^3 + \cdots \tag{2.12}$$

and

$$F(x,r(w)) = B_0(x) + [B_1(x)r_1]w + [B_1(x)r_2 + \frac{B_2(x)}{2!}r_1^2]w^2 \quad + \left[B_1(x)r_3 + B_2(x)r_1r_2 + \frac{B_3(x)}{3!}r_1^3\right]w^3 + \cdots, \quad (2.13)$$

where $B_0(x) = 1$. In addition, left hand sides of the equations (2.10) and (2.11) are, respectively,

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) =$$
$$1 + (\lambda+\mu+2\delta\xi)a_2 z + (\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + \left(1+\frac{6\delta}{2\lambda+1}\right)a_3\right]z^2 + \cdots \quad (2.14)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) =$$
$$1 - (\lambda+\mu+2\delta\xi)a_2 w + (\mu+2\lambda)\left[\left(\frac{\mu+3}{2}+\frac{12\delta}{2\lambda+1}\right)a_2^2 - \left(1+\frac{6\delta}{2\lambda+1}\right)a_3\right]w^2 + \cdots. \quad (2.15)$$

Here, by comparing the coefficients of the equations (2.12) and (2.14) we obtain

$$(\lambda+\mu+2\delta\xi)a_2 = B_1(x)p_1 \quad (2.16)$$

$$(\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + (1+\frac{6\delta}{2\lambda+1})a_3\right] = B_1(x)p_2 + \frac{B_2(x)}{2!}p_1^2. \quad (2.17)$$

Also, by similar point of view from the equations (2.13) and (2.15) we have

$$-(\lambda+\mu+2\delta\xi)a_2 = B_1(x)r_1 \quad (2.18)$$

$$(\mu+2\lambda)\left[\frac{\mu+3}{2} + \frac{12\delta}{2\lambda+1}a_2^2 - (1+\frac{6\delta}{2\lambda+1})a_3\right] = B_1(x)r_2 + \frac{B_2(x)}{2!}r_1^2. \quad (2.19)$$

Now, from the equations (2.16) and (2.18), it follows that

$$p_1 = -r_1, \quad (2.20)$$

$$2(\lambda+\mu+2\delta\xi)^2 a_2^2 = B_1^2(x)(p_1^2+r_1^2) \quad (2.21)$$

and

$$a_2^2 = \frac{B_1^2(x)(p_1^2+r_1^2)}{2(\lambda+\mu+2\delta\xi)^2}. \quad (2.22)$$

Summation of the expressions (2.17) and (2.19) imply that

$$(\mu+2\lambda)\left(\mu+1+\frac{12\delta}{2\lambda+1}\right)a_2^2 = B_1(x)(p_2+r_2) + \frac{B_2(x)}{2}(p_1^2+r_1^2). \quad (2.23)$$

Using equation (2.22) in (2.23) one can easily see that

$$a_2^2 = \frac{B_1^3(x)(p_2+r_2)}{B_1^2(x)(\mu+2\lambda)\left(\mu+1+\frac{12\delta}{2\lambda+1}\right) - B_2(x)(\lambda+\mu+2\delta\xi)^2}. \quad (2.24)$$

Since $|p_i| \le 1, |r_i| \le 1$ for $\forall i \in \mathbb{N}$, by using triangle inequality in (2.24) we can write that

$$|a_2| \le \frac{|B_1(x)|\sqrt{2|B_1(x)|}}{\sqrt{\left|B_1^2(x)(\mu+2\lambda)\left(\mu+1+\frac{12\delta}{2\lambda+1}\right) - B_2(x)(\lambda+\mu+2\delta\xi)^2\right|}}. \quad (2.25)$$

On the other hand, if we subtract (2.17) from (2.19), then we get

$$a_3 = a_2^2 + \frac{B_1(x)(p_2 - r_2)}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}. \tag{2.26}$$

If we replace (2.22) in (2.26), then we can write that

$$a_3 = \frac{B_1^2(x)(p_1^2 + r_1^2)}{2(\lambda + \mu + 2\delta\xi)^2} + \frac{B_1(x)(p_2 - r_2)}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}. \tag{2.27}$$

Now, using triangle inequality in the last equality we deduce

$$|a_3| \leq \frac{B_1^2(x)}{(\lambda + \mu + 2\delta\xi)^2} + \frac{|B_1(x)|}{\left|(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)\right|} \tag{2.28}$$

Finally, taking into account the equations (2.24) and (2.26), we can write that

$$\begin{aligned} a_3 - \eta a_2^2 =& \frac{(1-\eta)B_1^3(x)(p_2 + r_2)}{B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda + \mu + 2\delta\xi)^2} + \frac{B_1(x)(p_2 - r_2)}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)} \\ =& \left[\frac{(1-\eta)B_1^3(x)}{B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda + \mu + 2\delta\xi)^2} + \frac{B_1(x)}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right] p_2 \\ &+ \left[\frac{(1-\eta)B_1^3(x)}{B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda + \mu + 2\delta\xi)^2} - \frac{B_1(x)}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right] r_2 \\ =& B_1(x)\left\{\left[t(\eta) + \frac{1}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right] p_2 + \left[t(\eta) - \frac{1}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right] r_2\right\} \end{aligned}$$

for $\eta \in \mathbb{R}$, where $t(\eta) = \frac{(1-\eta)B_1^2(x)}{B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2}$. A straightforward calculation implies here that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq |B_1(x)| \left\{\left|t(\eta) - \frac{1}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right| + \left|t(\eta) + \frac{1}{2(\mu + 2\lambda)\left(1 + \frac{6\delta}{12\lambda+1}\right)}\right|\right\} \\ &= \begin{cases} \frac{|B_1(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{12\lambda+1}\right)}, & |1-\eta| \leq T(x,\lambda,\mu,\delta) \\ \frac{2|B_1(x)|^3|1-\eta|}{\left|B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2\right|}, & |1-\eta| \geq T(x,\lambda,\mu,\delta) \end{cases}. \end{aligned}$$

The proof is thus completed. □

Remark 2.4. Taking $\lambda = 1$ and $\delta = \mu = 0$ in Theorem 2.3 we obtain some bounds for the class $\mathcal{Y}_\Sigma^0(1,0)$ of bi-starlike functions as below:

$$|a_2| \leq \frac{B_1(x)\sqrt{2\,|B_1(x)|}}{\sqrt{|2B_1^2(x) - B_2(x)|}}, \tag{2.29}$$

$$|a_3| \leq B_1^2(x) + \frac{|B_1(x)|}{2} \tag{2.30}$$

and for some $\eta \in \mathbb{R}$

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{|B_1(x)|}{2}, & |1-\eta| \leq \frac{\left|2B_1^2(x) - B_2(x)\right|}{4B_1^2(x)} \\ \frac{2|B_1(x)|^3|1-\eta|}{\left|2B_1^2(x) - B_2(x)\right|}, & |1-\eta| \geq \frac{\left|2B_1^2(x) - B_2(x)\right|}{4B_1^2(x)}. \end{cases} \tag{2.31}$$

3 The Class $\mathcal{Y}_\Sigma(\zeta,\gamma)$

In this section we define a new function class $\mathcal{Y}_\Sigma(\zeta,\gamma)$ and determine some bounds for initial coefficients and the Fekete-Szegö functional.

Definition 3.1. Let $\gamma>0$ and $\zeta>0$. If the function $f(z)\in\Sigma$ of the form (1.1) satisfies the following conditions, then it is called in the class $\mathcal{Y}_\Sigma(\zeta,\gamma)$:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\zeta)z+\zeta zf'(z)}-1\right)\prec\frac{ze^{xz}}{e^z-1}, \tag{3.1}$$

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+w^2g''(w)}{(1-\zeta)w+\zeta wg'(w)}-1\right)\prec\frac{we^{xw}}{e^w-1}, \tag{3.2}$$

where the function g is of the form (1.2).

Remark 3.2. Taking $\zeta=\gamma=1$ in Definition 3.1 we obtain the class $\mathcal{Y}_\Sigma(1,1)$ of bi-convex functions and it satisfies the following subordinations:

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{ze^{xz}}{e^z-1}=F(x,z), \tag{3.3}$$

$$1+\frac{wg''(w)}{g'(w)}\prec\frac{we^{xw}}{e^w-1}=F(x,w). \tag{3.4}$$

Theorem 3.3. Suppose that $\zeta\neq 2,\ 0\leq\zeta<3,$ and $\gamma>0$. If $f\in\mathcal{Y}_\Sigma(\zeta,\gamma)$, then

$$|a_2|\leq\frac{\gamma\,|B_1(x)|\,\sqrt{|B_1(x)|}}{\sqrt{|(4\zeta^2-11\zeta+9)B_1^2(x)\gamma-B_2(x)2(2-\zeta)^2|}}, \tag{3.5}$$

$$|a_3|\leq\frac{B_1^2(x)\gamma^2}{4(2-\zeta)^2}+\frac{|B_1(x)|\,\gamma}{|3(3-\zeta)|} \tag{3.6}$$

and for $\eta\in\mathbb{R}$

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{\gamma^2|B_1(x)|}{3(3-\zeta)}, & |1-\eta|\leq\frac{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2-2B_2(x)(2-\zeta)^2]\right|}{3\gamma^2B_1^2(x)(3-\zeta)}\\ \frac{\gamma^2|B_1(x)|^3|1-\eta|}{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2-2B_2(x)(2-\zeta)^2]\right|}, & |1-\eta|\geq\frac{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2-2B_2(x)(2-\zeta)^2]\right|}{3\gamma^2B_1^2(x)(3-\zeta)}\end{cases}. \tag{3.7}$$

Proof . Let $\zeta\neq 2,\ 0\leq\zeta<3,\ \gamma>0$ and $f(z)\in\mathcal{Y}_\Sigma(\zeta,\gamma)$. By Definition 3.1, there are two Schwarz functions $u,v:\Delta\to\Delta$,

$$u(z)=u_1z+u_2z^2+u_3z^3+\ldots, \tag{3.8}$$

$$v(w)=v_1w+v_2w^2+v_3w^3\ldots \tag{3.9}$$

such that

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\zeta)z+\lambda zf'(z)}-1\right)=F(x,u(z)) \tag{3.10}$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+w^2g''(w)}{(1-\zeta)w+\zeta wg'(w)}-1\right)=F(x,v(w)), \tag{3.11}$$

where $z,w\in\Delta$. It is well-known the definition of Schwarz function that $|u_i|\leq 1$ and $|v_i|\leq 1$ for $\forall i\in\mathbb{N}$. A basic calculation yields that right hand sides of the equations (3.10) and (3.11) are, respectively,

$$F(x,u(z))=B_0(x)+[B_1(x)u_1]z+[B_1(x)u_2+\frac{B_2(x)}{2!}u_1^2]z^2+\left[B_1(x)u_3+B_2(x)u_1u_2+\frac{B_3(x)}{3!}u_1^3\right]z^3+\cdots \tag{3.12}$$

and

$$F(x, v(w)) = B_0(x) + [B_1(x)v_1]w + [B_1(x)v_2 + \frac{B_2(x)}{2!}v_1^2]w^2 + \left[B_1(x)v_3 + B_2(x)v_1v_2 + \frac{B_3(x)}{3!}v_1^3\right]w^3 + \cdots, \quad (3.13)$$

where $B_0(x) = 1$. In addition, left hand sides of the equations (3.10) and (3.11) are, respectively,

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2f''(z)}{(1-\zeta)z + \zeta zf'(z)} - 1\right) = 1 + \frac{2(2-\zeta)}{\gamma}a_2z + \frac{3(3-\zeta)a_3 - 4\zeta(2-\zeta)a_2^2}{\gamma}z^2 + \cdots \quad (3.14)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + w^2g''(w)}{(1-\zeta)w + \zeta wg'(w)} - 1\right) = 1 - \frac{2(2-\zeta)}{\gamma}a_2w + \frac{3(3-\zeta)(2a_2^2 - a_3) - 4\zeta(2-\zeta)a_2^2}{\gamma}w^2 + \cdots. \quad (3.15)$$

Here, by comparing the coefficients of the equations (3.12) and (3.14) we obtain

$$\frac{2(2-\zeta)}{\gamma}a_2 = B_1(x)u_1 \quad (3.16)$$

and

$$\frac{3(3-\zeta)a_3 - 4\zeta(2-\zeta)a_2^2}{\gamma} = B_1(x)u_2 + \frac{B_2(x)}{2!}u_1^2. \quad (3.17)$$

Also, by similar point of view from the equations (3.13) and (3.15) we have

$$-\frac{2(2-\zeta)}{\gamma}a_2 = B_1(x)v_1 \quad (3.18)$$

and

$$\frac{3(3-\zeta)(2a_2^2 - a_3) - 4\zeta(2-\zeta)a_2^2}{\gamma} = B_1(x)v_2 + \frac{B_2(x)}{2!}v_1^2. \quad (3.19)$$

Now, from the equations (3.16) and (3.18), it follows that

$$u_1 = -v_1, \quad (3.20)$$

$$8(2-\zeta)^2a_2^2 = B_1^2(x)(u_1^2 + v_1^2)\gamma^2 \quad (3.21)$$

and

$$a_2^2 = \frac{B_1^2(x)(u_1^2 + v_1^2)\gamma^2}{8(2-\zeta)^2}. \quad (3.22)$$

Summation of the expressions 3.17 and 3.19 imply that

$$\frac{(8\zeta^2 - 22\zeta + 18)}{\gamma}a_2^2 = B_1(x)(u_2 + v_2) + \frac{B_2(x)4(2-\zeta)^2a_2^2}{B_1^2(x)\gamma^2}. \quad (3.23)$$

Using equation (3.22) in (3.23) one can easily see that

$$a_2^2 = \frac{B_1^3(x)(u_2 + v_2)\gamma^2}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]}. \quad (3.24)$$

Since $|u_i| \le 1, |v_i| \le 1$ for $\forall i \in \mathbb{N}$, by using triangle inequality in (3.24) we can write that

$$|a_2| \le \frac{\gamma\,|B_1(x)|\sqrt{2\,|B_1(x)|}}{\sqrt{2\,|(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2|}}. \quad (3.25)$$

On the other hand, if we subtract (3.17) from (3.19), then we get

$$a_3 = a_2^2 + \frac{B_1(x)(u_2 - v_2)\gamma}{6(3-\zeta)}. \quad (3.26)$$

If we replace (3.22) in (3.26), then we can write that

$$a_3 = \frac{B_1^2(x)(u_1^2+v_1^2)\gamma^2}{8(2-\zeta)^2} + \frac{B_1(x)(u_2-v_2)\gamma}{6(3-\zeta)}. \tag{3.27}$$

Now, using triangle inequality in the last equality we deduce

$$|a_3| \le \frac{B_1^2(x)\gamma^2}{4(2-\zeta)^2} + \frac{|B_1(x)|\,\gamma}{|3(3-\zeta)|}. \tag{3.28}$$

Finally, taking into acount the equations (3.24) and (3.26), we can write that

$$\begin{aligned} a_3 - \eta a_2^2 =& \frac{(1-\eta)B_1^3(x)(u_2+v_2)\gamma^2}{2[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} + \frac{B_1(x)(u_2-v_2)\gamma}{6(3-\zeta)} \\ =& B_1(x)\gamma\left[\frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} + \frac{1}{6(3-\zeta)}\right]u_2 \\ &+ B_1(x)\gamma\left[\frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} - \frac{1}{6(3-\zeta)}\right]v_2 \\ =& B_1(x)\gamma\left\{\left[\kappa(\eta) + \frac{1}{6(3-\zeta)}\right]u_2 + \left[\kappa(\eta) - \frac{1}{6(3-\zeta)}\right]v_2\right\} \end{aligned}$$

for $\eta \in \mathbb{R}$, where $\kappa(\eta) = \frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]}$. A straightforward calculation implies here that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\le |B_1(x)|\,\gamma\left\{\left|\kappa(\eta) + \frac{1}{6(3-\zeta)}\right| + \left|\kappa(\eta) - \frac{1}{6(3-\zeta)}\right|\right\} \\ &\le \begin{cases} \frac{\gamma^2|B_1(x)|}{3(3-\zeta)}, & |1-\eta| \le \frac{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]\right|}{3\gamma^2 B_1^2(x)(3-\zeta)}, \\ \frac{\gamma^2|B_1(x)|^3|1-\eta|}{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]\right|}, & |1-\eta| \ge \frac{\left|[(4\zeta^2-11\zeta+9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]\right|}{3\gamma^2 B_1^2(x)(3-\zeta)}. \end{cases} \end{aligned}$$

□

Remark 3.4. Taking $\gamma = \zeta = 1$ in Theorem 3.3 we obtain some bounds for the class $\mathcal{Y}_\Sigma(1,1)$ of bi-convex functions as below:

$$|a_2| \le \frac{B_1(x)\sqrt{|B_1(x)|}}{\sqrt{2\left|B_1^2(x) - B_2(x)\right|}}, \tag{3.29}$$

$$|a_3| \le \frac{B_1^2(x)}{4} + \frac{|B_1(x)|}{6} \tag{3.30}$$

and

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{|B_1(x)|}{6}, & |1-\eta| \le \frac{\left|B_1^2(x) - B_2(x)\right|}{3B_1^2(x)} \\ \frac{|B_1(x)|^3|1-\eta|}{2\left|B_1^2(x) - B_2(x)\right|}, & |1-\eta| \ge \frac{\left|B_1^2(x) - B_2(x)\right|}{3B_1^2(x)} \end{cases}. \tag{3.31}$$

4 Conclusion

In the present investigation two new subclasses of analytic and bi-univalent functions are introduced by using Bernoulli polynomial. Also, some coefficient bounds are estimated for certain coefficients of functions belonging to these subclasses defined. In addition, the Fekete-Szegö problem are handled for the mentioned function subclasses. Finally, a few remarks are indicated for the certain function subclasses which are related to bi-starlike and bi-convex functions.

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